



EXISTENCE OF SOLUTION OF DIFFERENTIAL EQUATION VIA FIXED POINT IN COMPLEX VALUED *B*-METRIC SPACES

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Abstract. The concepts of new classes of mappings are introduced in the spaces which are more general space than the usual metric spaces. The obtained results are new and are extension of Banach contraction principle. The existence and uniqueness of common fixed points and fixed point results for the newly introduced classes of mappings are established in the setting of complete complex valued *b*-metric spaces. An illustration is given by establishing the existence of solution of periodic differential equations in the framework of a complete complex valued *b*-metric spaces. The results obtained in this work provide extension as well as substantial generalization and improvement of several well-known results on fixed point theory and its applications. The classes of mappings which are being considered in this paper are more general and the results are obtained in more broad spaces.

1. INTRODUCTION AND PRELIMINARIES

The pivot of theory of fixed point and its applications is traceable to Banach contraction principle [10]. The theory of fixed point plays an important role in nonlinear functional analysis (See e.g [4, 5] and the references therein). Fixed point theory is very useful for showing the existence and uniqueness theorems for nonlinear differential and integral equations. There are reports on the extension and generalization to the Banach contraction principle. Different spaces and classes of nonlinear mappings which are respectively more general than the metric spaces and class of contraction mappings have been investigated (See e.g [1, 7, 8, 12, 19, 21, 22, 27, 29, 34, 35] and the references therein). For instance, the notion of α -admissible mapping was introduced by Samet et al. [28].

Definition 1.1. [28] Let $\alpha : X \times X \rightarrow [0, \infty)$ be a function. We say that a self mapping $T : X \rightarrow X$ is α -admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.2. [28] Let $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow [0, \infty)$ be mappings. We say that T is a triangular α -admissible if

- (1) T is α -admissible and
- (2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$.

Samet et al. [28] obtained the fixed point results for this class of mappings in the axiom stated below.

Theorem 1.3. [28] Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an α -admissible mapping. Suppose that the following conditions hold:

- (1) for all $x, y \in X$, we have $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;
- (2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;
- (3) either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \geq 1$.

Then T has a fixed point.

In 2014, the notion of C -class function was introduced, some fixed point results were proved by using the concept of C -class function and it was also established that the C -class function is a generalization of a whole lot of contractive conditions (See Ansari [6]).

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Definition 1.4. [6] A mapping $F : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is called a C -class function if it is continuous and the following axioms hold:

- (1) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;
- (2) $F(s, t) = s$ implies either $s = 0$ or $t = 0$.

Example 1.5. The following are examples of C -class functions where $F : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ is defined for all $s, t \in [0, \infty)$ by:

- (1) $F(s, t) = s - t$, $F(s, t) = s$ implies $t = 0$;
- (2) $F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s$ implies $s = 0$;
- (3) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \rightarrow [0, 1)$ is a continuous function, $F(s, t) = s$ implies $s = 0$.

For details about C -class function see [6].

In 2015, the notion of \mathcal{Z} -contraction was introduced, which generalizes the well-known Banach contraction and a host of other contractive conditions (See Khojasteh et al. [18]).

Definition 1.6. Let $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

- (1) $\zeta(0, 0) = 0$;
- (2) $\zeta(t, s) < s - t$, for all $t, s > 0$;
- (3) If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$, then $\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by \mathcal{Z} .

Example 1.7. Suppose $\zeta_i : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2, 3, 4$ is defined as

- (1) $\zeta_1(t, s) = s - \phi(s) - t$ for all $t, s \in [0, \infty)$, where $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function such that $\phi(t) = 0$ if and only if $t = 0$.
- (2) $\zeta_2(t, s) = \eta(s) - t$ for all $t, s \in [0, \infty)$, where $\eta : [0, \infty) \rightarrow [0, \infty)$ is an upper semicontinuous mapping such that $\eta(t) < t$ for all $t > 0$ $\eta(t) = 0$ if and only if $t = 0$.
- (3) $\zeta_3(t, s) = \lambda s - t$ for all $t, s \in [0, \infty)$, where $0 < \lambda < 1$.
- (4) $\zeta_4(t, s) = \frac{s}{s+1} - t$ for all $t, s \in [0, \infty)$.

Definition 1.8. Let (X, d) be a metric space, $T : X \rightarrow X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a \mathcal{Z} -contraction with respect to ζ , if the following condition is satisfied

$$\zeta(d(Tx, Ty), d(x, y)) > 0,$$

for all distinct $x, y \in X$.

Theorem 1.9. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a \mathcal{Z} -contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ converges to the fixed point of T .*

A slightly modification to the notion of simulation function which strengthened and generalizes the definition of Khojasteh et al. in [18] was proposed by Antonio-Francisco et al. [23].

Definition 1.10. Let $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

- (1) $\zeta(0, 0) = 0$;
- (2) $\zeta(t, s) < s - t$, for all $t, s > 0$;
- (3) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < 0$.

An example is presented to show that every simulation function in the sense of Definition 1.6 is also a simulation function in the sense of Definition 1.10. However, the converse is not true.

Example 1.11. [23] Let $k \in \mathbb{R}$ be such that $k < 1$ and let $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ be the function defined by

$$\zeta(t, s) = \begin{cases} 2(s - t) & \text{if } s < t \\ ks - t & \text{if otherwise.} \end{cases}$$

In 2018, the notion of C -class function was engaged to generalize the concept of simulation function by Liu et al. [20].

Definition 1.12. A mapping $F : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ has the property C_F , if there exists a $C_F \geq 0$ such that

- (1) $F(s, t) > C_F \Rightarrow s > t$;
- (2) $F(t, t) \leq C_F$ for all $t \in [0, \infty)$.

Definition 1.13. A C_F simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$ satisfying the following conditions:

- (1) $\zeta(t, s) < F(s, t)$, for all $t, s > 0$, where F is a C -class function;
- (2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 0$ and $t_n < s_n$ for all $n \in \mathbb{N}$, then $\limsup_{n \rightarrow \infty} \zeta(t_n, s_n) < C_F$.

Some examples of a C -class functions that have property C_F are as follows:

- (1) $F(s, t) = s - t, C_F = r, r \in [0, \infty)$;
- (2) $F(s, t) = \frac{s}{1+kt}, k \geq 1, C_F = \frac{r}{1+k}, r \geq 2$.

Remark 1.14. It is easy to see that if $r \geq 2$ in the above example the required result will not hold. More so, it is worth mentioning that every simulation function in the sense of Definition 1.6 is also a C_F simulation function, but the converse is not true. This claim is easy to see using Example 1.11 with $F(s, t) = s - t$.

