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PROXIMAL POINT ALGORITHMS FOR FINDING COMMON FIXED POINTS OF A FINITE FAMILY OF NONEXPANSIVE MULTIVALUED MAPPINGS IN REAL HILBERT SPACES

AKINDELE ADEBAYO MEBAWONDU

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Abstract. We start by showing that the composition of fixed point of minimization problem and a finite family of multivalued nonexpansive mapping are equal to the common solution of the fixed point of each of the aforementioned problems, that is, $F(J_\lambda^f \circ T_i) = F(J_\lambda^f) \cap F(T_i) = \Gamma$. Furthermore, we then propose an iterative algorithm and prove weak and strong convergence results for approximating the common solution of the minimization problem and fixed point problem of a multivalued nonexpansive mapping in the framework of real Hilbert space. Our result extends and complements some related results in literature.

1. Introduction

Let H be a real Hilbert space and let K be a nonempty subset of H. Let $CB(K)$ be the collection of all nonempty, closed, and bounded subsets of K and let $C(K)$ be the collection of all nonempty compact subsets of K. Let H be a Pompeiu–Hausdorff metric induced by the metric d on $CB(K)$ defined by

$$H(A,B) = \max\{\sup_{x\in A} \operatorname{dist}(x,B), \sup_{y\in B} \operatorname{dist}(x,A)\} \qquad \text{for all } A,B \in CB(K).$$

Let $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is said to be a fixed point of T, if $x \in Tx$.

Definition 1.1. A multivalued mapping $T : K \to CB(K)$ is said to be

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(1) contraction, if there exists a constant $\alpha \in (0,1)$ such that

$$H(Tx,Ty) \le \alpha d(x,y) \qquad \text{for all } x,y \in K;$$

(2) nonexpansive, if

$$H(Tx,Ty) \le d(x,y) \qquad \text{for all } x,y \in K;$$

(3) quasi-nonexpansive, if $F(T) \neq \emptyset$ and

$$H(Tx,Tx^*) \le d(x,x^*) \qquad \text{for all } x \in K, x^* \in F(T);$$

(4) nonspreading [14] if

$$2H(Tx,Ty)^2 \le \operatorname{dist}(y,Tx)^2 + \operatorname{dist}(x,Ty)^2 \qquad \text{for all } x,y \in K;$$

(5) λ-hybrid [15], if there exists $\lambda \in \mathbb{R}$ such that, for all $x,y \in K$, it follows that

$$(1+\lambda)H(Tx,Ty)^2 \le (1-\lambda)\|x-y\|^2 + \lambda \operatorname{dist}(y,Tx)^2 + \lambda \operatorname{dist}(x,Ty)^2.$$

The minimization problem (MP) is one of the most important problems in nonlinear analysis and optimization theory. The MP is defined as follows: Finding $x \in H$ such that

$$f(x) = \min_{y \in H} f(y),$$

where $f : H \to (-\infty, \infty]$ is a proper and convex function. The set of all minimizers of f on H is denoted by $\operatorname{argmin}_{y\in H} f(y)$. In 1970, Martinet [10] introduced and studied the proximal point algorithm (PPA) for solving optimization problems. Thereafter the likes of Rockafellar [13], find a solution of the constrained convex minimization problem in the frame work of Hilbert space by using PPA. Let f be a proper convex and lower semicontinuous function on H. The PPA is defined as

$$\begin{cases} x_1 \in H \\ x_{n+1} = \operatorname{argmin}_{u\in H}[f(u) + \frac{1}{2\lambda_n}\|u_n - x_n\|^2], \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $\lambda_n > 0$. He established that if f has a minimizer and $\sum_{n=1}^{\infty} \lambda_n = \infty$, then the sequence $\{x_n\}$ converges weakly to a minimizer in f. In [6], it was shown that a PPA does not necessarily converges strongly. The fact that a PPA does not necessarily converges strongly have been overcome by researchers in this area by introducing a more general PPA in different spaces to obtain a weak and strong convergence (see [9] and the references therein). Over the years, researcher have been able to further extend the convex minimization problems by finding a common element of the set of solutions of various convex minimization problems and the set of fixed points for a single and multivalued mapping in Hilbert spaces and Banach spaces (see [2, 3] and the references therein).

