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ORIGINAL RESEARCH PAPER

An inertial iterative method for split generalized vector mixed equilibrium and fixed point problems.

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Abstract
In this paper, we introduce an inertial-type algorithm for approximating a common solution of split generalized mixed vector equilibrium and fixed point problems. In the framework of real Hilbert spaces, we state and prove a strong convergence theorem for obtaining a common solution of split generalized mixed vector equilibrium problem and fixed point of a finite family of nonexpansive mappings. Furthermore, we give some consequences of our main result and also report some numerical illustrations to display the performance of our method. The result obtained in this paper unifies and generalizes other corresponding results in the literature.

Keywords Nonexpansive mappings · Hilbert spaces · Split feasibility problem · Generalized vector mixed equilibrium · Fixed point problem

Mathematics Subject Classification 47H06 · 47H09 · 47J05 · 47J25

1 Introduction

The Vector Equilibrium Problem (VEP) is an important aspect of vector optimization problem that unifies other mathematical problems such as vector variational inequality problem, vector saddle point problem, complementarity problem, as well as fixed point problem etc. Let X, Y be two Hausdorff topological spaces, P be a proper, closed and convex cone of Y with $intP \neq \emptyset$ and $F : X \times X \rightarrow Y$ be a bifunction. A Strong Vector Equilibrium Problem (SVEP) is defined as : find $x \in X$ such that

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$$F(x,y) \in P, \quad \forall y \in X,$$

and the Weak Vector Equilibrium Problem (WVEP) is defined as find $x \in X$ such that

$$F(x,y) \notin -intP, \quad \forall y \in X.$$

If $P = \mathbb{R}$ in the above, then the VEP reduces to the classical Equilibrium Problem (EP) introduced by Blum and Oettli [7]. The EP consists of finding $x \in X$ such that

$$F(x,y) \geq 0, \quad \forall y \in X. \tag{1}$$

Peng and Yao [31] studied a generalized version of (1) called the Generalized Mixed Equilibrium Problem (GMEP). They formulated the GMEP as follows: find $x \in X$ such that

$$F(x,y) + \Phi(y) - \Phi(x) + \langle Tx, y - x \rangle \geq 0, \quad \forall y \in X, \tag{2}$$

where F is a bifunction, Φ is a mapping and T is a nonlinear mapping. Following Peng and Yao [31], Shan et al. [35] in 2012 introduced the Generalized Mixed Vector Equilibrium Problem (GMVEP) as follows: Let C be a nonempty, closed and convex subset of a Hilbert space, F be a bifunction, $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be a function and $T : X \to H$ be a nonlinear mapping. The GMVEP is formulated as: find $x \in X$ such that

$$F(x,y) + \Phi(y) - \Phi(x) + e\langle Tx, y - x \rangle \in C, \quad \forall y \in P, \tag{3}$$

where $e \in intP$. They denoted the solution set of (3) by

$$GMVEP(F,\Phi) := \{x \in F(x,y) + \Phi(y) - \Phi(x) + e\langle Tx, y - x \rangle \in C\}, \quad \forall y \in X.$$

Furthermore, they considered an auxiliary problem for the *GMVEP* and proved the existence and uniqueness of the solution for the auxiliary problem. Also, they introduced an iterative method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of *GMVEP* and the solution set of a variational inequality problem.

The following remark highlights some particular cases of (3).

Remark 1.1

(i) If $\Phi \equiv 0, T \equiv 0$ and $e = 1$, then (3) reduces to the classical EP (1).
(ii) If $Y = \mathbb{R}, P = [0, \infty)$ and $e = 1$, then (3) reduces to the generalized mixed equilibrium problem (2).
(iii) If $\Phi \equiv 0$ and $T \equiv 0$, then (3) reduces to the classical VEP.

The vector equilibrium problem and its other variant generalizations have been studied extensively in both Hilbert and Banach spaces (see [3, 7, 11, 15, 17–20, 26–28, 33] and other references therein).

Censor and Elfving [8] introduced the concept of Split Feasibility Problem (SFP) in finite dimensional Hilbert spaces. Let H_1 and H_2 be two Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. The SFP is formulated as

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q.$$

The SFP have notable real life applications in diverse areas such as signal processing, medical image reconstruction, intensity modulated radiation therapy, sensor network, antenna design, immaterial science, computerized tomography, data denoising and data compression (see [4–6, 9, 10] and other references therein). Due to this advantage, SFP for other optimization problems have also been developed and studied by numerous researchers. In 2013, Kazmi and Rizvi [21] introduced the Split Equilibrium Problem (SEP). Let $C \subset H_1$, $Q \subset H_2$ be nonempty, closed and convex sets and $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1 : C \times C \to H_1$ and $F_2 : Q \times Q \to H_1$ be two bifunctions. Then, the SEP is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C \tag{4}$$

and $y = Ax^* \in Q$ solves

$$F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{5}$$

Furthermore, they proposed the following viscosity iterative method for approximating the solutions of SEP and fixed point problem of nonexpansive semigroup in real Hilbert spaces:

$$\begin{cases} u_n = K_{r_n}^{F_1}(x_n + \delta A^*(K_{r_n}^{F_2} - I)Ax_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]\dfrac{1}{s_n}\displaystyle\int_0^{s_n} T(s)u_n ds, \end{cases} \tag{6}$$

where $r_n \in (0, \infty)$, f is a contraction with $\gamma \in (0,1)$, B is a strongly positive linear bounded self-adjoint operator on H_1 with constant $\eta > 0$ such that $0 < \eta < \frac{\eta}{\gamma} < \gamma + \frac{1}{\eta}$, $\{s_n\}$ is a positive real sequence diverging to $+\infty$, $\delta \in (0, \frac{1}{L})$, L is being the spectral radius of the operator A^*A and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$, $K_{r_n}^{F_1}$ and $K_{r_n}^{F_2}$ are resolvent mappings of F_1 and F_2 respectively. They established that the iterative method converges strongly under some mild assumptions.

In the same vein, Yao et al. [39] studied another modified viscosity iterative method for finding a common element of the sets of solutions of mixed equilibrium problem, nonexpansive mappings and variational inclusion problems. Precisely, they proposed the following iterative method:

$$\begin{cases} F(u_n, y) + \Phi(y_n) - \Phi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq 0, \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1-\beta_n)I - \alpha_n(I + \mu B)]W_n K_{R,\lambda}(z_n - rBz_n), \end{cases} \tag{7}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$ and W_n is a system of nonexpansive mappings. They obtained the convergence results of the proposed sequences under some mild conditions. Also, Withayarat et al. [38] introduced and studied an iterative method for finding a common element in the solution sets of mixed equilibrium problem, variational inclusion and fixed point problem of nonexpansive mappings. They proposed the following iterative method:

$$\begin{cases} F(u_n, y) + \Phi(y_n) - \Phi(u_n) + \dfrac{1}{r_n}\langle y - u_n, u_n - (x_n - rAx_n)\rangle \geq 0, \\ z_n = K_{R,s_2}(u_n - s_2 Bu_n), \\ y_n = K_{R,s_1}(z_n - s_1 Dz_n), \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1-\beta_n)I - \alpha_n(I + \mu B)]W_n K_{R,s_1}(z_n - rDz_n), \end{cases} \tag{8}$$

where $\{\alpha_n\}, \{\beta_n\} \in [0,1]$, B and D are inverse strongly monotone mappings, W_n is an infinite family of nonexpansive mappings and K_{R,s_1} and K_{R,s_2} are resolvent mappings. They obtained the convergence results of the proposed sequences under some mild conditions.

Motivated by the works of Censor and Elfving [8], and Kazmi and Rizvi [21], Kazmi et al. [19] extended the notion of split inverse problems to the framework of vector optimization problems in real Hilbert spaces. In particular, the authors [19] introduced and studied the following Split Generalized Vector Equilibrium Problems (SGVEP):

Let $F_1 : C \times C \to Y$ and $F_2 : Q \times Q \to Y$ be nonlinear bifunctions, let $\Phi : C \to Y$ and $\Psi : Q \to Y$ be nonlinear mappings, then the SGVEP is to find $x^* \in C$ such that

$$F_1(x^*, x) + \Phi(x) - \Phi(x^*) \in P, \quad \forall x \in C, \tag{9}$$

and such that $y^* = Ax^* \in Q$ solves

$$F_2(y^*, y) + \Psi(y) - \Psi(y^*) \in P, \quad \forall y \in Q. \tag{10}$$

Furthermore, the authors established the existence and uniqueness of solutions to SGVEP (9) and (10).

Remark 1.2 Observe that if $\Phi \equiv \Psi \equiv 0$ in (9) and (10) then the SGVEP becomes the Split Vector Equilibrium Problem (SVEP). If in addition $P = [0, \infty)$ in SVEP then it reduces to the classical SEP (4) and (5).

On the other hand, it is well known that incorporating inertial term in iterative methods speeds up the rate of convergence of the iterative methods. The inertial-type iterative method was first introduced by Polyak [32]. Consequently, a host of researchers have employed the inertial term to accelerate variant of iterative methods (see [1, 2, 14, 22, 23, 30] and other references therein).

Inspired by the works of Kazmi and Rizvi [21], Shan et al. [35], Rouhani et al. [34], Yao et al. [39] and Withayarat et al. [38], we present an inertial-type iterative method for approximating a common solution of a split generalized vector mixed equilibrium problem and fixed point problems of a finite family of nonexpansive mappings in real Hilbert spaces. Furthermore, we prove a strong convergence theorem of the proposed method in the framework of real Hilbert space. In addition, we also report some numerical illustrations to display the performance of our method. Our result extends and complements some results in the literature.

2 Preliminaries

We state some known and useful results which will be needed in the proof of our main result. In the sequel, we denote strong and weak convergence by $'' \to ''$ and $'' \rightharpoonup ''$ respectively.

Lemma 2.1 [12] *Let H_1 be a real Hilbert space and C be a nonempty, closed and convex subset of H_1. The following inequalities hold:*

(i) $||x-y||^2 = ||x||^2 - ||y||^2 - 2\langle x-y, y\rangle, \quad \forall x, y \in H_1,$
(ii) $||x+y||^2 \leq ||x||^2 + 2\langle x+y, y\rangle, \quad \forall x, y \in H_1,$
(iii) $||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2, \quad \forall \alpha \in [0,1] \quad \forall x, y \in H_1.$

The nearest projection P_C from H_1 to C assign to each $x \in H_1$, the unique point $P_C x$ satisfying the property

$$||x - P_C x|| = \min_{y \in C} ||x-y||.$$

The following is a very useful property of the nearest point mapping:

$$\langle x - P_C x, P_C x - y\rangle \geq 0, \quad \forall y \in C.$$

Let $T : C \to C$ be a mapping. We denote the set of all fixed points of T by $Fix(T)$, that is, $Fix(T) = \{x \in C : x = Tx\}$. The mapping T is said to be:

(i) a contraction, if $T : C \to C$ is called λ-contraction, if there exists a constant $\lambda \in (0,1)$ such that

$$||Tx - Ty|| \leq \lambda||x-y||, \quad \forall x, y \in C,$$

(ii) nonexpansive, if

$$||Tx - Ty|| \leq ||x-y||, \quad \forall x, y \in C.$$

It is well known that the metric projection P_C is nonexpansive.