Remark 1.15. It is natural to ask if we can further generalize the notion C_F simulation function.

One of the interesting generalization of metric spaces is the concept of b -metric spaces which was introduced by Czerwik in [13]. Banach contraction principle was established with the fact that b need not be continuous. Thereafter, several results have been extended from metric spaces to b -metric spaces. Indeed, a lot of results on the fixed point theory of various classes of mappings in the frame work of b -metric spaces has been established by different researchers in this area (see [11, 13, 25] and the references therein). For example in [30], Sintunavarat introduced the concept of α -admissible mapping type S as a generalization of α -admissible mapping [28].

Definition 1.16. [30] Let X be a nonempty set and $s \geq 1$ be a given real number. Let $\alpha : X \times X \rightarrow [0, \infty)$ and $T : X \rightarrow X$ be mappings. The mapping T is said to be an α -admissible mapping type S if for all $x, y \in X$

$$\alpha(x, y) \geq s \Rightarrow \alpha(Tx, Ty) \geq s.$$

Remark 1.17. Clearly, if $s = 1$, we obtain Definition 1.1.

Remark 1.18. It is also natural to ask, if the notion of α -admissible mapping type S can further be generalized.

Coming up with new algebraic structures to improve and extend the obtained results in the literature is always a worthwhile research effort. In [9], Azam et al. introduce the notion of complex valued metric space and established some common fixed point results for mapping satisfying generalized contractive conditions. Thereafter, several results and applications has been extended from metric spaces to complex valued metric spaces. Furthermore, a lot of results on the fixed point theory and common fixed point results of various classes of mappings in the framework of complex valued metric spaces has been established by different researchers in this area (see[31, 32, 33] and the references therein).

The following symbols, notation and definition can be found in [9] will be useful in this study. Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$$z_1 \preceq z_2 \quad \text{if and only if} \quad Re(z_1) \leq Re(z_2), \quad Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \preceq z_2$ if one of the following conditions is satisfied:

- (1) $Re(z_1) = Re(z_2), \quad Im(z_1) < Im(z_2);$
- (2) $Re(z_1) < Re(z_2), \quad Im(z_1) = Im(z_2);$
- (3) $Re(z_1) < Re(z_2), \quad Im(z_1) < Im(z_2);$
- (4) $Re(z_1) = Re(z_2), \quad Im(z_1) \leq Im(z_2).$

In particular, we write $z_1 \succsim z_2$ if $z_1 \neq z_2$ and one of (1), (2) and (3) is satisfied and we write $z_1 \prec z_2$ if only (3) is satisfied. Note that

- (1) $a, b \in \mathbb{R}$ and $a \leq b$ implies that $az \succsim bz$ for all $z \in \mathbb{C}$;
- (2) $0 \succsim z_1 \succsim z_2$ implies that $|z_1| < |z_2|$;
- (3) $z_1 \succsim z_2$ and $z_2 \prec z_3$ implies that $z_1 \prec z_3$.

Definition 1.19. Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$, satisfies:

- (1) $0 \succsim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, y) \succsim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric and (X, d) is called a complex valued metric space.

Example 1.20. Let $X = \mathbb{C}$, $z_i \in X$, $i = 1, 2, 3$ and $d : X \times X \rightarrow \mathbb{C}$ be defined as

- (1) $d(z_1, z_2) = |z_1 - z_2|$ for all $z_1, z_2 \in X$;
- (2) $d(z_1, z_2) = e^{ik}|z_1 - z_2|$ for all $z_1, z_2 \in X$ and $k \in \mathbb{R}$;
- (3) $d(z_1, z_2) = e^{i\theta}|z_1 - z_2|$ for all $z_1, z_2 \in X$ and $\theta \in (0, \frac{\pi}{2})$.

Motivated by the concept of b -metric spaces and complex valued metric spaces [13, 9], Rao et al. in [24], introduced the notion of complex valued b -metric spaces and established some common fixed point results. Thereafter, several results and applications has been extended from metric spaces, b -metric spaces and complex valued metric spaces to complex valued b -metric spaces (see [24, 14] and the reference therein). The notion of complex valued b -metric spaces generalizes, improves and unifies the results in metric spaces, b -metric spaces and complex valued metric spaces.

Definition 1.21. Let X be a nonempty set and $s \geq 1$ be a given real number. Suppose that the mapping $d_b : X \times X \rightarrow \mathbb{C}$, satisfies:

- (1) $0 \succsim d_b(x, y)$ for all $x, y \in X$ and $d_b(x, y) = 0$ if and only if $x = y$;
- (2) $d_b(x, y) = d_b(y, x)$ for all $x, y \in X$;
- (3) $d_b(x, y) \succsim s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$, where d retains its usual definition.

Then d_b is called a complex valued b metric and (X, d_b) is called a complex valued metric space.

Example 1.22. [24] Let $X = \mathbb{C}$ defined the mapping $d_b : X \times X \rightarrow \mathbb{C}$ by $d_b(z_1, z_2) = |z_1 - z_2|^2 + i|z_1 - z_2|^2$ for all $z_1, z_2 \in X$.

Definition 1.23. Suppose that (X, d_b) is a complex valued b -metric space and $\{z_n\}$ is a sequence in X , then the sequence $\{z_n\}$

- (1) converges to an element $z_0 \in X$ if for every $0 \prec c \in \mathbb{C}$, there exist an integer \mathbb{N} such that $d_b(z_n, z_0) \prec c$ for all $n \in \mathbb{N}$.
- (2) is a Cauchy sequence if for every $0 \prec c \in \mathbb{C}$, there exist an integer \mathbb{N} such that $d_b(z_n, z_m) \prec c$ for all $n, m \in \mathbb{N}$.

Definition 1.24. Suppose that (X, d_b) is a complex valued b -metric space, the space (X, d_b) is said to be complete if every Cauchy sequence in X converges to a point in X .

Definition 1.25. [17] Let X be a nonempty set and $S, T : X \rightarrow X$ be any two mappings.

- (1) A point $x \in X$ is called:
 - (a) coincidence point of S and T if $Sx = Tx$,
 - (b) common fixed point of S and T if $x = Sx = Tx$.
- (2) If $y = Sx = Tx$ for some $x \in X$, then y is called the point of coincidence of S and T .
- (3) A pair (S, T) is said to be:
 - (a) commuting if $TSx = STx$ for all $x \in X$,
 - (b) weakly compatible if they commute at their coincidence points, that is $STx = TSx$, whenever $Sx = Tx$.