Chang, Wu, Wang, and Wang [5] introduced and studied the PPA-Ishikawa iteration process, which is defined as

$$\begin{cases} y_n = \operatorname{argmin}_{u\in H}[f(u) + \frac{1}{2\lambda_n}\|u_n - x_n\|^2], \\ z_n = (1-\beta_n)x_n + \beta_n w_n, \quad w_n T y_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n v_n, \quad v_n T z_n, \ n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, K is a nonempty closed convex subset of a Hilbert space H, $f : K \to (\infty, \infty]$ is a proper convex and lower semicontinuous function, and $T : K \to C(K)$ is a nonspreading multivalued mapping. Using (1.2), they proved weak convergence and strong convergence theorems for minimizers of proper convex and lower semicontinuous functions and fixed points of nonspreading multivalued mappings in Hilbert spaces.

Lerkchaiyaphum and Phuengraitana [8] introduced and studied a new iteration process called PPA-S-iteration, which has to do by combining the PPA and S-iteration process together. The iterative process is defined as follows:

$$\begin{cases} y_n = \operatorname{argmin}_{u\in H}[f(u) + \frac{1}{2\lambda_n}\|u_n - x_n\|^2], \\ z_n = (1-\beta_n)x_n + \beta_n w_n, \quad w_n T y_n, \\ x_{n+1} = (1-\beta_n)w_n + \beta_n v_n, \quad v_n T z_n, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, K is a nonempty closed convex subset of a Hilbert space H, $f : K \to (\infty, \infty]$ is a proper convex and lower semicontinuous function, and $T : K \to C(K)$ is a nonspreading multivalued mapping. Using (1.3), they proved a weak convergence theorem for minimizers of proper convex and lower semicontinuous functions and fixed points of λ-hybrid multivalued mappings in Hilbert spaces.

Very recently, Phuengraitana and Tiammee [12] introduced and studied a new iteration process defined as

$$\begin{cases} u_n = y_n^{(0)} = \operatorname{argmin}_{u\in H}[f(u) + \frac{1}{2\lambda_n}\|u_n - x_n\|^2], \\ y_n^{(1)} = (1-\alpha_n^1)x_n + \alpha_n^1 w_n^1, \\ y_n^{(2)} = (1-\alpha_n^2)x_n + \alpha_n^2 w_n^2, \\ \vdots \\ y_n^{(m-1)} = (1-\alpha_n^{m-1})x_n + \alpha_n^{m-1} w_n^{m-1} \\ x_{n+1} = y_n^m = (1-\alpha_n^m)x_n + \alpha_n^m w_n^m, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $w_n^i \in T_i y_n^{i-1}, \{\alpha_n^i\}$, is a sequence in $[0,1]$, C is a nonempty closed convex subset of a Hilbert space H, $f : K \to (\infty, \infty]$ is a proper convex and lower semicontinuous function, and $T : K \to CB(K)$ is a multivalued mapping satisfying condition (E). Using (1.4), they proved weak and strong convergence theorems for minimizers of proper convex and lower semicontinuous functions and fixed points of multivalued mappings satisfying condition (E) in Hilbert spaces.

In the research described above, we note that in the iterative processes, we will arrive at a point say $T(J_\lambda^f x_n)$, where $J_\lambda^f = \operatorname{argmin}_{u\in H}\left[f(u) + \frac{1}{2\lambda}\|u - x\|^2\right]$ for all $x \in H$. In view of this, the authors assume that $F(T \circ J_\lambda^f) = F(T) \cap F(J_\lambda^f)$.

Question 1: Is it always true that $F(T \circ J_\lambda^f) = F(T) \cap F(J_\lambda^f)$, regardless of the nonlinear mappings in question?