(iii) β-inverse strongly monotone, if there exists a constant $\beta > 0$ such that

$$\langle Ax - Ay, x-y\rangle \geq \beta||Ax - Ay||^2, \quad \forall x, y \in C.$$

If $\beta = 1$, then A is firmly nonexpansive.

An operator B is strongly positive, if there exists a constant $\mu > 0$ with the property

$$\langle Bx, x\rangle \geq \mu||x||^2, \quad \forall x \in H_1.$$

Lemma 2.2 [29] *Assume B is a strong positive linear bounded operator on a real Hilbert space H_1 with a coefficient $\mu > 0$ and $0 < \rho \leq ||B||^{-1}$. Then $||I - \rho B|| \leq 1 - \rho\mu$.*

Definition 2.3 A mapping $T : H_1 \to H_1$ is said to be an averaged mapping, if and only if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T = (1 - \alpha)I + \alpha S, \tag{11}$$

where α is a number (0, 1) and $S : H_1 \to H_1$ is nonexpansive. More precisely, when (11) holds, we say that T is α-averaged. Thus firmly nonexpansive mappings (in particular projections) are $\frac{1}{2}$-average maps.

Lemma 2.4 [6, 13] *Let the operators $S, T, G : H_1 \to H_1$ be given.*

- (i) If $T = (1 - \alpha)S + \alpha G$ for some $\alpha \in (0, 1)$ and if S is averaged and G is nonexpansive, then S is averaged.
- (ii) T is firmly nonexpansive, if and only if the complement $I - T$ is firmly nonexpansive.
- (iii) If $T = (1 - \alpha)I + \alpha G$ for some $\alpha \in (0, 1)$, S is firmly nonexpansive and G is nonexpansive, then T is averaged.

Definition 2.5 Let C be a nonempty, closed and convex subset of a Hilbert space H_1. Let $T_i : C \to H_1, i = 0, 1, 2, \ldots, N$ be a finite family of nonexpansive mappings. Then, we define the mapping $W_n : C \to C \forall x \in C$ as follows:

$$\begin{aligned} U_{n,0} &= I \\ U_{n,1} &= \lambda_{n,1} T_1 U_{n,0} + (1 - \lambda_{n,1})I \\ U_{n,2} &= \lambda_{n,2} T_2 U_{n,1} + (1 - \lambda_{n,2})I \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1} T_{N-1} U_{n,N-2} + (1 - \lambda_{n,N-1})I \\ W_n = U_{n,N} &= \lambda_{n,N} T_N U_{n,N-1} + (1 - \lambda_{n,N})I, \end{aligned}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real number such that $0 < \lambda_n \leq 1$ for all $n \geq 1$.

Remark 2.6 [39] W_n mapping is nonexpansive.

Lemma 2.7 [36] *Let C be a nonempty, closed and convex subset of a real Hilbert space H_1. Let $\{T_n\}_{n=1}^N$ be a finite family of nonexpansive mapping $T_n : H_1 \to H_1$ such that $\cap_{n=1}^N Fix(T_n) \neq \emptyset$. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_n \leq b < 1$ for all $n \in N$. Then the following statements hold*:

- (i) for all $x \in H$ and $k \in N$, then the limit $\lim_{n\to\infty} U_{n,k}x$ exists,

(ii) $Fix(W) = \cap_{n=1}^{N} Fix(T_n)$, where $Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,N} x$ for all $x \in C$,

(iii) for any bounded sequence $\{x_n\}$ in H_1, $\lim_{n\to\infty} ||Wx_n - W_n x_n|| = 0$.

Lemma 2.8 [36] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\beta_n\}$ be a sequence* [0, 1] *such that* $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. *Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$, for all $n \ge 0$ and* $\limsup_{n\to\infty}(||z_{n+1} - z_n|| - ||x_{n+1} - x_n||) \le 0$. *Then* $\lim_{n\to\infty} ||z_n - x_n|| = 0$.

Lemma 2.9 [24, 25] *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \le (1-\alpha_n) a_n + \delta_n, \quad n \ge 0,$$

where $\{\alpha_n\}$ and $\{\delta_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0$ or $\sum_{n=0}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Definition 2.10 [28, 37] Let X and Y be two Hausdorff topological spaces, E be a nonempty convex subset of X and P be a proper, closed and convex cone of Y with *int* P $\neq \emptyset$. Let 0 be the zero point of Y, $\mathbb{U}(0)$ be the neighborhood set of 0, $\mathbb{U}(x_0)$ be the neighborhood set of x_0 and $h : E \to Y$ be a mapping:

1. If for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x)$ such that

$$h(x) \in h(x_0) + V + P, \quad \forall x \in U \cap E,$$

then h is called upper P-continuous on x_0. If h is upper P-continuous for all $x \in E$, then h is called upper P-continuous on E.

2. If for any $V \in \mathbb{U}(0)$ in Y, there exists $U \in \mathbb{U}(x)$ such that

$$h(x) \in h(x_0) + V - P, \quad \forall x \in U \cap E,$$

then h is called lower P-continuous on x_0. If h is lower P-continuous for all $x \in E$, then h is called lower P-continuous on E.

3. If for any $x, y \in E$ and $\alpha \in [0,1]$, the mapping h satisfies

$$h(x) \in h(\alpha x + (1-\alpha)y) + P \text{ or } h(y) \in h(\alpha x + (1-\alpha)y) + P,$$

then h is called proper P-quasi-convex.

4. If for any $x_1, x_2 \in E$ and $\alpha \in [0,1]$, the mapping h satisfies

$$\alpha h(x_1) + (1-\alpha) h(x_2) \in h(\alpha x_1 + (1-\alpha) x_2) + P,$$

then h is called P-convex.

Lemma 2.11 [16] *Let X and Y be two Hausdorff topological spaces, E be a nonempty, compact and convex subset of X and P is a proper, closed and convex cone of Y with int* $\mathrm{P} \neq \emptyset$. *Assume that $F : E \times E \to Y$ are $\Phi : E \to Y$ are two vector mappings. Suppose that F and Φ satisfy*

(i) $F(x,x) \in \mathrm{P}, \forall x \in E$,
(ii) Φ is upper P -continuous on E,
(iii) $F(\cdot, y)$ is lower P -continuous, $\forall y \in E$,
(iv) $F(x,\cdot) + \Phi(\cdot)$ is proper P -quasi-convex, $\forall x \in E$. Then there exists a point $x \in X$ satisfying

$$\bar{F}(x,y) \in P \setminus \{0\}, \forall x \in E$$

where

$$\bar{F}(x,y) = F(x,y) + \Phi(y) - \Phi(x), \quad \forall x, y \in E.$$

To solve the generalized mixed equilibrium problem, the following assumptions are required.

Let $F : X \times X \to Y$ and $\Phi : X \to Y$ be two mappings. For any $z \in H_1$, define a mapping $F : X \times X \to Y$ as follows

$$F_z(x,y) = F(x,y) + \Phi(y) - \Phi(x) + \frac{e}{r}\langle y - x, x - z\rangle,$$

where r is a positive number in $\mathbb{R}$ and $e \in P$. Then $F_z, \bar{F}, \Phi$ satisfy the following conditions

(A1) for al $x \in X$, $F(x,x) = 0$,
(A2) F is monotone, that is $F(x,y) + F(y,x) \in -P, \forall x, y \in X$,
(A3) $F(x, .)$ is continuous, $\forall y \in X$,
(A4) $F(x, .)$ is weakly continuous and P-convex, that is

$$\alpha F(x, y_1) + (1-\alpha)F(x, y_2) \in F(x, \alpha y_1 + (1-\alpha)y_2) + P, \forall x, y_1, y_2 \in X,$$

(A5) $F_z(.,y)$ is lower continuous, $\forall y \in X$ and H_1,
(A6) Φ is P-convex and weakly continuous,
(A7) $F_z(x,.)$ is proper P-quasi-convex, $\forall x \in X$ and $z \in H_1$.

Lemma 2.12 [35] *Let F and Φ satisfy assumptions* $(A1)$-(A7). Define the mapping $K_r : H_1 \to X$ as follows

$$K_r^F = \{F(x,y) + \Phi(y) - \Phi(x) + \frac{e}{r}\langle y - x, x - z\rangle \in P \forall y \in X\}.$$

Then

1. $K_r^F(z) \neq \emptyset, \forall z \in H_1$,
2. K_r^F is singlevalued,

3. K_r^F is firmly nonexpansive, that is

$$||K_r^F z_1 - K_r^F z_2||^2 \leq \langle K_r^F z_1 - K_r^F z_2, z_1 - z_2\rangle, \forall z_1, z_2 \in H_1,$$

4. $Fix(K_r^F) = GMVEP(F, \Phi)$,
5. $GMVEP(F, \Phi)$ is closed and convex.

Following the ideas in [19, 35], we introduce the split generalized vector mixed equilibrium problem (SGMVEP) as follows:

Let H_1 and H_2 be two Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Let $F_1 : C \times C \to Y$ and $F_2 : Q \times Q \to Y$ be nonlinear bifunctions, let $\Phi_1 : C \to Y$, $\Psi_2 : Q \to Y$, $L_1 : X \to H_1$ and $L_2 : Y \to H_2$ be nonlinear mappings, then the SGMVEP is to find $x^* \in C$ such that

$$F_1(x, x^*) + \Phi_1(x^*) - \Phi_1(x) + e\langle L_1 x, x^* - x\rangle \in C, \quad \forall x^* \in P, \tag{12}$$

and such that $y^* = Ax^* \in Q$ solves

$$F_2(y, y^*) + \Psi_2(y^*) - \Psi_2(y) + e\langle L_2 y, y^* - y\rangle \in Q, \quad \forall y^* \in P. \tag{13}$$

We denote the solution set of SGMVEP (12) and (13) by $SGMVEP(F_1, \Phi_1, \Psi_1, F_2, \Phi_2, \Psi_2) = \{x^* \in GMVEP(F_1, \Phi_1, \Psi_i) : Ax^* \in GMVEP(F_2, \Phi_2, \Psi_2)\}$. It follows from [8] (see also [19, 35]) that the SGMVEP ((12) and (13)) are well-defined and the solution set $SGVMEP(F_1, \Phi_1, \Psi_1, F_2, \Phi_2, \Psi_2)$ is closed and convex.