Motivated by the works of Samet et al. [28], Liu et al. [20], Khojasteh et al. [18], Antonio-Francisco [23] and the current research interest in this direction, the purpose of this work is to introduce new concepts which are based on simulation functions in the framework of complex valued b -metric spaces. The notions of b - C_F simulation function, $\alpha_s S$ -admissible mapping, $\alpha_s S$ - \mathcal{Z}_F -contraction type I mappings and $\alpha_s S$ - \mathcal{Z}_F -contraction type II mappings with respect to the simulation function, ζ are introduced. Moreover, some common fixed point results and fixed point results are established for these newly introduced classes of mappings in the framework of complete complex valued b -metric spaces. The consequence of our results is the establishment of existence of solutions of periodic differential equations.

2. MAIN RESULTS

In this section, the notions of b - C_F simulation function, $\alpha_s S$ -admissible mapping, $\alpha_s S$ - \mathcal{Z}_F -contraction type I mappings and $\alpha_s S$ - \mathcal{Z}_F -contraction type II mappings with respect to ζ in the framework of complex valued b -metric spaces are introduced. The existence and uniqueness results of the common fixed

point and fixed point results for these classes of mappings in the framework of a complete complex valued b -metric spaces are established.

Let the common fixed point of mappings T and S be denoted by $C(T, S)$ and define $\mathbb{S} = \{z \in \mathbb{C} : 0 \preceq z\}$. As a start, the below definition is given in the framework of complex valued metric spaces which is due to Liu et al. [20].

Definition 2.1. A mapping $F : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ has the property C_F , if there exists a $0 \preceq C_F$ such that

- (1) $F(s, t) \succeq C_F \Rightarrow t \prec s$;
- (2) $F(t, t) \preceq C_F$ for all $t \in \mathbb{S}$.

We propose the following definitions which are based on simulation functions in the framework of complex valued b -metric spaces.

Definition 2.2. A C_F simulation function is a mapping $\zeta : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ satisfying the following conditions:

- (1) $\zeta(t, s) \prec F(s, t)$, for all $t, s > 0$, where F is a C -class function;
- (2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $0 \prec \lim_{n \rightarrow \infty} |t_n| = \lim_{n \rightarrow \infty} |s_n|$ and $|t_n| \prec |s_n|$ for all $n \in \mathbb{N}$,

then $\limsup_{n \rightarrow \infty} \zeta(|t_n|, |s_n|) \prec C_F$.

Definition 2.3. A b - C_F simulation function is a mapping $\zeta : \mathbb{S} \times \mathbb{S} \rightarrow \mathbb{C}$ satisfying the following conditions:

- (1) $\zeta(t, s) \prec F(s, t)$, for all $t, s > 0$, where F is a C -class function;
- (2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $0 \prec \lim_{n \rightarrow \infty} |t_n| \prec \liminf_{n \rightarrow \infty} |s_n| \prec \limsup_{n \rightarrow \infty} |s_n| \leq b \lim_{n \rightarrow \infty} |t_n| \prec \infty$ and $|t_n| \prec |s_n|$ for all $n \in \mathbb{N}$,

then

$$\limsup_{n \rightarrow \infty} \zeta(b|t_n|, |s_n|) \prec C_F.$$

Remark 2.4. It is easy to see that if $b = 1$, Definition 2.2 is obtained.

Definition 2.5. Let X be a nonempty set with $s \geq 1$ a given real number, $T : X \rightarrow X$ and $\alpha, \beta : X \times X \rightarrow \mathbb{S}$ be mappings. Then T is called $\alpha_s S$ -admissible type mapping if for all $x, y \in X$ with

$$\alpha(Sx, Sy) \succeq s \Rightarrow \alpha(Tx, Ty) \succeq s.$$

Remark 2.6.

- (1) If $Sx = x$, Definition 1.16 is obtained in the framework of complex valued b -metric spaces.
- (2) If $Sx = x$ and $s = 1$, Definition 1.1 is obtained in the framework of complex valued metric spaces.

Definition 2.7. Let X be a nonempty set with $s \geq 1$ a given real number, $T : X \rightarrow X$ and $\alpha : X \times X \rightarrow \mathbb{S}$ be mappings. We say that T is a triangular $\alpha_s S$ -admissible if

- (1) T is $\alpha_s S$ -admissible and
- (2) $\alpha(Sx, Sy) \succeq s$ and $\alpha(Sy, Sz) \succeq s \Rightarrow \alpha(Sx, Sz) \succeq s$ for all $x, y, z \in X$.

Lemma 2.8. Let X be a nonempty set with $s \geq 1$ a given real number and T be a triangular $\alpha_s S$ -admissible and there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \succeq s$. Suppose that the sequence $\{Sx_n\}$ is defined by $Sx_{n+1} = Tx_n$, then $\alpha(Sx_m, Sx_n) \succeq s$ for all $n, m \in \mathbb{N} \cup \{0\}$, with $m < n$.

Proof. Suppose that T is triangular $\alpha_s S$ -admissible and there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \succeq s$, we then have that $\alpha(Sx_0, Tx_0) = \alpha(Sx_0, Sx_1) \succeq s$ which implies that $\alpha(Tx_0, Tx_1) = \alpha(Sx_1, Sx_2) \succeq s$. Continuing the process, we obtain that $\alpha(Sx_n, Sx_{n+1}) \succeq s$. For all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$, observe that since $\alpha(Sx_m, Sx_{m+1}) \succeq s$ and $\alpha(Sx_{m+1}, Sx_{m+2}) \succeq s$, we obtain $\alpha(Sx_m, Sx_{m+2}) \succeq s$. Also, since $\alpha(Sx_m, Sx_{m+2}) \succeq s$ and $\alpha(Sx_{m+2}, Sx_{m+3}) \succeq s$ we obtain $\alpha(Sx_m, Sx_{m+3}) \succeq s$. Continuing the process, we have that

$$\alpha(Sx_m, Sx_n) \succeq s.$$

□

Definition 2.9. Let (X, d_b) be a complex valued b -metric space with $s \geq 1$, $\alpha : X \times X \rightarrow \mathbb{S}$ be functions and S, T be a self map on X . The mapping T is said to be $\alpha_s S$ - \mathcal{Z}_F -contraction type I mapping with respect to ζ , if

$$\alpha(Sx, Sy) \succeq s \Rightarrow \zeta(sd_b(Tx, Ty), d_b(Sx, Sy)) \succeq C_F \quad (2.1)$$

for all distinct $x, y, z \in X$.

Remark 2.10. Suppose $s = 1$ and $C_F = 0$, a new type of generalized \mathcal{Z} -contraction with respect to ζ is obtained

$$\alpha(Sx, Sy) \succeq 1 \Rightarrow \zeta(d_b(Tx, Ty), d_b(Sx, Sy)) \succeq 0 \quad (2.2)$$

for all distinct $x, y \in X$. It is easy to see that (2.2) is a generalization of Definition 1.8.