Motivated by the above work and the research in this direction, we propose a new proximal point algorithm which is a modification of (1.1) for finding a common element of the set of common fixed points of a finite family of nonexpansive multivalued mappings and the set of minimizers of convex and lower semicontinuous functions. We provide an affirmative answer to the above question raised for a finite family of a multivalued nonexpansive mapping and prove weak and strong convergence of the proposed algorithm to a common element of the set of common fixed points of a finite family of nonexpansive multivalued mappings and the set of minimizers of convex and lower semicontinuous functions in real Hilbert spaces.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We denote the strong convergence by " $\longrightarrow$ " and the weak convergence by " $\rightharpoonup$ " of any arbitrary sequence, say $\{x_n\}$ to a point $x \in H$. It is known [11] that a Hilbert space satisfies Opial's condition.

Let $f : H \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. For any $\lambda > 0$, the Moreau–Yosida resolvent of f in H is defined as:

$$J_\lambda^f = \operatorname{argmin}_{u \in H}\left[f(u) + \frac{1}{2\lambda}\|u - x\|^2\right] \quad \text{for all } x \in H.$$

It is an established result (see[6, 7]) that the set of fixed points of the J_λ associated with f coincides with the set of minimizers of f and that J_λ^f is nonexpansive.

Lemma 2.1. [1] *Let H be a real Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Then for all $x, y \in H$ and $\lambda > 0$, the following subdifferential inequality holds:*

$$\frac{1}{2\lambda}\|J_\lambda x - y\|^2 - \frac{1}{2\lambda}\|x - y\|^2 + \frac{1}{2\lambda}\|x - J_\lambda x\|^2 \le f(y) - f(J_\lambda x).$$

Lemma 2.2. [7] *Let H be a real Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Then for all $x \in H$ and $\lambda > \mu_n > 0$, the following identity holds:*

$$J_\lambda x = J_\mu\left(\frac{\lambda - \mu}{\lambda} J_\lambda x + \frac{\mu}{\lambda} x\right).$$

Lemma 2.3. [4] *Let H be a real Hilbert space and let $T : H \to H$ be a nonexpansive mapping. If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ and that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$, then $Tx = x$.*

Lemma 2.4. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^n$ be a finite family of a multivalued nonexpansive mapping of K into $CB(K)$. If $\{x_n\}$ is a sequence in H such that $x_n \rightharpoonup x$ and that $\lim_{n\to\infty} \operatorname{dist}(x_n, T_i x_n) = 0$, then $x \in Tx$.*

Proof. The proof is similar to that of Lemma 2.3, thus we omit it. □

Lemma 2.5. *Let K be a nonempty closed convex subset of a real Hilbert space H and let $\{T_i\}_{i=1}^N$ be a finite family of a multivalued nonexpansive mapping of H into $CB(H)$ such that $T_i x = \{x\}$ for all $x \in \cap_{i=1}^N F(T_i)$. also let $f : K \to (-\infty, \infty]$*

be a proper convex and lower semicontinuous function and let J_λ^f be the resolvent for any $\lambda > 0$. Then $F(T_i) \cap F(J_\lambda^f) = F(T_i \circ J_\lambda^f)$.

Proof. Firstly, we note that $T_i \circ J_\lambda^f$ is a multivalued mapping. It is clear that $F(T_i) \cap F(J_\lambda^f) \subseteq F(T_i \circ J_\lambda^f)$. Thus, we only need to show that $F(T_i \circ J_\lambda^f) \subseteq F(T_i) \cap F(J_\lambda^f)$. To do this let $x \in F(T_i \circ J_\lambda^f)$ and $y \in F(T_i) \cap F(J_\lambda^f)$. We have

$$\begin{aligned} \|x - y\|^2 &\leq H(T_i(J_\lambda^f x), T_i(J_\lambda^f y))^2 \\ &\leq \|J_\lambda^f x - J_\lambda^f y\|^2 \\ &= \|J_\lambda^f x - y\|^2. \end{aligned} \tag{2.1}$$