Remark 2.13 If $\Phi \equiv \Psi \equiv 0$ in (12) and (13), we obtain the vector split variational inequality problem as introduced by Giannessi [15]. The SGVMEP is related to the Split Mixed Vector Variational Inequality Problem (SMVVIP) if $F_1 = F_2 = 0, e = 1$ such that Φ_1 and Ψ_2 are bifunctions. See [34] for details on the SMVVIP.

3 Main results

Theorem 3.1 *Let C and Q be nonempty, compact, convex subset of real Hilbert spaces H_1 and H_2 respectively. Assume that P and D are closed, convex cones of real Hausdorff topological spaces Y and Z with e and d fixed points in P and D respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $T_i : C \to C$ be a finite family of nonexpansive mappings, $i = 1, 2, \ldots, N$. Let $F_1 : C \times C \to Y$, $F_2 : Q \times Q \to Z$, $\Phi_1 : C \to Y$ and $\Phi_2 : Q \to Z$ be functions satisfying assumptions* (A1)-(A7). Let $\Psi_1 : C \to H_1$ and $\Psi_2 : Q \to H_2$ be β_1 and β_2 inverse strongly monotone mappings respectively. Let f be a contraction of H_1 into itself with coefficient $\lambda \in (0, 1)$ and B be a strongly positive linear bounded operator defined as in Lemma 2.2. Assume that $\Gamma = \cap_{i=1}^N Fix(T_i) \cap SGMVEP(F_1, \Phi_1, \Psi_1, F_2, \Phi_2, \Psi_2) \neq \emptyset$. Let the sequences $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by $u, x_0, x_1 \in H_1$ and

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1-\beta_n)I - \alpha_n(I + \eta B)]W_n y_n, \end{cases} \tag{14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in [0, 1], $r, s > 0, \gamma < \frac{(1+\eta)\mu}{\lambda}$ and $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n < \liminf_{n\to\infty} \beta_n < 1$

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, and $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* = P_\Gamma(u + \gamma f(x^*) - \eta B x^*) \in \Gamma$.

Proof First, we show that $A^*(I - K_s^{F_2})A$ is $\frac{1}{L}$-inverse strongly monotone. From Lemma 2.12(3) and the fact that $I - K_s^{F_2}$ is firmly nonexpansive (see Lemma 2.4(ii)), we have

$$\begin{aligned} ||A^*(I - K_s^{F_2})Ax - A^*(I - K_s^{F_2})Ay||^2 &= ||A^*(I - K_s^{F_2})A(x-y)||^2 \\ &= \langle A^*(I - K_s^{F_2})A(x-y), A^*(I - K_s^{F_2})A(x-y)\rangle \\ &= \langle (I - K_s^{F_2})A(x-y), AA^*(I - K_s^{F_2})A(x-y)\rangle \\ &\le L\langle (I - K_s^{F_2})A(x-y), (I - K_s^{F_2})A(x-y)\rangle \\ &= L||(I - K_s^{F_2})A(x-y)||^2 \\ &\le L\langle (I - K_s^{F_2})A(x-y), A(x-y)\rangle \\ &= L\langle (x-y, A^*(I - K_s^{F_2})A(x-y)\rangle \\ &= L\langle (x-y, A^*(I - K_s^{F_2})Ax - A^*(I - K_s^{F_2})Ay\rangle, \quad \forall x, y \in H. \end{aligned}$$

This implies that $A^*(I - K_s^{F_2})A$ is $\frac{1}{L}$ inverse strongly monotone. Since $\xi \in (0, \frac{1}{L})$, it follows that $I - \xi A^*(I - K_s^{F_2})A$ is nonexpansive. We divide the rest of the proof into 5 steps.

Step 1 We show that $\{x_n\}$ is bounded. Let $p \in \Gamma$, then we have $p = K_r^{F_1}p$ and $p = (I - \xi A^*(I - K_s^{F_2})A)p$. By nonexpansivity of $I - \xi A^*(I - K_s^{F_2})A$, it implies that

$$\begin{aligned} ||y_n - p|| &= ||K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n - K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)p|| \\ &\le ||(I - \xi A^*(I - K_s^{F_2})A)w_n - (I - \xi A^*(I - K_s^{F_2})A)p|| \\ &\le ||w_n - p|| \\ &= ||x_n + \theta_n(x_n - x_{n-1}) - p|| \\ &\le ||x_n - p|| + \theta_n||x_n - x_{n-1}||. \end{aligned} \tag{15}$$

From (14), (15) and Remark 2.6, we have

$$
\begin{aligned}
||x_{n+1}-p|| &= ||\alpha_n(u+\gamma f(x_n))+\beta_n x_n+[(1-\beta_n)I-\alpha_n(I+\eta B)]W_n y_n-p|| \\
&= ||\alpha_n u+\alpha_n(\gamma f(x_n))-(I+\eta B)p)+\beta_n(x_n-p)+(1-\beta_n)(W_n y_n-p) \\
&\quad -\alpha_n(I+\eta B)(W_n y_n-p)|| \\
&= ||\alpha_n u+\alpha_n(\gamma f(x_n))-(I+\eta B)p)+\beta_n(x_n-p)+[(1-\beta_n)I-\alpha_n(I+\eta B)](W_n y_n-p)|| \\
&\le \alpha_n||u||+\alpha_n||(\gamma f(x_n))-(I+\eta B)p)||+\beta_n||x_n-p||+(1-\beta_n)-\alpha_n(1+\eta\mu)||W_n y_n-p|| \\
&\le \alpha_n||u||+\alpha_n\gamma\lambda||x_n-p||+\alpha_n||\gamma f(p)-(I+\eta B)p)||+\beta_n||x_n-p|| \\
&\quad +(1-\beta_n)-\alpha_n(1+\eta\mu)||y_n-p|| \\
&\le \alpha_n||u||+\alpha_n\gamma\lambda||x_n-p||+\alpha_n||\gamma f(p)-(I+\eta B)p)||+\beta_n||x_n-p|| \\
&\quad +(1-\beta_n)-\alpha_n(1+\eta\mu)\big[||x_n-p||+\theta_n||x_n-x_{n-1}||\big] \\
&= \alpha_n||u||+\alpha_n\gamma\lambda||x_n-p||+\alpha_n||\gamma f(p)-(I+\eta B)p)||+\beta_n||x_n-p|| \\
&\quad +(1-\beta_n)-\alpha_n(1+\eta\mu)||x_n-p||+\theta_n(1-\beta_n)-\alpha_n(1+\eta\mu)||x_n-x_{n-1}|| \\
&\le (1-\alpha_n(1+\eta\mu-\gamma\lambda))||x_n-p||+\alpha_n||u||+\alpha_n||\gamma f(p)-(I+\eta B)p)|| \\
&\quad -\alpha_n\theta_n(1+\eta\mu)||x_n-x_{n-1}||+\theta_n||x_n-x_{n-1}|| \\
&= (1-\alpha_n(1+\eta\mu-\gamma\lambda))||x_n-p||+\alpha_n\big[||u||+||\gamma f(p)-(I+\eta B)p)|| \\
&\quad -\theta_n(1+\eta\mu)||x_n-x_{n-1}||+M\big],
\end{aligned}
\tag{16}
$$

where $M>0$ is a constant such that

$$
\frac{\theta_n}{\alpha_n}||x_n-x_{n-1}||\le M, \quad \forall n\ge 1.
$$

This implies that

$$
\begin{aligned}
||x_{n+1},p|| &= (1-\alpha_n(1+\eta\mu-\gamma\lambda))||x_n-p||+\alpha_n\big[||u||+||\gamma f(p)-(I+\eta B)p)|| \\
&\quad -\big(\theta_n(1+\eta\mu)\big)||x_n-x_{n-1}||+M\big] \\
&\le \max\left\{||x_n-p||,\frac{||u||+||\gamma f(p)-(I+\eta B)p)||-\big(\theta_n(1+\eta\mu)\big)||x_n-x_{n-1}||+M}{\alpha_n(1+\eta\mu-\gamma\lambda)}\right\} \\
&\ \ \vdots \\
&\le \max\left\{||x_0-p||,\frac{||u||+||\gamma f(p)-(I+\eta B)p)||-\big(\theta_n(1+\eta\mu)\big)||x_n-x_{n-1}||+M}{\alpha_n(1+\eta\mu-\gamma\lambda)}\right\}.
\end{aligned}
\tag{17}
$$

Therefore $\{x_n\}$ is bounded, and consequently, $\{w_n\}$ and $\{y_n\}$ are also bounded.

Step 2 We show that $\lim_{n\to\infty}||x_{n+1}-x_n||=0$. Define $x_{n+1}=\beta_n x_n+(1-\beta_n)v_n$, then $v_n=\frac{x_{n+1}-\beta_n x_n}{1-\beta_n}$.

$$
\begin{aligned}
v_{n+1} - v_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}(u + \gamma f(x_{n+1})) + \beta_{n+1}x_{n+1} + [(1 - \beta_{n+1})I - \alpha_{n+1}(I + \eta B)]W_{n+1}y_{n+1} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{\alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n(I + \eta B)]W_n y_n - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(u + \gamma f(x_{n+1})) \\
&\quad - \frac{\alpha_n}{1 - \beta}(u + \gamma f(x_n)) + \frac{(1 - \beta_{n+1})I - \alpha_{n+1}(I + \eta B)]W_{n+1}y_{n+1}}{1 - \beta_{n+1}} \\
&\quad - \frac{(1 - \beta_n)I - \alpha_n(I + \eta B)]W_n y_n}{1 - \beta_n} \\
&\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\Big[(u + \gamma f(x_{n+1})) - (I + \eta B)]W_{n+1}y_{n+1}\Big] \\
&\quad - \frac{\alpha_n}{1 - \beta_n}\Big[(u + \gamma f(x_n)) - (I + \eta B)]W_n y_n\Big] \\
&\quad + ||W_{n+1}y_{n+1} - W_{n+1}y_n|| + ||W_{n+1}y_n - W_n y_n|| \\
&\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\Big[(u + \gamma f(x_{n+1})) - (I + \eta B)]W_{n+1}y_{n+1}\Big] \\
&\quad - \frac{\alpha_n}{1 - \beta_n}\Big[(u + \gamma f(x_n)) - (I + \eta B)]W_n y_n\Big] \\
&\quad + ||y_{n+1} - y_n|| + ||W_{n+1}y_n - W_n y_n||.
\end{aligned} \tag{18}
$$