Theorem 2.11. Let (X, d_b) be a complete complex valued b -metric space with $s \geq 1$ and $T : X \rightarrow X$ be an $\alpha_s S$ - \mathcal{Z}_F -contraction type I mapping with respect to ζ . Suppose the following conditions hold:

- (1) T is triangular $\alpha_s S$ -admissible,
- (2) $T(X) \subseteq S(X)$,
- (3) there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \succsim s$,
- (4) $T(X)$ is complete in $S(X)$,

- (5) if for any sequence $\{Sx_n\}$ in X with $\alpha(Sx_n, x_{n+1}) \gtrsim s$ for all $n \geq 0$ and $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$, then $\alpha(Sx_n, Sx) \gtrsim s$.

Then the pair (T, S) has a coincidence point in X .

In addition, if the pair (T, S) is weakly compatible. Then (T, S) have a common fixed point.

Proof. Let $Sx_0 \in X$ be such that $\alpha(Sx_0, Tx_0, Tx_0) \succeq s$, using condition (ii), we define the sequence $\{Sx_n\}$ in $T(X)$ by $Sx_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $Sx_{n+1} = Sx_n Tx_n$, for some $n \in \mathbb{N} \cup \{0\}$, we have that x_n is a coincidence point. Now, suppose that $Sx_{n+1} \neq Sx_n$ for all $n \in \mathbb{N} \cup \{0\}$. From Lemma 2.8, it is easy to see that

$$\alpha(Sx_n, Sx_{n+1}) \succeq s$$

for all $n \in \mathbb{N} \cup \{0\}$. Using $\zeta_*(i), \eta(i)$ and from (2.1), we have that

$$\begin{aligned} C_F &\lesssim \zeta(sd_b(Tx_n, Tx_{n+1}), d_b(Sx_n, Sx_{n+1})) \\ &= \zeta(sd_b(Sx_{n+1}, Sx_{n+2}), d_b(Sx_n, Sx_{n+1})) \\ &\prec F(d_b(Sx_n, Sx_{n+1}), sd_b(Sx_{n+1}, Sx_{n+2})). \end{aligned} \quad (2.3)$$

From (2.3), we obtain

$$F(d_b(Sx_n, Sx_{n+1}), sd_b(Sx_{n+1}, Sx_{n+2})) \gtrsim C_F,$$

which implies that

$$sd_b(Sx_{n+1}, Sx_{n+2}) \prec d_b(Sx_n, Sx_{n+1}).$$

That is

$$|d_b(Sx_{n+1}, Sx_{n+2})| \leq |sd_b(Sx_{n+1}, Sx_{n+2})| < |d_b(Sx_n, Sx_{n+1})|. \quad (2.4)$$

It is easy to see from (2.4) that the sequence $\{|d_b(Sx_n, Sx_{n+1})|\}$ is monotonically decreasing and nonnegative. More so, inductively, we have that $\{|d_b(Sx_n, Sx_{n+1})|\}$ is bounded. Therefore, there exists $c \geq 0$ such that

$$\lim_{n \rightarrow \infty} |d_b(Sx_n, Sx_{n+1})| = c.$$

Suppose that $c > 0$, clearly $\lim_{n \rightarrow \infty} |d_b(Sx_{n+1}, Sx_{n+2})| = c$. Since T is an $\alpha_s S$ - \mathcal{Z}_F -contraction type I mapping with respect to $\zeta \in \mathcal{Z}$ and using $\zeta_*(ii)$, we have

$$C_F \leq \limsup_{n \rightarrow \infty} \zeta(s|d_b(Sx_{n+1}, Sx_{n+2})|, |d_b(Sx_n, Sx_{n+1})|) < C_F.$$

This is a contradiction, thus $c = 0$ and so we have that

$$\lim_{n \rightarrow \infty} |d_b(Sx_n, Sx_{n+1})| = 0. \quad (2.5)$$

In what follows, we will show that $\{Sx_n\}$ is bounded.

Suppose that $\{Sx_n\}$ is not a bounded sequence, then there exists a subsequence

$\{Sx_{n_k}\}$ of $\{Sx_n\}$ such that for $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d_b(Sx_{n_k}, Sx_{n_{k+1}}) \succeq 1 \text{ and } d_b(Sx_{n_k}, Sx_m) \lesssim 1 \quad (2.6)$$

for $n_k \leq m \leq n_{k+1} - 1$. Using (2.6), we have

$$\begin{aligned} 1 \prec d_b(Sx_{n_k}, Sx_{n_{k+1}}) &\lesssim sd_b(Sx_{n_k}, Sx_{n_{k+1}-1}) + sd_b(x_{n_{k+1}-1}, Sx_{n_{k+1}}) \\ &\lesssim sd_b(Sx_{n_{k+1}-1}, Sx_{n_{k+1}}) + s. \end{aligned}$$

Letting $k \rightarrow \infty$ and using (2.5), we obtain

$$1 \leq \liminf_{k \rightarrow \infty} |d_b(Sx_{n_k}, Sx_{n_{k+1}})| \leq \limsup_{k \rightarrow \infty} |d_b(Sx_{n_k}, Sx_{n_{k+1}})| \leq s. \quad (2.7)$$

From (2.4), we deduce that

$$\begin{aligned} sd_b(Sx_{n_k}, Sx_{n_{k+1}}) &\lesssim d_b(Sx_{n_k-1}, Sx_{n_{k+1}-1}) \\ &\lesssim sd_b(Sx_{n_k-1}, Sx_{n_k}) + sd_b(Sx_{n_k}, Sx_{n_{k+1}-1}) \\ &\lesssim sd_b(Sx_{n_k-1}, Sx_{n_k}) + s. \end{aligned} \quad (2.8)$$

Letting $k \rightarrow \infty$, using (2.5) and (2.7), we obtain that

$$\lim_{n \rightarrow \infty} |d_b(Sx_{n_k-1}, Sx_{n_{k+1}-1})| = s.$$

From Lemma 2.8, it is easy to see that $\alpha(Sx_{n_k-1}, Sx_{n_{k+1}-1}) \succeq s$ and by definition of $\alpha_s S$ - \mathcal{Z}_F -contraction type I mapping with respect to ζ , and by $\zeta_*(ii)$, we obtain

$$C_F \leq \limsup_{k \rightarrow \infty} \zeta(s|d_b(Sx_{n_k}, Sx_{n_{k+1}})|, |d_b(Sx_{n_k-1}, Sx_{n_{k+1}-1})|) < C_F.$$

This is a contradiction. Thus $\{Sx_n\}$ is bounded.

We now establish that $\{Sx_n\}$ is Cauchy.