From Lemma 2.1, since $f(y) \leq f(J_\lambda^f x)$, we have

$$\|J_\lambda^f x - y\|^2 \leq \|x - y\|^2 - \|J_\lambda^f x - x\|^2. \tag{2.2}$$

Using (2.1) implies that (2.2) becomes

$$\begin{aligned} \|J_\lambda^f x - y\|^2 &\leq \|x - y\|^2 - \|J_\lambda^f x - x\|^2 \\ &\leq \|J_\lambda^f x - y\|^2 - \|J_\lambda^x - x\|^2. \end{aligned}$$

Clearly, $\|J_\lambda^f x - x\| = 0$, which implies that

$$J_\lambda^f x = x. \tag{2.3}$$

Keeping in mind that $T_i x = \{x\}$, for all $x \in \cap_{i=1}^n F(T_i)$, we have

$$x = J_\lambda^f x \in T_i(J_\lambda^f x) = T_i x.$$

Thus, $x \in F(T_i) \cap F(J_\lambda^f)$ for all $i = 1, 2, \ldots, N$. This complete the proof. $\square$

3. Main result

In this section, we prove some weak and strong convergence theorems for common fixed points of a finite family of nonexpansive multivalued mappings and minimizers of convex and lower semicontinuous functions in real Hilbert spaces.

Theorem 3.1. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be finite family of nonexpansive multivalued mapping of K into $CB(K)$ and let $f : K \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Suppose that $\Gamma = \cap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{u \in K} f(u) \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in \cap_{i=1}^N F(T_i)$. For $x_1 \in K$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = J_\lambda^f x_n, \\ z_n = \gamma_n^0 x_n + \sum_{i=1}^N \gamma_n^i w_n^i \\ x_{n+1} = \alpha_n^0 z_n + \sum_{i=1}^N \alpha_n^i T_i z_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.1}$$

where $w_n^i \in T_i y_n, \sum_{i=0}^N \alpha_n^i = \sum_{i=0}^N \gamma_n^i = 1, \{\alpha_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $(0,1)$ for all $i = 1, 2 \ldots, N$ and $\lambda_n > \lambda > 0$. Then the sequence $\{x_n\}$ converges weakly to an element of Γ.

Proof. Since $T_i p = \{p\}$, using (3.1) and the fact that J_λ^f is nonexpansive, it holds that

$$\begin{aligned}
\|z_n - p\| &\leq \|\gamma_n^0 x_n + \sum_{i=1}^{N} \gamma_n^i w_n^i - p\| \\
&\leq \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \|w_n^i - p\| \\
&= \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \mathrm{dist}(w_n^i, T_i p) \\
&\leq \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i H(T_i y_n, T_i p) \qquad (3.2) \\
&\leq \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \|y_n - p\| \\
&=\leq \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \|J_\lambda^f x_n - p\| \\
&\leq \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

Using (3.1) and (3.2), we obtain

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n^0 \|z_n - p\| + \sum_{i=1}^{N} \alpha_n^i \|T_i z_n - p\| \\
&\leq \alpha_n^0 \|z_n - p\| + \sum_{i=1}^{N} \alpha_n^i \|z_n - p\| \\
&= \|z_n - p\| \qquad (3.3) \\
&\leq \|x_n - p\|.
\end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is decreasing and bounded below; therefore, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Gamma$.

Now suppose that $\lim_{n\to\infty} \|x_n - p\| = c$ for some $c > 0$. From (3.2), we have