We obtain from (18) that

$$
\begin{aligned}
||v_{n+1} - v_n|| - ||x_{n+1} - x_n|| \le{}& \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\Big[(u + \gamma f(x_{n+1})) - (I + \eta B)]W_{n+1}y_{n+1}\Big] \\
&- \frac{\alpha_n}{1 - \beta_n}\Big[(u + \gamma f(x_n)) - (I + \eta B)]W_n y_n\Big] \\
&+ ||y_{n+1} - y_n|| + ||W_{n+1}y_n - W_n y_n|| - ||x_{n+1} - x_n||.
\end{aligned} \tag{19}
$$

From (14) and nonexpansivity of $K_r^{F_1}$ and $(I - \xi A^*(I - K_s^{F_2})$, we obtain that

$$
\begin{aligned}
||y_{n+1} - y_n|| &= \left|\left|K_r^{F_1}\left(I - \xi A^*\left(I - K_s^{F_2}\right)A\right)w_{n+1} - K_r^{F_1}\left(I - \xi A^*\left(I - K_s^{F_2}\right)A\right)w_n\right|\right| \\
&\le ||w_{n+1} - w_n||
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&= ||x_{n+1} + \theta_{n+1}(x_{n+1} - x_n) - x_n - \theta_n(x_n - x_{n-1})|| \\
&\leq ||x_{n+1} - x_n|| + \theta_{n+1}||x_{n+1} - x_n|| + \theta_n||x_{n-1} - x_n|| \\
&= ||x_{n+1} - x_n|| + \theta_{n+1}||x_{n+1} - x_n|| + \theta_n||x_n - x_{n-1}|| \\
&\leq ||x_{n+1} - x_n|| + \theta_{n+1}||x_{n+1} - x_n|| + \theta_n||x_n - x_{n-1}|| + \theta_{n-1}||x_{n-1} - x_{n-2}|| \\
&\leq ||x_{n+1} - x_n|| + \sum_{i=1}^{n+1} \theta_i||x_i - x_{i-1}||.
\end{aligned}
\tag{21}
$$

It follows from Definition 2.5 and Remark 2.6 that

$$
\begin{aligned}
||W_{n+1}y_n - W_n y_n|| &= ||\lambda_{n+1,N}T_N U_{n+1,N-1}y_n + (1 - \lambda_{n+1,N})y_n - \lambda_{n,N}T_N U_{n,N-1}y_n + (1 - \lambda_{n,N})y_n|| \\
&\leq ||\lambda_{n+1,N}T_N U_{n+1,N-1}y_n - \lambda_{n,N}T_N U_{n,N-1}y_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|||y_n|| \\
&\leq ||\lambda_{n+1,N}T_N U_{n+1,N-1}y_n - \lambda_{n+1,N}T_N U_{n,N-1}y_n|| \\
&\quad + ||\lambda_{n+1,N}T_N U_{n,N-1}y_n - \lambda_{n,N}T_N U_{n,N-1}y_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|||y_n|| \\
&\leq \lambda_{n+1,N}||U_{n+1,N-1}y_n - U_{n,N-1}y_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|||T_N U_{n,N-1}y_n|| \\
&\quad + |\lambda_{n+1,N} - \lambda_{n,N}|||y_n|| \\
&\leq \lambda_{n+1,N}||U_{n+1,N-1}y_n - U_{n,N-1}y_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|M_1,
\end{aligned}
\tag{22}
$$

where M_1 is a constant such that $M_1 \geq \max\{\sup_{n\geq 1}||y_n||, \sup_{n\geq 1}||T_N U_{n,N-1}y_n||\}$. From (22), we have

$$
\begin{aligned}
||U_{n+1,N-1}y_n - U_{n,N-1}y_n|| &= ||\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n + (1 - \lambda_{n+1,N-1})y_n \\
&\quad - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n + (1 - \lambda_{n,N-1})y_n|| \\
&\leq ||\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n|| \\
&\quad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|||y_n|| \\
&\leq ||\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}y_n - \lambda_{n+1,N-1}T_{N-1}U_{n,N-2}y_n|| \\
&\quad + ||\lambda_{n+1,N-1}T_{N-1}U_{n,N-2}y_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}y_n|| \\
&\quad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|||y_n|| \\
&\leq \lambda_{n+1,N-1}||U_{n+1,N-2}y_n - U_{n,N-2}y_n|| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|||T_N U_{n,N-2}y_n|| \\
&\quad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|||y_n|| \\
&\leq \lambda_{n+1,N-1}||U_{n+1,N-2}y_n - U_{n,N-2}y_n|| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|M_2,
\end{aligned}
\tag{23}
$$

where M_2 is a constant such that $M_2 \geq \max\{\sup_{n\geq 1}||y_n||, \sup_{n\geq 1}||T_{N-1}U_{n,N-2}y_n||\}$. Following same argument in (22) and (23), we obtain

$$
||U_{n+1,N-1}y_n - U_{n,N-1}y_n|| \leq \sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|M_3, \tag{24}
$$

where M_3 is a constant such that $M_2 \geq \max\{\sup_{n\geq 1}||y_n||, \sup_{n\geq 1}||T_i U_{i,i-1}y_n||\}$.

Substituting (24) into (22), we have

$$\begin{aligned}||W_{n+1}y_n - W_ny_n|| &\leq |\lambda_{n+1,N} - \lambda_{n,N}|M_1 + \lambda_{n+1,N}\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|M_3 \\ &\leq \sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|M_4,\end{aligned} \tag{25}$$

where M_4 is a constant such that $M_4 \geq \max\{M_1, M_3\}$. Substituting (21) and (25) in (19), we have

$$\begin{aligned}||v_{n+1} - v_n|| - ||x_{n+1} - x_n|| \leq & \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left[(u + \gamma f(x_{n+1})) - (I + \eta B)\right]W_{n+1}y_{n+1}\Big] \\ & - \frac{\alpha_n}{1-\beta_n}\left[(u + \gamma f(x_n)) - (I + \eta B)\right]W_ny_n\Big] \\ & + ||x_{n+1} - x_n|| + \sum_{i=1}^{n+1}\theta_i||x_i - x_{i-1}|| \\ & + \sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|M_4 - ||x_{n+1} - x_n||.\end{aligned}$$

By conditions (C1), (C2) and (C3), we obtain that

$$\limsup_{n\to\infty}(||v_{n+1} - v_n|| - ||x_{n+1} - ||x_n||) \leq 0.$$

Hence, by Lemma 2.8, we have that

$$\lim_{n\to\infty}||v_n - x_n|| = 0.$$

But

$$\begin{aligned}\lim_{n\to\infty}||v_n - x_n|| &= \left\|\frac{x_{n+1} - \beta_n x_n}{1-\beta_n} - x_n\right\| \\ &= \left\|\frac{x_{n+1} - x_n}{1-\beta_n}\right\|.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = 0. \tag{26}$$

Step 3 We show that $\lim_{n\to\infty}\left\|Aw_n - K_s^{F_2}Aw_n\right\| = 0$ and $\lim_{n\to\infty}||x_n - y_n|| = 0$. From (14), we have

$$\begin{aligned}||x_{n+1}-p||^2 &= ||\alpha_n(u+\gamma f(x_n))+\beta_n x_n+[(1-\beta_n)I-\alpha_n(I+\eta B)]W_n y_n-p||^2\\ &= ||\alpha_n(u+\gamma f(x_n)-(I+\eta B)p)+\beta_n(x_n-p)\\ &\quad +(1-\beta_n)(W_n y_n-p)-\alpha_n(I+\eta B)(W_n y_n-p)||^2\\ &= ||\alpha_n(u+\gamma f(x_n)-(I+\eta B)p)+\beta_n(x_n-p)+\beta_n(x_n-W_n y_n)\\ &\quad +(W_n y_n-p)-\alpha_n(I+\eta B)(W_n y_n-p)||^2\\ &= ||\alpha_n(u+\gamma f(x_n)-(I+\eta B)p)+\beta_n(x_n-p)+\beta_n(x_n-W_n y_n)\\ &\quad +(I-\alpha_n(I+\eta B)(W_n y_n-p)||^2\\ &= ||\alpha_n(u+\gamma f(x_n)-(I+\eta B)p)+\beta_n(x_n-W_n y_n)+(I-\alpha_n(I+\eta B)(W_n y_n-p)||^2\\ &\le ||(I-\alpha_n(I+\eta B)(W_n y_n-p)+\beta_n(x_n-W_n y_n)||^2\\ &\quad +2\alpha_n\langle(u+\gamma f(x_n)-(I+\eta B)p),x_{n+1}-p\rangle\\ &\le \Big[||(I-\alpha_n(I+\eta B))||||y_n-p||+\beta_n||x_n-W_n y_n||\Big]^2\\ &\quad +2\alpha_n\langle(u+\gamma f(x_n)-(I+\eta B)p),x_{n+1}-p\rangle\\ &\le \Big[||(I-\alpha_n(I+\eta B))||||y_n-p||+\beta_n||x_n-W_n y_n||\Big]^2\\ &\quad +2\alpha_n||(u+\gamma f(x_n)-(I+\eta B)p)||||x_{n+1}-p||\\ &\le (1-\alpha_n(I+\eta\mu)^2||y_n-p||^2+2(1-\alpha_n(I+\eta\mu)^2||y_n\\ &\quad -p||\beta_n||x_n-W_n y_n||+\beta_n^2||x_n-W_n y_n||^2\\ &\quad +2\alpha_n||(u+\gamma f(x_n)-(I+\eta B)p)||||x_{n+1}-p||.\end{aligned}$$

Since $(1-\alpha_n(I+\eta\mu)<1$, it implies that

$$\begin{aligned}&||x_{n+1}-p||^2||y_n-p||^2+2(1-\alpha_n(I+\eta\mu)^2||y_n-p||\beta_n||x_n-W_n y_n||+\beta_n^2||x_n-W_n y_n||^2\\ &\quad +2\alpha_n||(u+\gamma f(x_n)-(I+\eta B)p)||||x_{n+1}-p||.\end{aligned}\tag{27}$$