Suppose that $C_n = \max\{d_b(Sx_i, Sx_j, x_j) : i, j \succeq n\}$, $n \in \mathbb{N}$. Since $\{Sx_n\}$ is bounded, we have that $C_n < \infty$ for all $n \in \mathbb{N}$, as such $\{C_n\}$ is a positive monotonically decreasing sequence which converges. That is $\lim_{n \rightarrow \infty} C_n = C \geq 0$. Suppose that $C > 0$, then by definition of C_n , for every $k \in \mathbb{N}$, we can find n_k, m_k such that $m_k > n_k > k$ and

$$C_n - \frac{1}{K} \prec d_b(Sx_{m_k}, Sx_{n_k}) \lesssim C_k,$$

letting $k \rightarrow \infty$, we obtain

$$\lim_{k \rightarrow \infty} |d_b(Sx_{m_k}, Sx_{n_k})| = C. \quad (2.9)$$

From (2.4) and using the definition of C_n , we deduce that

$$sd_b(Sx_{m_k}, Sx_{n_k}) \lesssim d_b(Sx_{m_k-1}, Sx_{n_k-1}) \lesssim C_{k-1}.$$

Letting $k \rightarrow \infty$ and using (2.9), we obtain

$$sC \leq \liminf_{k \rightarrow \infty} |d_b(Sx_{m_k-1}, Sx_{n_k-1})| \leq \limsup_{k \rightarrow \infty} |d_b(Sx_{m_k-1}, Sx_{n_k-1})| \leq C. \quad (2.10)$$

It is easy to see from Lemma 2.8 that $\alpha(Sx_{m_k-1}, Sx_{n_k-1}) \succeq s$, so by definition of $\alpha_s S$ - \mathcal{Z}_F -contraction type I with respect to ζ and using $\zeta_*(ii)$, we have that

$$C_F \leq \limsup_{n \rightarrow \infty} \zeta(s|d_b(Sx_{m_k}, Sx_{n_k}), |d_b(Sx_{m_k-1}, Sx_{n_k-1})|) < C_F.$$

This is a contradiction, thus $C = 0$. Hence, $\{Sx_n\}$ is a Cauchy sequence. Finally, we show the existence of a common fixed point of the pair (T, S) .

Since $T(X)$ is precomplete in $S(X)$, there exists $x \in X$ such that $\lim_{n \rightarrow \infty} Sx_n = Sx$. Using condition (5), we have that $\alpha(Sx_n, Sx) \succeq s$, and since T is $\alpha_s S$ - \mathcal{Z}_F -contraction type I mapping with respect to ζ and using $\eta(i)$, we have that

$$\begin{aligned} C_F &\preceq \zeta(sd_b(Tx_n, Tx), d_b(Sx_n, Sx)) \\ &= \zeta(sd_b(Sx_{n+1}, Tx), d_b(Sx_n, Sx)) \\ &\prec F(d_b(Sx_n, Sx), sd_b(Sx_{n+1}, Tx)), \end{aligned}$$

that is $F(d_b(Sx_n, Sx), sd_b(Sx_{n+1}, Tx)) \succeq C_F$, which implies that $sd_b(Sx_{n+1}, Tx) \prec d_b(Sx_n, Sx)$, so that

$$|d_b(Sx_{n+1}, Tx)| \leq |sd_b(Sx_{n+1}, Tx)| < |d_b(Sx_n, Sx)|,$$

taking limit as $n \rightarrow \infty$, we have that

$$d_b(Sx, Tx) \leq 0 \Rightarrow Sx = Tx$$

Hence, x a coincidence point for the pair (T, S) .

Now suppose that $y = Tx = Sx$, using condition (6), we have that

$$Ty = T(Sx) = S(Tx) = Sy.$$

It is easy to see that $\alpha(Sx, Sy) \succeq s$, and since T is $\alpha_b S$ - \mathcal{Z}_F -contraction with respect to ζ and using $\eta(ii)$, we obtain

$$\begin{aligned} C_F &\preceq \zeta(sd_b(Tx, Ty), d_b(Sx, Sy)) \\ &= \zeta(sd_b(y, Ty), d_b(y, Sy)) \prec C_F, \end{aligned}$$

a contradiction. Hence, we have

$$y = Ty = Sy.$$

Hence y is a common fixed point of the pair (T, S) . □

Theorem 2.12. *Suppose that the hypothesis of Theorem 2.11 holds and in addition suppose $\alpha(x, y) \succeq s$ for all $x, y \in C(T, S)$, where $C(T, S)$ is the set of common fixed point of the pair (T, S) . Then (T, S) has a unique fixed common fixed point.*

Proof. Let $x, y \in C(T, S)$, that is $x = Tx = Sx$ and $y = Ty = Sy$ such that $x \neq y$. Using our hypothesis, we have $\alpha(x, y) \succeq sb$, we obtain from (2.1) that

$$\begin{aligned} C_F &\preceq \zeta(sd_b(Tx, Ty), d_b(Sx, Sy)) \\ &= \zeta(sd_b(x, y), d_b(x, y)) \\ &\prec F(d_b(x, y), sd_b(x, y)). \end{aligned}$$

It follows that $F(d_b(x, y), sd_b(x, y)) \succeq C_F$, which implies that

$$sd_b(x, y) < d_b(x, y)$$

which is a contradiction, as such, we must have that $d_b(x, y) = 0 \Rightarrow x = y$. Hence (T, S) has a unique common fixed point. \square

Definition 2.13. Let (X, d_b) be a complex valued b -metric space with $s \geq 1$, $\alpha; X \times X \rightarrow \mathbb{S}$ be functions and S, T be a self map on X . The mapping T is said to be $\alpha_s S$ - \mathcal{Z}_F -contraction type II mapping with respect to ζ , if

$$\zeta(\alpha(Sx, Sy)d_b(Tx, Ty), d_b(Sx, Sy)) \succeq C_F \quad (2.11)$$

for all distinct $x, y, z \in X$.

Remark 2.14. If we suppose that $\alpha(Sx, Sy) = 1$ and $C_F = 0$, we obtain a new type of generalized \mathcal{Z} -contraction with respect to ζ ,

$$\zeta(d_b(Tx, Ty), d_b(Sx, Sy)) \succeq 0 \quad (2.12)$$

for all distinct $x, y \in X$. It is easy to see that (2.12) is a generalization of Definition 1.8.