$$\|z_n - p\| \leq \|x_n - p\|,$$

which implies that

$$\limsup_{n\to\infty} \|z_n - p\| \leq c. \qquad (3.4)$$

Also, form (3.3), we have

$$\|x_{n+1} - p\| \leq \|z_n - p\|,$$

which implies that

$$c \leq \liminf_{n\to\infty} \|z_n - p\|. \tag{3.5}$$

Using (3.4) and (3.5) implies

$$\lim_{n\to\infty} \|z_n - p\| = c. \tag{3.6}$$

In addition, we have

$$\|y_n - p\| = \|J_\lambda^f x_n - p\| \leq \|x_n - p\|,$$

which implies that

$$\limsup_{n\to\infty} \|y_n - p\| \leq c. \tag{3.7}$$

From (3.2), it follows that

$$\begin{aligned}
\|z_n - p\| &\leq \gamma_n^0 \|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \|y_n - p\| \\
&= (1 - \sum_{i=1}^{N} \gamma_n^i)\|x_n - p\| + \sum_{i=1}^{N} \gamma_n^i \|y_n - p\| \\
&\Rightarrow \|x_n - p\| \leq \frac{1}{\sum_{i=1}^{N} \gamma_n^i}[\|x_n - p\| + \|z_n - p\|] + \|y_n - p\|,
\end{aligned}$$

which implies that

$$c \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{3.9}$$

Using Lemma 2.1 and since $f(p) \leq f(y_n)$, we get

$$\|x_n - y_n\|^2 \leq \|x_n - p\|^2 - \|y_n - p\|^2,$$

and hence

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{3.10}$$

Furthermore

$$
\begin{aligned}
\|z_n - p\|^2 &= \|\gamma_n^0 x_n + \sum_{i=1}^{N} \gamma_n^i w_n^i - p\|^2 \\
&= \|\gamma_n^0 (x_n - p) + (1-\gamma_n^0)(w_n^i - p)\|^2 \\
&= \gamma_n^0 \|x_n - p\|^2 + (1-\gamma_n^0)\|w_n^i - p\|^2 - \gamma_n^0(1-\gamma_n^0)\|x_n - w_n^i\|^2 \\
&= \gamma_n^0 \|x_n - p\|^2 + (1-\gamma_n^0)\mathrm{dist}(w_n^i, T_i p)^2 - \gamma_n^0(1-\gamma_n^0)\|x_n - w_n^i\|^2 \\
&\leq \gamma_n^0 \|x_n - p\|^2 + (1-\gamma_n^0) H(T_i y_n^i, T_i p)^2 - \gamma_n^0(1-\gamma_n^0)\|x_n - w_n^i\|^2 \\
&\leq \|x_n - p\|^2 - \gamma_n^0(1-\gamma_n^0)\|x_n - w_n^i\|^2 \\
&\Rightarrow \|x_n - w_n^i\|^2 \leq \frac{1}{\gamma_n^0(1-\gamma_n^0)}[\|x_n - p\|^2 - \|z_n - p\|^2].
\end{aligned}
$$

Using (3.6) yields

$$
\lim_{n\to\infty} \|x_n - w_n^i\| = 0 \quad \text{for all } i = 1, 2, \dots, N. \tag{3.11}
$$

Using (3.11) and (3.10), we get

$$
\begin{aligned}
\mathrm{dist}(x_n, T_i x_n) &\leq \|x_n - w_n^i\| + \mathrm{dist}(w_n^i - T_i x_n) \\
&\leq \|x_n - w_n^i\| + H(T_i y_n - T_i x_n) \\
&\leq \|x_n - w_n^i\| + \|y_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.12}
$$

Using (3.10) and Lemma 2.2, we arrive at

$$
\begin{aligned}
\|x_n - J_\lambda^f x_n\| &\leq \|x_n - y_n\| + \|y_n - J_\lambda^f x_n\| \\
&= \|x_n - y_n\| + \|J_\lambda^f x_n - J_\lambda^f x_n\| \\
&= \|x_n - y_n\| + \|J_\lambda^f \left(\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n}^f x_n + \frac{\lambda}{\lambda_n} x_n\right) - J_\lambda^f x_n\| \\
&\leq \|x_n - y_n\| + \|\frac{\lambda_n - \lambda}{\lambda_n} J_{\lambda_n}^f x_n + \frac{\lambda}{\lambda_n} x_n - x_n\| \\
&\leq \|x_n - y_n\| + \left(1 - \frac{\lambda}{\lambda_n}\right)\|y_n - x_n\| \\
&\leq \left(2 - \frac{\lambda}{\lambda_n}\right)\|y_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned} \tag{3.13}
$$