Again from (14) and Lemma 2.12(3), we have

$$\begin{aligned}||y_n-p||^2 &= \left|\left|K_r^{F_1}\left(I-\xi A^*\left(I-K_s^{F_2}\right)A\right)w_n-K_r^{F_1}p\right|\right|^2\\ &\le \left|\left|w_n-p-\xi A^*\left(I-K_s^{F_2}\right)Aw_n\right|\right|^2\\ &\le ||w_n-p||^2-2\xi\left\langle w_n-p,A^*\left(I-K_s^{F_2}\right)Aw_n\right\rangle+\xi^2\left|\left|A^*\left(I-K_s^{F_2}\right)Aw_n\right|\right|^2\\ &= \le ||w_n-p||^2-2\xi\left\langle Aw_n-Ap,\left(I-K_s^{F_2}\right)Aw_n\right\rangle+\xi^2\left|\left|A^*\left(I-K_s^{F_2}\right)Aw_n\right|\right|^2.\end{aligned}\tag{28}$$

If we simplify the second term in (28), we have

$$\begin{aligned}\left\langle Aw_n-Ap,\left(I-K_s^{F_2}\right)Aw_n\right\rangle &= \left\langle Aw_n-K_s^{F_2}Aw_n,\left(Aw-K_s^{F_2}\right)Aw_n\right\rangle+\left\langle K_s^{F_2}Aw_n-Ap,\left(Aw-K_s^{F_2}\right)Aw_n\right\rangle\\ &= \left|\left|Aw_n-K_s^{F_2}Aw_n\right|\right|^2+\langle K_s^{F_2}Aw_n-Ap,Aw-K_s^{F_2}Aw_n\rangle\\ &= \left|\left|Aw_n-K_s^{F_2}Aw_n\right|\right|^2.\end{aligned}\tag{29}$$

Hence, (28) becomes

$$\begin{aligned}||y_n - p||^2 &\le ||w_n - p||^2 - 2\xi\big|\big|Aw_n - K_s^{F_2}Aw_n\big|\big|^2 + \xi^2||A^*(I - K_s^{F_2})Aw_n\big|\big|^2\\ &\le ||w_n - p||^2 - 2\xi\big|\big|Aw_n - K_s^{F_2}Aw_n\big|\big|^2 + \xi^2||A||^2\big|\big|Aw_n - K_s^{F_2}Aw_n\big|\big|^2\\ &\le ||w_n - p||^2 - \xi(2 - \xi||A||^2)\big|\big|Aw_n - K_s^{F_2}Aw_n\big|\big|^2.\end{aligned} \tag{30}$$

Substituting (30) in (27), we have

$$\begin{aligned}||x_{n+1} - p||^2 \le{}& ||w_n - p||^2 - \xi(2 - \xi^2||A||^2)\big|\big|Aw_n - K_s^{F_2}Aw_n\big|\big|^2\\ &+ 2(1 - \alpha_n(I + \eta\mu)^2||y_n - p||\beta_n||x_n - W_ny_n||\\ &+ \beta_n^2||x_n - W_ny_n||^2 + 2\alpha_n||(u + \gamma f(x_n) - (I + \eta B)p)||||x_{n+1} - p||.\end{aligned}$$

This implies

$$\begin{aligned}\xi(2 - \xi||A||^2)&\big|\big|Aw_n - K_s^{F_2}Aw_n\big|\big|^2 \le ||w_n - p||^2\\ &+ 2(1 - \alpha_n(I + \eta\mu)^2||y_n - p||\beta_n||x_n - W_ny_n||\\ &+ \beta_n^2||x_n - W_ny_n||^2 + 2\alpha_n||(u + \gamma f(x_n) - (I + \eta B)p)||||x_{n+1} - p||\\ &- ||x_{n+1} - p||^2\\ \le{}& ||w_n - x_{n+1}(||w_n - p|| + ||x_{n+1} - p||)\\ &+ 2(1 - \alpha_n(I + \eta\mu)^2||y_n - p||\beta_n||x_n - W_ny_n||\\ &+ \beta_n^2||x_n - W_ny_n||^2 + 2\alpha_n||(u + \gamma f(x_n)\\ &- (I + \eta B)p)||||x_{n+1} - p||.\end{aligned} \tag{31}$$

Now, we write $x_{n+1} = \alpha_n(u + \gamma f(x_n) - (I + \xi B)W_ny_n) + \beta_n(x_n - W_ny_n) + W_ny_n$. Then

$$\begin{aligned}||x_n - W_ny_n|| &\le ||x_n - x_{n+1}|| + ||x_{n+1} - W_ny_n||\\ &\le ||x_n - x_{n+1}|| + \alpha_n||u + \gamma f(x_n) - (I + \xi B)W_ny_n|| + \beta_n||x_n - W_ny_n||\\ &\le \frac{1}{1 - \beta_n}||x_n - x_{n+1}|| + \frac{\alpha_n}{1 - \beta_n}||u + \gamma f(x_n) - (I + \xi B)W_ny_n||.\end{aligned}$$

From (26) and condition (C1) we obtain that

$$\lim_{n\to\infty} ||x_n - W_ny_n|| = 0. \tag{32}$$

Also,

$$\begin{aligned}||w_n - x_{n+1}|| &\le ||w_n - x_n|| + ||x_n - x_{n+1}||\\ &= ||\theta_n(x_n - x_{n-1})|| + ||x_n - x_{n+1}||\\ &\le \theta_n||x_n - x_{n-1}|| + ||x_n - x_{n+1}||.\end{aligned}$$

Observe from condition (C3) that

$$\lim_{n\to\infty} ||w_n - x_n|| = 0. \tag{33}$$

Then, from (26) and (33), we obtain that

$$\lim_{n\to\infty} ||w_n - x_{n+1}|| = 0. \tag{34}$$

Thus by (34), (32) and condition (C1), we obtain that

$$\lim_{n\to\infty} \left|\left|Aw_n - K_s^{F_2}Aw_n\right|\right| = 0. \tag{35}$$

Again,

$$\begin{aligned}
||y_n - p||^2 &= \left|\left|K_r^{F_1}w_n - \xi A^*\left(I - K_s^{F_2}\right)Aw_n - K_r^{F_1}p\right|\right|^2 \\
&\leq \langle K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n - K_r^{F_1}p, (I - \xi A^*(I - K_s^{F_2})A)w_n - p\rangle \\
&= \langle K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n - p, (w_n - \xi A^*(I - K_s^{F_2})Aw_n) - p\rangle \\
&= \langle y_n - p, w_n - p - \xi A^*(I - K_s^{F_2})Aw_n\rangle \\
&\leq \frac{1}{2}\left[||y_n - p||^2 + ||w_n - p||^2 - ||y_n - w_n||^2 + \xi A^*(I - K_s^{F_2})Aw_n||^2\right] \\
&\leq \frac{1}{2}\Big[||y_n - p||^2 + ||w_n - p||^2 - \big(||y_n - w_n||^2 + 2\xi\langle y_n - w_n, A^*(I - K_s^{F_2})Aw_n\rangle \\
&\quad + \xi^2||A^*(I - K_s^{F_2})Aw_n||^2\big)\Big].
\end{aligned}$$

This implies

$$||y_n - p||^2 \leq ||w_n - p||^2 - ||y_n - w_n||^2 + 2\xi\langle w_n - y_n, A^*(I - K_s^{F_2})Aw_n\rangle. \tag{36}$$

Substituting (36) in (27), we have

$$\begin{aligned}
||x_{n+1} - p||^2 \leq\; &||w_n - p||^2 - ||y_n - w_n||^2 \\
&+ 2\xi\langle w_n - y_n, A^*(I - K_s^{F_2})Aw_n\rangle \\
&+ 2(1 - \alpha_n(I + \eta\mu)^2||y_n - p||\beta_n||x_n - W_ny_n|| + \beta_n^2||x_n - W_ny_n||^2 \\
&+ 2\alpha_n||(u + \gamma f(x_n) - (I + \eta B)p)||||x_{n+1} - p||.
\end{aligned}$$

This implies that

$$\begin{aligned}
||y_n - w_n||^2 \leq\; &||w_n - x_{n+1}||\big(||w_n - p|| + ||x_{n+1} - p||\big) \\
&+ 2\xi\big\langle w_n - y_n, A^*\big(I - K_s^{F_2}\big)Aw_n\big\rangle \\
&+ 2(1 - \alpha_n(I + \eta\mu)^2||y_n - p||\beta_n||x_n - W_ny_n|| + \beta_n^2||x_n - W_ny_n||^2 \\
&+ 2\alpha_n||(u + \gamma f(x_n) - (I + \eta B)p)||||x_{n+1} - p||.
\end{aligned}$$

Therefore, from (35), (32), (34) and condition (C1), we obtain that

$$\lim_{n\to\infty} ||y_n - w_n|| = 0. \tag{37}$$

Also,

$$||y_n - W_n y_n|| \leq ||y_n - w_n|| + ||w_n - x_n|| + ||x_n - W_n y_n||.$$

Thus, from (32), (33) and (37), we obtain

$$\lim_{n\to\infty} ||y_n - W_n y_n|| = 0. \tag{38}$$

From Lemma 2.7 and (38), we have

$$||Wy_n - y_n|| \leq ||Wy_n - W_n y_n|| + ||W_n y_n - y_n|| \to 0, \text{ as } n \to \infty. \tag{39}$$

Finally, from (32) and (38), we obtain

$$||x_n - y_n|| \leq ||x_n - W_n y_n|| + ||W_n y_n - y_n|| \to 0, \text{ as } n \to \infty. \tag{40}$$

Step 4 We show that $\limsup_{n\to\infty}\langle u + \gamma f(p) - (I + \eta B)p, x_n - p\rangle = \lim_{k\to\infty}\langle u + \gamma f(p) - (I + \eta B)p, x_{n_k} - p\rangle$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to x^*. Then by (40), we get that $y_{n_k} \rightharpoonup x^*$. Thus, by (39), we get that $x^* \in F(W)$.