Theorem 2.15. *Let (X, d_b) be a complete complex valued b -metric space with $s \geq 1$ and $T : X \rightarrow X$ be an $\alpha_s S$ - \mathcal{Z}_F -contraction type II mapping with respect to ζ . Suppose the following conditions hold:*

- (1) T is triangular $\alpha_s S$ -admissible,
- (2) $T(X) \subseteq S(X)$,
- (3) there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \succeq s$,
- (4) $T(X)$ is complete in $S(X)$,
- (5) if for any sequence $\{Sx_n\}$ in X with $\alpha(Sx_n, x_{n+1}) \succeq s$ for all $n \geq 0$ and $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$, then $\alpha(Sx_n, Sx) \succeq s$.

Then the pair (T, S) has a coincidence point in X .

- (6) *In addition, if the pair (T, S) is weakly compatible.*

Then (T, S) have a common fixed point.

Proof. The proof is similar to Theorem 2.11, as such we omit it. \square

Theorem 2.16. *Suppose that the hypothesis of Theorem 2.15 holds and in addition suppose $\alpha(x, y) \succeq s$ for all $x, y \in C(T, S)$, where $C(T, S)$ is the set of common fixed point of the pair (T, S) . Then (T, S) has a unique fixed common fixed point.*

Proof. The proof is similar to Theorem 2.12, as such we omit it. \square

3. CONSEQUENCES OF MAIN RESULT

In this section, we present some consequences of our main result.

Corollary 3.1. *Let (X, d_b) be a complete complex valued b -metric spaces space with $s \geq 1$ and $S, T : X \rightarrow X$ be a mapping satisfying*

$$\alpha(Sx, Sy) \succeq s \Rightarrow \zeta(sd_b(Tx, Ty), \lambda d_b(Sx, Sy)) \succeq 0, \quad (3.1)$$

for all distinct $x, y, z \in X$, where $\lambda \in (0, 1)$. Suppose the following conditions hold:

- (1) T is triangular $\alpha_s S$ -admissible,
- (2) $T(X) \subseteq S(X)$,
- (3) there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \succeq s$,
- (4) $T(X)$ is complete in $S(X)$,
- (5) if for any sequence $\{Sx_n\}$ in X with $\alpha(Sx_n, x_{n+1}) \succeq s$ for all $n \geq 0$ and $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$, then $\alpha(Sx_n, Sx) \succeq s$.
Then the pair (T, S) has a coincidence point in X .
- (6) In addition, if the pair (T, S) is weakly compatible.

Then (T, S) have a common fixed point.

Proof. The result follows from Theorem 2.11. Since by taking $C_F = 0$, and defining $\zeta(t, s) = s - t$, for all $s, t \geq 0$, we obtain

$$\alpha(Sx, Sy) \succeq s \Rightarrow sd_b(Tx, Ty) \preceq \lambda d_b(Sx, Sy).$$

\square

Remark 3.2. Corollary 3.1 can be seen as a generalization of the well-known Banach contraction principle [10] in the framework of complete complex valued d_b -metric spaces.

Corollary 3.3. *Let (X, d_b) be a complete complex valued b -metric spaces space with $s \geq 1$ and $S, T : X \rightarrow X$ be a mapping satisfying*

$$\alpha(Sx, Sy) \succeq s \Rightarrow \zeta(sd_b(Tx, Ty), d_b(Sx, Sy) - \psi(d_b(Sx, Sy))) \geq 0, \quad (3.2)$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a lower semicontinuous function with $\psi^{-1}(0) = (0)$. Suppose the following conditions hold:

- (1) T is triangular $\alpha_s S$ -admissible,

- (2) $T(X) \subseteq S(X)$,
- (3) there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \gtrsim s$,
- (4) $T(X)$ is complete in $S(X)$,
- (5) if for any sequence $\{Sx_n\}$ in X with $\alpha(Sx_n, x_{n+1}) \gtrsim s$ for all $n \geq 0$ and $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$, then $\alpha(Sx_n, Sx) \gtrsim s$.
Then the pair (T, S) has a coincidence point in X .
- (6) In addition, if the pair (T, S) is weakly compatible.

Then (T, S) have a common fixed point.

Proof. The result follows from Theorem 2.11. Since by taking $C_F = 0$, and defining $\zeta(t, s) = \lambda s - \psi(s) - t$, for all $s, t \gtrsim 0$, we obtain

$$\alpha(Sx, Sy) \succeq s \Rightarrow sd_b(Tx, Ty) \lesssim d_b(Sx, Sy) - \psi(d_b(Sx, Sy)).$$

□

Remark 3.4. Corollary 3.3 can be seen as a generalization of Rhoades fixed point result [22] in the framework of complete complex valued b -metric spaces.

Corollary 3.5. Let (X, d_b) be a complete complex valued b -metric space with $s \geq 1$ and $T : X \rightarrow X$ be a mapping satisfying

$$\alpha(x, y) \succeq 1 \Rightarrow \zeta(sd_b(Tx, Ty), d_b(x, y)) \geq 0, \quad (3.3)$$

for all distinct $x, y \in X$. Suppose the following conditions hold:

- (1) T is triangular α -admissible mapping,
- (2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \succeq 1$,
- (3) if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \succeq 1$, for all $n \geq 0$ and $x_n \rightarrow x$ as $n \rightarrow \infty$, then $\alpha(x_n, x) \succeq 1$.

Then T has a fixed point.

Proof. The result follow similar argument as in Theorem 2.11, by taking $C_F = 0$. Since by defining $\zeta(t, s) = s - t$, for all $s, t \geq 0$, we obtain

$$\alpha(x, y, z) \succeq 1 \Rightarrow sd_b(Tx, Ty) \lesssim d_b(x, y).$$

□

Corollary 3.6. Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$ be a mapping satisfying

$$\alpha(Sx, Sy) \gtrsim 1 \Rightarrow \zeta(d(Tx, Ty), d(Sx, Sy)) \geq 0, \quad (3.4)$$

for all distinct $x, y, z \in X$. Suppose the following conditions hold:

- (1) T is triangular α -admissible,
- (2) $T(X) \subseteq S(X)$,
- (3) there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \gtrsim 1$,
- (4) $T(X)$ is complete in $S(X)$,

- (5) if for any sequence $\{Sx_n\}$ in X with $\alpha(Sx_n, x_{n+1}) \gtrsim 1$ for all $n \geq 0$ and $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$, then $\alpha(Sx_n, Sx) \gtrsim 1$.

Then the pair (T, S) has a coincidence point in X .

- (6) In addition, if the pair (T, S) is weakly compatible.

Then (T, S) have a common fixed point.