We have established that $\{x_n\}$ is bounded, so that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p$. Using (3.12) and Lemma 2.4 implies that $p \in T_i p$ for all $i = 1, 2, \dots, N$. Thus $p \in \cap_{i=1}^N F(T_i)$. More so, since J_λ^f is single valued and nonexpasive, using (3.13) and Lemma 2.3, then $p \in F(J_\lambda^f)$, and by Lemma 2.5, we have that $p \in F(T_i \circ J_\lambda^f)$. Thus $p \in \Gamma$. We need to establish that x_n has a unique weak limit, say, $p \in \Gamma$. Let p and q be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. Using similar argument, it is easy to see that $q \in \Gamma$. In what follows, we establish uniqueness. We have shown above that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \Gamma$. Now, suppose that $p \neq q$; then by Opial's condition, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\|x_n - p\| &= \lim_{k\to\infty}\|x_{n_k} - p\| \\
&< \lim_{k\to\infty}\|x_{n_k} - q\| \\
&= \lim_{n\to\infty}\|x_n - q\| \\
&= \lim_{j\to\infty}\|x_{n_j} - q\| \\
&< \lim_{j\to\infty}\|x_{n_j} - p\| \\
&= \lim_{n\to\infty}\|x_n - p\|.
\end{aligned}$$

This is a contradiction, so $p = q$ and $\{x_n\}$ converges weakly to a point in Γ. $\square$

Theorem 3.2. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be finite family of nonexpansive multivalued mapping of K into $CB(K)$ and let $f : K \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Suppose that $\Gamma = \cap_{i=1}^N F(T_i) \cap \operatorname{argmin}_{u\in K} f(u) \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in \cap_{i=1}^N F(T_i)$. For $x_1 \in K$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = J_\lambda^f x_n, \\ z_n = \gamma_n^0 x_n + \sum_{i=1}^N \gamma_n^i w_n^i \\ x_{n+1} = \alpha_n^0 z_n + \sum_{i=1}^N \alpha_n^i T_i z_n, \quad n \in \mathbb{N}, \end{cases} \tag{3.14}$$

where $w_n^i \in T_i y_n$, $\sum_{i=0}^N \alpha_n^i = \sum_{i=0}^N \gamma_n^i = 1$, $\{\alpha_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $(0,1)$ for all $i = 1, 2 \ldots, N$ and $\lambda_n > \lambda > 0$. Then the sequence $\{x_n\}$ converges strongly to a point in Γ if and only if $\liminf_{n\to\infty} \operatorname{dist}(x_n, \Gamma) = 0$.

Proof. Suppose that $\{x_n\}$ converges to a fixed point, say, $p \in \Gamma$. Then, $\lim_{n\to\infty}\|x_n - p\| = 0$ and since

$$0 \leq \operatorname{dist}(x_n, \Gamma) \leq \|x_n - p\|.$$

It follows that $\lim_{n\to\infty} \operatorname{dist}(x_n, \Gamma) = 0$. Therefore, the $\liminf_{n\to\infty} \operatorname{dist}(x_n, \Gamma) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} \operatorname{dist}(x_n, \Gamma) = 0$. From Theorem 3.1, we have established that $\lim_{n\to\infty}\|x_n - p\|$ exists for all $p \in \Gamma$, consequently $\lim_{n\to\infty} \operatorname{dist}(x_n, \Gamma)$ exists and by our hypothesis that $\liminf_{n\to\infty} dist(x_n, \Gamma) = 0$, we therefore have $\lim_{n\to\infty} \operatorname{dist}(x_n, \Gamma) = 0$. Suppose that $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and that $\{p_k\}$ is a sequence in Γ such that

$$\|x_{n_k} - p_k\| \leq \frac{1}{2^k}$$

for all $n, k \in \mathbb{N}$. From (3.3), we have

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| < \frac{1}{2^k}.$$

Thus,

$$\begin{aligned}\|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &\leq \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}.\end{aligned}$$