Now, we show that $x^* \in GVMEP(F_1, \Psi_1, \Phi_1, C)$. Since $y_n = K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n$, we have by Lemma 2.12 that there exists $y \in C$ such that

$$F_1(y, y_n) + e\langle \Psi_1(w_n), y - y_n\rangle + \Phi_1(y) - \Phi_1(y_n) + \frac{e}{r}\langle y - y_n, y_n - w_n\rangle \in P. \tag{41}$$

By the monotonicity of F_1, it follows that

$$0 \in F_1(y, y_n) - \left\{\Phi_1(y) - \Phi_1(y_n) + e\langle \Psi_1(w_n), y - y_n\rangle + \frac{e}{\xi}\langle y - y_n, y_n - w_n\rangle\right\} + P.$$

Now, let $y_t = (1 - t)x^* + ty$ for all $t \in (0, 1]$. Since $y \in X$ and $x^* \in X$, we obtain that $y_t \in X$. Then by (41), we have

$$\begin{aligned} e\langle \Psi_1(y_t), y_t - y_{n_k}\rangle \in\ & F_1(y_t, y_{n_k}) - \left(\Phi_1(y_t) - \Phi_1(y_{n_k})\right) + e\langle \Psi_1(y_t), y_t - y_{n_k}\rangle \\ & - e\langle \Psi_1(w_{n_k}), y_t - y_{n_k}\rangle \\ & - \frac{e}{r}\langle y_t - y_{n_k}, y_{n_k} - w_{n_k}\rangle + P, \forall y \in X \\ =\ & F_1(y_t, y_{n_k}) + e\langle \Psi_1(y_t) - \Psi_1(w_{n_k}), y_t - y_{n_k}\rangle \\ & - \left(\Phi_1(y_t) - \Phi_1(y_{n_k})\right) \\ & - \frac{e}{r}\langle y_t - y_{n_k}, y_{n_k} - w_{n_k}\rangle + P. \end{aligned}$$

This implies that

$$\begin{aligned} e\langle \Psi_1(y_t), y_t - y_{n_k}\rangle =\ & F_1(y_t, y_{n_k}) + e\langle \Psi_1(y_t) \\ & - \Psi_1(y_{n_k}), y_t - y_{n_k}\rangle + e\langle \Psi_1(y_{n_k}) - \Psi_1(w_{n_k}), y_t - y_{n_k}\rangle \\ & - \frac{e}{r}\langle y_t - y_{n_k}, y_{n_k} - w_{n_k}\rangle - (\Phi_1(y_t) - \Phi_1(y_{n_k})) + P. \end{aligned}$$

Using the properties of F_1, Ψ_1 and the fact that $||y_{n_k} - w_{n_k}|| \to 0, ||\Psi_1(y_{n_k}) - \Psi_1(w_{n_k})|| \to 0$ and $\frac{||\Psi_1(y_{n_k}) - \Psi_1(w_{n_k})||}{r} \to 0$ as $k \to \infty$, we have

$$e\langle \Psi_1(y_t), y_t - x^*\rangle \in F_1(y_t, x^*) + \Phi_1(x^*) - \Phi_1(y_t) + P. \tag{42}$$

Using conditions (A1), (A4) and (A6), we get

$$\begin{aligned} &tF_1(y_t, y) + (1-t)F_1(y_t, x^*) + t\Phi_1(y) + (1-t)\Phi_1(x^*) - \Phi_1(y_t) \\ &\quad \in F_1(y_t, y_t) + \Phi_1(y_t) - \Phi_1(y_t) + P = P, \end{aligned}$$

which gives rise to

$$-t\big[F_1(y_t, y) + \Phi_1(y) - \Phi_1(y_t)\big] - (1-t)\big[F_1(y_t, x^*) + \Phi_1(x^*) - \Phi_1(y_t)\big] \in -P.$$

Using this and (42), we have

$$\begin{aligned} -t\big[F_1(y_t, y) + \Phi_1(y) - \Phi_1(y_t)\big] &\in (1-t)\big[F_1(y_t, x^*) + \Phi_1(x^*) - \Phi_1(y_t)\big] - P \\ &\in (1-t)e\langle \Psi_1(y_t), y_t - x^*\rangle - P \end{aligned}$$

and

$$-t\big[F_1(y_t, y) + \Phi_1(y) - \Phi_1(y_t)\big] - t(1-t)e\langle \Psi_1(y_t), y_t - x^*\rangle \in -P.$$

Therefore, it follows that

$$F_1(y_t, y) + \Phi_1(y) - \Phi_1(y_t) + (1-t)e\langle \Psi_1(y_t), y - x^*\rangle \in P.$$

Letting $t \to 0$, we get

$$F_1(x^*, y) + \Phi_1(y) - \Phi_1(x^*) + e\langle \Psi_1(x^*), y - x^*\rangle \in P,$$

and so, $x^* \in GVMEP \in (F_1, \Psi_1, \Phi_1, C)$.

Since A is a bounded linear operator, we have that $x_{n_k} \rightharpoonup x^*$ implies that $Ax_{n_k} \rightharpoonup Ax^*$. It follows from (35) and (33) that $K_s^{F_2}Aw_{n_k} \rightharpoonup Ax^*$ as $k \to \infty$. By definition of $K_s^{F_2}Aw_{n_k}$, we get by Lemma 2.12 that there exists a point $y \in H_2$ such that

$$\begin{aligned} &F_2\big(K_s^{F_2}Aw_n, y\big) + \big\langle \Psi_2\big(K_s^{F_2}Aw_n\big), y - Aw_{n_k}\big\rangle + \Phi_2(y) - \Phi_2(Aw_{n_k}) \\ &\quad + \frac{e}{s}\big\langle y - K_s^{F_2}Aw_{n_k}, K_s^{F_2}Aw_{n_k} - Aw_{n_k}\big\rangle \in Q, \forall y \in H. \end{aligned}$$

Since F_2 is upper semicontinuous, following the same argument as above, we have by (35) that

$$F_2(Ax^*, y) + \Phi_2(y) - \Phi_2(Ax^*) + e\langle \Psi_2(Ax^*), y - Ax^*\rangle \in D, \forall y \in H_2.$$

This implies that $Ax^* \in GVMEP(F_2, \Psi_2, \Phi_2, Q)$. Therefore $x^* \in SGVMEP$ and hence $x^* \in \Gamma$.

Since $p = P_\Gamma(u + \gamma f(p) - \eta Bp)$, then we have

$$\begin{aligned} \limsup_{n\to\infty}\langle u + (\gamma f - (I + \eta B))p, x_n - p\rangle &= \lim_{k\to\infty}\langle u + (\gamma f - (I + \eta B))p, x_{n_k} - p\rangle. \\ &= \langle u + (\gamma f - (I + \eta B))p, x^* - p\rangle \le 0. \end{aligned}$$

Step 5 We show that $\{x_n\}$ converges strongly to p. Observe from (14) and Lemma 2.1 that

$$
\begin{aligned}
||x_{n+1}-p||^2 &= ||\alpha_n(u+\gamma f(x_n)-(I+\eta B)p)+\beta_n(x_n-p)\\
&\quad+\big[(1-\beta_n)I-\alpha_n(I+\eta B)\big](W_ny_n-p)||^2\\
&\leq ||\beta_n(x_n-p)+\big[(1-\beta_n)I-\alpha_n(I+\eta B)\big](W_ny_n-p)||^2\\
&\quad+2\alpha_n\langle u+\gamma f(x_n)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[\beta_n||x_n-p||\\
&\quad+||(1-\beta_n)I-\alpha_n(I+\eta B)(W_ny_n-p)||\big]^2+2\alpha_n\langle\gamma f(x_n)-\gamma f(p),x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[\beta_n||x_n-p||+||(1-\beta_n)I-\alpha_n(I+\eta B)(W_ny_n-p)||\big]^2\\
&\quad+2\alpha_n\gamma\lambda\langle x_n-p,x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[\beta_n||x_n-p||+||(1-\beta_n)I-\alpha_n(I+\eta B)(W_ny_n-p)||\big]^2\\
&\quad+2\alpha_n\gamma\lambda\langle x_n-p,x_{n+1}-p\rangle\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[\beta_n||x_n-p||+(1-\beta_n)-\alpha_n(1+\eta\mu)||W_ny_n-p||\big]^2\\
&\quad+2\alpha_n\gamma\lambda||x_n-p||||x_{n+1}-p||\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[\beta_n||x_n-p||+(1-\beta_n)-\alpha_n(1+\eta\mu)||w_n-p||\big]^2\\
&\quad+2\alpha_n\gamma\lambda||x_n-p||||x_{n+1}-p||\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[\beta_n||x_n-p||+(1-\beta_n)-\alpha_n(1+\eta\mu)(||x_n-p||\\
&\quad+\theta_n||x_n-x_{n-1}||)\big]^2\\
&\quad+\alpha_n\gamma\lambda\big[||x_n-p||^2+||x_{n+1}-p||^2\big]\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[(1-\alpha_n(1+\eta\mu))||x_n-p||\\
&\quad+\theta_n(1-\beta_n-\alpha_n(1+\eta\mu))||x_n-x_{n-1}||\big]^2\\
&\quad+\alpha_n\gamma\lambda\big[||x_n-p||^2\\
&\quad+||x_{n+1}-p||^2\big]+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq \big[(1-\alpha_n(1+\eta\mu))||x_n-p||+\theta_n||x_n-x_{n-1}||\big]^2+\alpha_n\gamma\lambda\big[||x_n-p||^2\\
&\quad+||x_{n+1}-p||^2\big]\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n(1+\eta\mu))^2||x_n-p||^2\\
&\quad+2\theta_n(1-\alpha_n(1+\eta\mu))\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2||x_n-x_{n-1}||^2\\
&\quad+\alpha_n\gamma\lambda\big[||x_n-p||^2+||x_{n+1}-p||^2\big]\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle\\
&\leq (1-\alpha_n(1+\eta\mu))^2||x_n-p||^2\\
&\quad+2\theta_n(1-\alpha_n(1+\eta\mu))||x_n-p||||x_n-x_{n-1}||+\theta_n^2||x_n-x_{n-1}||^2\\
&\quad+\alpha_n\gamma\lambda\big[||x_n-p||^2+||x_{n+1}-p||^2\big]\\
&\quad+2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p,x_{n+1}-p\rangle.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(1-\alpha_n\gamma\lambda)||x_{n+1}-p||^2 &\le (1-\alpha_n(1+\eta\mu))^2||x_n-p||^2\\
&\quad + 2\theta_n(1-\alpha_n(1+\eta\mu))||x_n-p||||x_n-x_{n-1}||\\
&\quad + \theta_n^2||x_n-x_{n-1}||^2 + \alpha_n\gamma\lambda||x_n-p||^2\\
&\quad + 2\alpha_n\langle u+\gamma f(p)-(I+\eta B)p, x_{n+1}-p\rangle\\
&= \frac{1-\alpha_n(1+\eta\mu)^2}{1-\alpha_n\gamma\lambda}||x_n-p||^2\\
&\quad + \frac{2\theta_n(1-\alpha_n(1+\eta\mu))}{1-\alpha_n\gamma\lambda}||x_n-p||||x_n-x_{n-1}|| + \frac{\theta_n^2}{1-\alpha_n\gamma\lambda}||x_n-x_{n-1}||^2\\
&\quad + \frac{\alpha_n\gamma\lambda}{1-\alpha_n\gamma\lambda}||x_n-p||^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\lambda}\langle u+\gamma f(p)-(I+\eta B)p, x_{n+1}-p\rangle\\
&\le 1-\frac{2(\alpha_n(1+\eta\mu)-\gamma\lambda)}{1-\alpha_n\gamma\lambda}||x_n-p||^2\\
&\quad + \frac{(\alpha_n(1+\eta)\mu)^2}{1-\alpha_n\gamma\lambda}||x_n-p||^2 + \frac{2\theta_n(1-\alpha_n(1+\eta\mu))}{1-\alpha_n\gamma\lambda}||x_n-p||||x_n-x_{n-1}||\\
&\quad + \frac{\theta_n^2}{1-\alpha_n\gamma\lambda}||x_n-x_{n-1}||^2 + \frac{2\alpha_n}{1-\alpha_n\gamma\lambda}\langle u+\gamma f(p)-(I+\eta B)p, x_{n+1}-p\rangle\\
&\le 1-\frac{2(\alpha_n(1+\eta\mu)-\gamma\lambda)}{1-\alpha_n\gamma\lambda}||x_n-p||^2 + \frac{2(\alpha_n(1+\eta\mu)-\gamma\lambda)}{1-\alpha_n\gamma\lambda}\\
&\quad \times\left[\frac{(\alpha_n(1+\eta)\mu)^2}{2\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-p||^2 + \frac{\theta_n(1-\alpha_n(1+\eta\mu))}{\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-p||||x_n-x_{n-1}||\right.\\
&\quad \left. + \frac{\theta_n^2}{2\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-x_{n-1}||^2 + \frac{1}{\alpha_n(1+\eta)\mu-\gamma\lambda}\langle u+\gamma f(p)-(I+\eta B)p, x_{n+1}-p\rangle\right]\\
&= 1-\frac{2(\alpha_n(1+\eta\mu)-\gamma\lambda)}{1-\alpha_n\gamma\lambda}||x_n-p||^2 + \frac{2(\alpha_n(1+\eta\mu)-\gamma\lambda)}{1-\alpha_n\gamma\lambda}\left[\frac{\alpha_n(1+\eta)^2\mu^2}{2(1+\eta)\mu-\gamma\lambda}M_6\right.\\
&\quad + \frac{\theta_n(1-\alpha_n(1+\eta\mu))}{\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-p||||x_n-x_{n-1}|| + \frac{\theta_n^2}{2\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-x_{n-1}||^2\\
&\quad \left. + \frac{1}{\alpha_n(1+\eta)\mu-\gamma\lambda}\langle u+\gamma f(p)-(I+\eta B)p, x_{n+1}-p\rangle\right]\\
&= (1-\sigma_n)||x_n-p||^2 + \sigma_n\kappa_n,
\end{aligned}
$$