Corollary 3.7. Let (X, d) be a complete complex valued metric space and $S, T : X \rightarrow X$ be a mapping satisfying

$$\zeta(\alpha(Sx, Sy)d(Tx, Ty), d(Sx, Sy)) \geq 0, \quad (3.5)$$

for all distinct $x, y, z \in X$. Suppose the following conditions hold:

- (1) T is triangular α -admissible,
- (2) $T(X) \subseteq S(X)$,
- (3) there exists $Sx_0 \in X$ such that $\alpha(Sx_0, Tx_0) \gtrsim 1$,
- (4) $T(X)$ is complete in $S(X)$,
- (5) if for any sequence $\{Sx_n\}$ in X with $\alpha(Sx_n, x_{n+1}) \gtrsim 1$ for all $n \geq 0$ and $Sx_n \rightarrow Sx$ as $n \rightarrow \infty$, then $\alpha(Sx_n, Sx) \gtrsim 1$.

Then the pair (T, S) has a coincidence point in X .

- (6) In addition, if the pair (T, S) is weakly compatible.

Then (T, S) have a common fixed point.

4. APPLICATION

In this section, the application is presented to establish the existence of a solution of periodic differential equation.

$$\begin{aligned} u'(t) &= f(t, u(t)), \quad t \in I = [0, 1] \\ u(0) &= u(1), \end{aligned} \quad (4.1)$$

where $f : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function. It is easy to see that (4.1) can be rewritten as

$$\begin{aligned} u'(t) + 2u(t) &= f(t, u(t)) + 2u(t), \quad t \in I = [0, 1] \\ u(0) &= u(1), \end{aligned} \quad (4.2)$$

which is equivalent to

$$u(t) = \int_0^1 G(t, s)[f(s, u(s)) + 2u(s)]ds.$$

The Green function $G(t, s)$ associated with (4.1) is given by

$$G(t, s) = \begin{cases} \frac{e^{2(1+s-t)}}{e^2-1} & \text{if } 0 \leq s \leq t \leq 1 \\ \frac{e^{2(s-t)}}{e^2-1} & \text{if } 0 \leq t \leq s \leq 1. \end{cases}$$

It is easy to see that $\max_{t \in [0,1]} \int_0^1 G(t,s)ds = \frac{1}{2}$. Let $X = C([0,1], \mathbb{R}^n)$ be the space of continuous function, $u : [0,1] \rightarrow \mathbb{R}^n$ and $\|(u_1, u_2, \dots, u_n)\| = \max\{|u_1|, |u_2|, \dots, |u_n|\}$ and $d : X \times X \rightarrow \mathbb{C}$ be defined as

$$d_b(u, v) = \left[\max_{t \in [0,1]} \|u(t) - v(t)\| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2$$

where $s = 2$. It is well-known that (X, d_b) is a complete complex valued b -metric space. Define $T : X \rightarrow X$ as

$$Tu(t) = \int_0^1 G(t,s)[f(s, u(s)) + 2u(s)]ds$$

Theorem 4.1. *Suppose the following conditions hold:*

- (1) *there exists $\beta : X \times X \rightarrow \mathbb{S}$, such that $t \in I$ and $a, b \in X$ with $\beta(a, b) \succ 1$,*

$$\|f(t, u) + 2u(s) - f(t, v) + 2v(s)\| \preceq \|u(s) - v(s)\|; \quad (4.3)$$

- (2) *there exists $u_0 \in X$ such that for all $t \in I$, we have*

$$\beta\left(u_0, \int_0^1 G(t,s)f(s, u_0(s))ds\right) \succ 1;$$

- (3) *for all $t \in I$ and $u, v \in X$*

$$\beta(u(t), v(t)) \succ 1 \Rightarrow \beta\left(\int_0^1 G(t,s)f(s, u(s))ds, \int_0^1 G(t,s)f(s, v(s))ds\right) \succ 1;$$

- (4) *if $u_n \rightarrow u \in X$ and $\beta(u_{n+1}, u_n) \succ 1$ for all $n \in \mathbb{N}$ then $\beta(u_n, u) \succ 1$ for all $n \in \mathbb{N}$.*

Then Equation 4.1 has a solution.

Proof. It is well-known that $u \in X$ is a fixed point of T if and only if u is a solution of problem (4.1). We define α as follows:

$\alpha : X \times X \times X \rightarrow \mathbb{S}$ is defined by

$$\alpha(u, v) = \begin{cases} 1 & \text{if } \beta(u(t), v(t)) \succeq 1 \forall t \in I \\ 0 & \text{if otherwise.} \end{cases}$$

It is easy to see that

$$\alpha(u, v) \succeq 1 \Rightarrow \beta(u(t), v(t)) \Rightarrow \beta(Tu(t), Tv(t)) \succeq 1 \Rightarrow \alpha(Tu, Tv) \succeq 1,$$

Thus T is α -admissible mapping. Also, for all $u, v \in X$, we have that

$$\begin{aligned}
d(Tx, Ty) &= \left[\max_{t \in [0,1]} \|Tu(t) - Tv(t)\| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \\
&= \left[\max_{t \in [0,1]} \left\| \int_0^1 G(t, s) [f(s, u(s)) + 2u(s) - f(s, v(s)) - 2v(s)] ds \right\| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \\
&\lesssim \left[\max_{t \in [0,1]} \int_0^1 G(t, s) |u(s) - v(s)| ds \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \\
&= \left[\max_{t \in [0,1]} |u(s) - v(s)| \sqrt{1 + a^2 e^{i \tan^{-1} a}} \right]^2 \left(\max_{t \in [0,1]} \int_0^1 G(t, s) ds \right)^2 \\
&= d(u, v) \frac{1}{4}.
\end{aligned} \tag{4.4}$$

Thus, we have that $2d(Tu, Tv) \lesssim 4d(Tu, Tv) \lesssim d(u, v)$. Clearly, all conditions in Corollary 3.5 are satisfied and guarantees the existence of the fixed point $x \in X$. Thus, x is the solution of the integral equation 4.1. \square

CONCLUSION

The notions of b - C_F simulation function, $\alpha_s S$ -admissible, $\alpha_s S$ - \mathcal{Z}_F -contraction type I mapping and $\alpha_s S$ - \mathcal{Z}_F -contraction type II mapping were introduced with respect to ζ in the framework of complex valued b -metric spaces. Furthermore, some common fixed point and fixed point results for these newly introduced classes of mappings were established. The results were applied to establish the existence of a solution of periodic differential equation. The obtained results in this paper generalize, unify and improve the fixed point results of Samet et al. [28], Liu et al., [20], Antonio-Francisco et al. [23], Khojasteh et al. [18] and other results in this direction, which are in the literature. The classes of mappings which are being considered in this paper are more general and the results are obtained in a more broad space.