Then, $\{p_k\}$ is a Cauchy sequence in K. Now, suppose that $\lim_{n\to\infty} p_k = q$. Therefore

$$[\mathrm{dist}](\mathrm{q}, \mathrm{T_i q}) = \lim_{\mathrm{n}\to\infty} \mathrm{dist}(\mathrm{p_k}, \mathrm{T_i q}) \leq \lim_{\mathrm{n}\to\infty} \mathrm{H}(\mathrm{T_i p_k}, \mathrm{T_i q}) \leq \lim_{\mathrm{n}\to\infty} \|\mathrm{p_k} - \mathrm{q}\| = 0,$$

for all $i = 1, 2, \ldots, N$ and also

$$\|q - J_\lambda^f q\| = \lim_{n\to\infty} \|p_k - J_\lambda^f q\| = \lim_{n\to\infty} \|J_\lambda^f p_k - J_\lambda^f q\| \leq \lim_{n\to\infty} \|p_k - q\| = 0.$$

Hence, $q \in \Gamma$ and $\{x_n\}$ converges strongly to q. Since $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \Gamma$, then $\{x_n\}$ converges strongly to $q \in \Gamma$. □

Theorem 3.3. *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive multivalued mapping of K into $CB(K)$ and lety $f : K \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Suppose that $\Gamma = \cap_{i=1}^N F(T_i) \cap argmin_{u\in K} f(u) \neq \emptyset$ and that $T_i p = \{p\}$ for all $p \in \cap_{i=1}^N F(T_i)$. For $x_1 \in K$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = J_\lambda^f x_n, \\ z_n = \gamma_n^0 x_n + \sum_{i=1}^N \gamma_n^i w_n^i \\ x_{n+1} = \alpha_n^0 z_n + \sum_{i=1}^N \alpha_n^i T_i z_n, \ n \in \mathbb{N}, \end{cases} \tag{3.15}$$

where $w_n^i \in T_i y_n, \sum_{i=0}^N \alpha_n^i = \sum_{i=0}^N \gamma_n^i = 1, \{\alpha_n^i\}$ and $\{\gamma_n^i\}$ are sequences in $(0,1)$ for all $i = 1, 2 \ldots, N$ and $\lambda_n > \lambda > 0$. Suppose that there is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$, $f(r) > 0$ for all $r > 0$, and $f(\mathrm{dist}(x_n, \Gamma)) \leq \mathrm{dist}(x_n, T_i x_n)$. Then $\{x_n\}$ converges strongly to an element of Γ.

Proof. We have established in Theorem 3.1 that $\mathrm{dist}(x_n, T_i x_n) = 0$ for all $i = 1, 2, \ldots, N$. By our hypothesis, we obtain that

$$\lim_{n\to\infty} f(\mathrm{dist}(x_n, \Gamma)) \leq \lim_{n\to\infty} \mathrm{dist}(x_n, T_i x_n) = 0.$$

Since f is a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, so $\liminf_{n\to\infty} \mathrm{dist}(x_n, \Gamma) = 0$. Hence, $\{x_n\}$ converges strongly to $p \in \Gamma$ by Theorem 3.2. □

4. Conclusion and open problem

In this work, we have established that $F(J_\lambda^f \circ T_i) = F(J_\lambda^f) \cap F(T_i)$ for a finite family of a multivalued nonexpansive mapping and minimization problem, in addition, we prove a weak and strong convergence result for approximating the common solution of the minimization problem and fixed point problem of a multivalued nonexpansive mapping in the framework of real Hilbert space, using the iterative process (3.1).

On the other hand, we have only shown that $F(J_\lambda^f \circ T_i) = F(J_\lambda^f) \cap F(T_i)$ for a finite family of a multivalued nonexpansive mapping. It is well known that there are other nonlinear mappings more general than nonexpansive mappings. A question of interest is can we extend Lemma 2.5 to mappings that are more general than nonexpansive mapping. This question is open for researcher interested in this field.

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School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa

E-mail address: `dele@aims.ac.za; 216028272@stu.ukzn.ac.za`