where $M_6 = \sup_{n\ge 1}\{||x_n-p||^2\}$, $\sigma_n = \frac{2(\alpha_n(1+\eta\mu)-\gamma\lambda)}{1-\alpha_n\gamma\lambda}$ and

$$
\begin{aligned}
\kappa_n = &\left[\frac{\alpha_n(1+\eta)^2\mu^2}{2(1+\eta)\mu-\gamma\lambda}M_6 + \frac{\theta_n(1-\alpha_n(1+\eta\mu))}{\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-p||||x_n-x_{n-1}||\right.\\
&+ \frac{\theta_n^2}{2\alpha_n(1+\eta)\mu-\gamma\lambda}||x_n-x_{n-1}||^2\\
&\left. + \frac{1}{\alpha_n(1+\eta)\mu-\gamma\lambda}\langle u+\gamma f(p)-(I+\eta B)p, x_{n+1}-p\rangle\right].
\end{aligned}
$$

Therefore, by Lemma 2.9, we conclude that the sequence $\{x_n\}$ converges strongly to the point $p \in \Gamma$. This completes the proof. □

Next, we give some consequences of our main result.

If $\Psi_1 = \Psi_2 \equiv 0$, we obtain the following result:

Corollary 3.2 *Let C and Q be nonempty, compact, convex subset of real Hilbert spaces H_1 and H_2 respectively. Assume that P and D are closed, convex cones of real Hausdorff topological spaces Y and Z with e and d fixed points in P and D respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $T_i \to C$ be a finite family of nonexpansive mappings, $i = 1, 2, \ldots, N$. Let $F_1 : C \times C \to Y$, $F_2 : Q \times Q \to Z$, $\Phi_1 : C \to Y$ and $\Phi_2 : Q \to Z$ be functions satisfying assumptions* (A1)-(A7). Let f be a contraction of H_1 into itself with coefficient $\lambda \in (0, 1)$ and B be a strongly positive linear bounded operator. Assume that $\Gamma = \cap_{i=1}^N Fix(T_i) \cap SGMVEP(F_1, \Phi_1, F_2, \Phi_2) \neq \emptyset$. Let the sequences $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by $u, x_0, x_1 \in H_1$ and

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n(I + \eta B)]W_n y_n, \end{cases} \tag{43}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in [0, 1], $r, s > 0, \gamma < \frac{(1+\eta)\mu}{\lambda}$ and $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n < \liminf_{n\to\infty} \beta_n < 1$

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, and $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* = P_\Gamma(u + \gamma f(x^*) - \eta Bx^*) \in \Gamma$.

If $\gamma = 0$ in Algorithm 14, we obtain the following result:

Corollary 3.3 *Let C and Q be nonempty, compact, convex subset of real Hilbert spaces H_1 and H_2 respectively. Assume that P and D are closed, convex cones of real Hausdorff topological spaces Y and Z with e and d fixed points in P and D respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $T_i \to C$ be a finite family of nonexpansive mappings, $i = 1, 2, \ldots, N$. Let $F_1 : C \times C \to Y$, $F_2 : Q \times Q \to Z$, $\Phi_1 : C \to Y$ and $\Phi_2 : Q \to Z$ be functions satisfying assumptions* (A1)-(A7). Let $\Psi_1 : C \to H_1$ and $\Psi_2 : Q \to H_2$ be β_1 and β_2 inverse strongly monotone mappings respectively and B be a strongly positive linear bounded operator defined as in Lemma 2.2. Assume that $\Gamma = \cap_{i=1}^N Fix(T_i) \cap SGMVEP(F_1, \Phi_1, \Psi_1, F_2, \Phi_2, \Psi_2) \neq \emptyset$. Let the sequences $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by $u, x_0, x_1 \in H_1$ and

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n \\ x_{n+1} = \alpha_n u + \beta_n x_n + [(1-\beta_n)I - \alpha_n(I + \eta B)]W_n y_n, \end{cases} \tag{44}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in [0, 1], $r, s > 0$ and $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n < \liminf_{n\to\infty} \beta_n < 1$

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, and $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* = P_\Gamma(u - \eta B x^*) \in \Gamma$.

For approximating a common solution of split generalized mixed equilibrium and fixed point of finite family of nonexpansive mappings in real Hilbert spaces. We set $Y = Z = \mathbb{R}, P = D = [0, \infty)$ and $e = 1$, in Theorem 3.1. We state the following theorem and omit the proof.

Theorem 3.4 *Let C and Q be nonempty, convex subset of real Hilbert spaces H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint A^* and $T_i \to C$ be a finite family of nonexpansive mappings, $i = 1, 2, \ldots, N$. Let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$, $\Phi_1 : C \to \mathbb{R}$ and $\Phi_2 : Q \to \mathbb{R}$ be functions satisfying assumptions* (A1)-(A7). Let $\Psi_1 : C \to H_1$ and $\Psi_2 : Q \to H_2$ be β_1 and β_2 inverse strongly monotone mappings respectively. Let f be a contraction of H_1 into itself with coefficient $\lambda \in (0, 1)$ and B be a strongly positive linear bounded operator defined as in Lemma 2.2. Assume that $\Gamma = \cap_{i=1}^{N} Fix(T_i) \cap SGMVEP(F_1, \Phi_1, \Psi_1, F_2, \Phi_2, \Psi_2) \neq \emptyset$. Let the sequences $\{w_n\}$, $\{y_n\}$ and $\{x_n\}$ be generated iteratively by $u, x_0, x_1 \in H_1$ and

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}) \\ y_n = K_r^{F_1}(I - \xi A^*(I - K_s^{F_2})A)w_n \\ x_{n+1} = \alpha_n(u + \gamma f(x_n)) + \beta_n x_n + [(1-\beta_n)I - \alpha_n(I + \eta B)]W_n y_n, \end{cases} \tag{45}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in [0, 1], $r, s > 0, \gamma < \frac{(1+\eta)\mu}{\lambda}$ and $\{\theta_n\} \subset [0, \theta]$ with $\theta \in [0, 1)$ satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $0 < \liminf_{n\to\infty} \beta_n < \liminf_{n\to\infty} \beta_n < 1$

(C3) $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, and $\sum_{n=1}^{\infty} \theta_n ||x_n - x_{n-1}|| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to a point $x^* = P_\Gamma(u + \gamma f(x^*) - \eta B x^*) \in \Gamma$.

4 Numerical example

In this section, we give some numerical illustrations to support Theorem (3.4).