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REFERENCES

- [1] C. T. Aage and J. N. Salunke, *Fixed points for weak contractions in G -metric spaces*, Appl. Math. E-Notes, 12, (2012), 23–28.
- [2] H. A. Abass, A. A. Mebawondu and O.T. Mewomo, *Some results for a new three steps iteration scheme in Banach spaces*, Bull. Transilv. Univ. Brasov Ser. III, 11(60) (2018), 1–18.
- [3] A. Aghajani, M. Abbas and J. Roshan, *Common fixed points of generalized weak contractive mappings in partially ordered G_b -metric spaces*, Filomat, 28, (2014), 1087–1101.
- [4] M. O. Aibinu and C. E. Chidume, *Existence and uniqueness of solutions of integral equations of Hammerstein type*, M.Sc Thesis, African University of Science and Technology, Abuja, Nigeria, (2013).
- [5] M. O. Aibinu and P. Pillay, *Iterative algorithms for solutions of nonlinear equations in Banach spaces*, Ph.D Thesis, University of KwaZulu-Natal, South Africa, (2019).
- [6] A. H. Ansari, *Note on φ - ψ -contractive type mappings and related fixed point*, in *Proceedings of the 2nd Regional Conference on Mathematics and Applications*, Payame Noor University, Tonekabon, Iran, (2014), 377–380.
- [7] H. Ayadi, T. Wongyat and W. Sintunavaret, *On new evolutions of Ris result via w -distance and the study on the solution for nonlinear integral equations and fractional differential equations*, Advances in Differences Equations, (2018), 2018:132.
- [8] H. Ayadi, E. Karapinar and W. Shatanawi, *Coupled fixed point results for (ψ, ϕ) -weakly contractive condition in ordered partial metric spaces*, Comput. Math. Appl., 62 (2011), 4449–4460.
- [9] A. Azam, B. Fisher and M. Khan, *Common fixed point theorems in complex valued metric spaces*, Numer. Funct. Anal. Optim. 32 (3), (2011), 243–253.
- [10] S. Banach, *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales*, Fund. Math. 3, (1922), 133–181.
- [11] M. Boriceanu, M. Bota and A. Petrusel, *Multivalued fractals in b -metric spaces*, Cent. Eur. J. Math., 8, (2010), 367–377.
- [12] Y. J. Cho, B. E. Rhoades, R. Saadait, B. Samet and W. Shatanawi, *Nonlinear coupled fixed point theorems in ordered generalized metric space with integral type*, Fixed point theory Appl. 2012, Article ID (2012).
- [13] S. Czerwik, *Contraction mappings in b -metric spaces*, Acta Math. Inform. Univ. Ostraviensis, 1, (1993), 5–11.
- [14] A. K. Dubey and M. Tripathi, *Common Fixed Point Theorem in Complex Valued b -metric Space for Rational Contractions*, J. Inf. Math. Sci., 7 (2015), 149–161.
- [15] N. Hussain, M. A. Kutbi and P. Salimi, *Fixed point theory in α -complete metric spaces with applications*, Abstr. Appl. Anal., 2014, (2014), Article ID 280817.
- [16] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly, 83 (4) (1976), 261–263.
- [17] G. Jungck and N. Hussain, *Compatible maps and invariant approximations*, J. Math. Anal. Appl. 325 (2) (2007), 1003-1012.
- [18] F. Khojasteh, S. Shukla and S. Radenovi, *A new approach to study fixed point theorems via simulation functions*, Filomat, 29 (6), (2015), 1189–1194.
- [19] P. Kumam, D. Gopal and L. Budhiyi, *A new fixed point theorem under Suzuki type \mathcal{Z} -contraction mappings*, J. Math. Anal., 8 (1), (2017), 113–119.
- [20] X. L. Liu, A. H. Ansari, S. Chandok and S. Radenovic, *On some results in metric spaces using auxillary simulation functions via new functions*, J. Comput. Anal. Appl. 24 (6) (2018), 1103–1114.

- [21] J. J. Nieto and R. Rodríguez-Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equation*, Acta Math. Sin., 23 (2007), 2205–2212.
- [22] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal., 47 (4) (2001), 2683–2693.
- [23] A. F. Roldan Lopex de Hierro, E. Karapinar and D. O'Regan, *Coincidence point theorems on metric spaces via simulation functions*, J. Comput. Appl. Math. 275 (2015), 345–355.
- [24] J. K. P. R. Rao, P. R. Swamy and J. R. Prasad, *A common fixed point theorem in complex valued b-metric spaces*, Bull. Maths. Stat. res., 1 (2013), 1–8.
- [25] J. R. Roshan, V. Parvaneh and Z. Kadelburg, *Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces*, J. Nonlinear Sci. Appl., 7, (2014), 229–245.
- [26] P. Salimi, A. Latif and N. Hussain, *Modified α - ψ -contraction mapping with applications*, Fixed Point Theory Appl., 2013, (2013), 1–19.
- [27] P. Salimi and V. Pasquale, *A result of Suzuki type in partial G-metric spaces*, Acta Math. Sci. Ser. B (Engl. Ed.) 34 (2), (2014), 274–284.
- [28] B. Samet, C. Vetro and P. Vetro, *Fixed point theorem for α - ψ -contractive type mappings*, Nonlinear Anal., 75, (2012), 2154–2165.
- [29] S. L. Singh, R. Kamal, M. De la Sen M. and R. Chugh, *A fixed point theorem for generalized weak contractions*, Filomat, 29 (7), (2015), 1481–1490.
- [30] W. Sintunavarat, *Nonlinear integral equations with new admissibility types in b-metric spaces*, J. Fixed Point Theory Appl., DOI 10.1007/s11784-015-0276-6, 1–20.
- [31] W. Sintunavarat, Y. J. Cho and P. Kumam, *Urysohn integral equations approach by common fixed point in complex valued metric spaces*, J. Adv. Difference Equ. 2013, 2013:49, 1–14.
- [32] W. Sintunavarat and P. Kumam, *Generalized common fixed point theorems in complex valued metric spaces and applications*, J. Inequal. Appl. 2012, 2012:84, 1–12.
- [33] W. Sintunavarat, Z. M. Bahadur and M. Sarwar, *Common solution of Urysohn integral equations with the help of common fixed point results in complex valued metric spaces*, Rev. R. Acad. Cienc. Exactas F's. Nat. Ser. A Math. RACSAM, 111 (2), (2017), 531–545.
- [34] W. Sintunavarat and A. Al-Rawashdeh, *Common fixed point of almost generalized (ψ, ϕ) -contractive mappings in ordered metric space*, Fixed point theory Appl., 2012, Article, ID80(2012).
- [35] W. Sintunavarat, *Some fixed point results for a generalized ψ -weak contraction mappings in orbitally metric space*, Chaos Solitons Fractals 45 (2012), 520–526.