Example 4.1 Let $H_1 = H_2 = C = Q = \ell_2$ be the space of all square summable sequences of scalars, i.e

$$\ell_2 = \left\{ x = (x_1, x_2, \ldots, x_i, \ldots) \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\},$$

when an inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$, where $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in \ell_2$ and $|| \cdot || : \ell_2 \to \mathbb{R}$ defined by $||x||_2 = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{\frac{1}{2}}$, where $x = \{x_i\}_{i=1}^{\infty} \in \ell_2$. Let the mapping $A : \ell_2 \to \ell_2$ be defined by $Ax = \left(\frac{x_1}{5}, \frac{x_2}{5}, \ldots, \frac{x_i}{5} \ldots \right)$ for all $x = \{x_i\}_{i=1}^{\infty} \in \ell_2$, then $Ay = \left(\frac{y_1}{5}, \frac{y_2}{5}, \ldots, \frac{y_i}{5} \ldots \right)$ for all $y = \{y_i\}_{i=1}^{\infty} \in \ell_2$. Let $F_1, F_2 : \ell_2 \times \ell_2 \to \mathbb{R}$ be defined by

$$F_1(x, y) = -x^2 + y^2, \forall x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in \ell_2,$$

and

$$F_1(x, y) = -2x^2 + xy + y^2, \forall x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in \ell_2.$$

Let the mapping $\Psi_1, \Psi_2 : \ell_2 \to \ell_2$ be defined by $\Psi_1 x = \left(\frac{x_1}{5}, \frac{x_2}{5}, \ldots, \frac{x_i}{5} \ldots \right) \forall x = \{x_i\}_{i=1}^{\infty} \in \ell_2$ and $\Psi_2 x = \left(\frac{x_1}{4}, \frac{x_2}{4}, \ldots, \frac{x_i}{4} \ldots \right) \forall x = \{x_i\}_{i=1}^{\infty} \in \ell_2$ respectively. Let the mapping $T_i : \ell_2 \to \ell_2, i = 1, 2, \ldots, N,$ be defined by $T_i x = \left(\frac{5ix_1}{7i+1}, \frac{5ix_2}{7i+1}, \ldots, \frac{5ix_i}{7i+1} \ldots \right) \forall x = \{x_i\}_{i=1}^{\infty} \in \ell_2$ and $r = 1, s = 0.5, \eta = 0.5$. Let $x_0 = (x_0^1, x_0^2, \ldots, x_0^i \ldots), x_1 = (x_1^1, x_1^2, \ldots, x_1^i \ldots), y_n = (y_n^1, y_n^2, \ldots, y_n^i, \ldots),$ and $u = (u_1, u_2, \ldots, u_i \ldots) \in \ell_2$. By the definition of $y_n \in C$ and Lemma 2.12 we obtain

$$\begin{aligned} 0 &\leq F_1(y_n, y) + \langle \Psi_1(y_n), y - y_n \rangle + \Phi(y_n) - \Phi(y) + \frac{1}{r} \langle y - y_n, y_n - w_n \rangle \\ &= -y_n^2 + y^2 + \frac{y_n}{5}(y - y_n) + (y_n - w_n) \\ &= -y_n^2 - \frac{y_n^2}{5} - y_n^2 + \frac{y_n y}{5} + y_n y + y_n w_n + y^2 + y w_n \\ &= y^2 + \left(\frac{6y_n}{5} - w_n \right) y - \frac{11 y_n^2}{5} - w_n y_n. \end{aligned} \tag{46}$$

Suppose (46) is a quadratic inequality such that $a = 1, b = \left(\frac{6z_n}{5} - w_n \right)$ and $c = \frac{11y_n^2}{5} - w_n z_n$. Then, the discriminant $\Delta = b^2 - 4ac$ is

$$\begin{aligned}\Delta &= \left(\frac{6y_n}{5} - w_n\right)^2 - 4\left(-\frac{11y_n^2}{5} - w_n y_n\right)\\ &= \frac{256}{25}y_n^2 - \frac{32}{5}y_n w_n + w_n^2\\ &= \left(\frac{16}{5}y_n - w_n\right)^2.\end{aligned}$$

This implies that $y_n = \frac{5}{16}w_n$. Now, we proceed to compute $v_n = K_s^{F_2}Aw_n$. Again, by Lemma 2.12, we have

$$0 \leq F_2(v_n, y) + \langle \Psi_2(v_n), y - v_n\rangle + \Phi_2(y) - \Phi_2(v_n) + \frac{1}{s}\langle y - v_n, v_n - Aw_n\rangle, \forall y \in Q.$$

Given that

$$f_2(u, v) = 2u - 10uv + 10u^2 - 2v, \Psi_2(u) = \frac{u}{5}, \text{ and } \Psi_2(u) = \frac{u}{2},$$

then

$$\begin{aligned}0 &\leq 2v_n - 10v_n y + 10v_n^2 2y + \langle \frac{v_n}{5}, y - v_n\rangle + \Phi_2(y) - \Phi_2(v_n) + \frac{1}{s}\langle y - v_n, v_n - Aw_n\rangle\\ &= 2v_n - 10v_n y + 10v_n^2 2y + \frac{v_n}{5}(y - v_n) + y - v_n + \frac{1}{s}(y - v_n)(v_n - Aw_n)\\ &= -2(y - v_n) - 10v_n(y - v_n) + \frac{v_n}{5}(y - v_n) + y - v_n + \frac{1}{s}(y - v_n)(v_n - Aw_n)\\ &= (-2 - 10v_n)(y - v_n) + \frac{v_n}{5}(y - v_n) + y - v_n + \frac{1}{s}(y - v_n)(v_n - Aw_n).\end{aligned}$$

This implies

$$\begin{aligned}(-2 - 10v_n) + \frac{v_n}{5} + 1 + \frac{v_n - Aw_n}{s} &= 0\\ -2 - 10v_n + \frac{v_n + 5}{5} + \frac{v_n + Aw_n}{s} &= 0\\ \frac{-10 - 50v_n + v_n + 5}{5} + \frac{v_n - Aw_n}{s} &= 0\\ \frac{-49v_n - 5}{5} + \frac{v_n - Aw_n}{s} &= 0\\ -49v_n s - 5s + v_n - Aw_n &= 0.\end{aligned} \tag{47}$$

Hence, we obtain from (47) that $v_n = \frac{Aw_n + 5s}{1 - 49s}$. Take $B = I, \gamma = 1, f = \frac{1}{10}x, \alpha_n = \frac{1}{2n^2+2}, \beta_n = \frac{1}{3n+1}$ and $\theta_n = \frac{n-1}{3n+1}$. Let $\{w_n\}, \{y_n\}$ and $\{x_n\}$ be generated by Algorithm 45 as follows

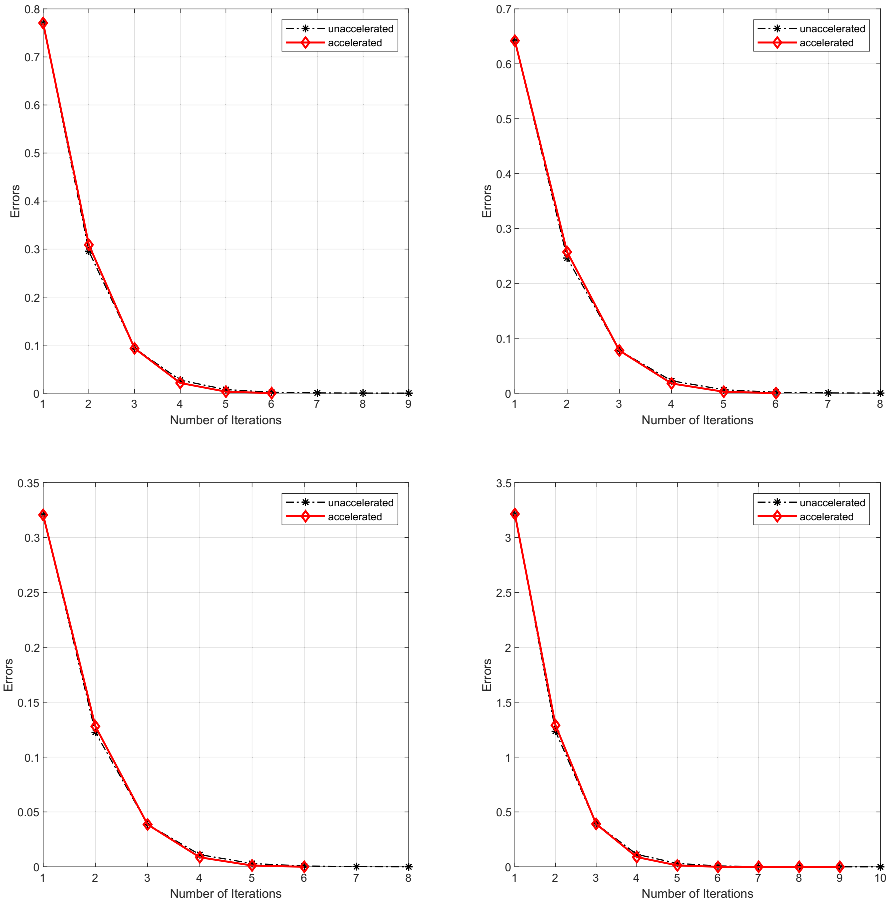

Fig. 1 Errors vs iteration numbers (n): Case 1 (top left); Case 2 (top right); Case 3 (bottom left); Case 4 (bottom right)

$$\begin{cases} w_n = x_n + \dfrac{n-1}{3n+1}(x_n - x_{n-1}) \\ y_n = K_{0.5}^{F_1}(I - 0.5A^*(I - K_1^{F_2})A)w_n \\ x_{n+1} = \dfrac{1}{2n^2+2}(u + \dfrac{1}{10}x_n) + \dfrac{1}{3n+1}x_n + [(1 - \dfrac{1}{3n+1}x_n)I - \dfrac{1}{2n^2+2}(I + 0.3I)]W_n y_n, \end{cases} \tag{48}$$

for all $n \geq 1$, $w_n = (w_n^1, w_n^2 \ldots, w_n^i, \ldots), y_n = (y_n^1, y_n^2 \ldots, y_n^i, \ldots)$ and $x_n = (x_n^1, x_n^2 \ldots, x_n^i, \ldots)$. Clearly, F_1, F_2, Ψ_1, Ψ_2, and f satisfying the assumptions in Theorem 3.4. Also, we have $\cap_{i=1}^N Fix(T_i) \cap SGVMEP(F_1, \Phi_1, \Psi_1, F_2, \Phi_2, \Psi_2) = \{0\}$, we conclude that $\{w_n\}, \{y_n\}$ and $\{x_n\}$ converge strongly to 0. We shall omit the computer programming in this instance.

Example 4.2 Let $H_1 = H_2 = C = Q = \mathbb{R}$ with an an inner product $\langle \cdot, \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\langle x, y \rangle = x \cdot y$ where $x, y \in \mathbb{R}$. Let the mapping $A : \mathbb{R} \to \mathbb{R}$ be defined by $Ax = 2x - 1$, for all $x \in \mathbb{R}$, then $A^*z = 2z + 1 \forall z \in \mathbb{R}$. Let $T_i : \mathbb{R} \to \mathbb{R}, i = 1, 2, \ldots, N$ be defined by $T_i = \frac{2ix}{3i+1}, \forall x \in \mathbb{R}$. Let $F_1, F_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$F_1(x, y) = -x^2 + y^2, \forall x, y \in \mathbb{R},$$

and

$$F_1(x, y) = -2x^2 + xy + y^2, \forall x, y \in \mathbb{R}.$$

Let the mapping $\Psi_1, \Psi_2 : \mathbb{R} \to \mathbb{R}$ be defined by $\Psi_1 x = \frac{x}{5} \forall x \in \mathbb{R}$ and $\Psi_2 x = \frac{x}{4} \forall x \in \mathbb{R}$ respectively. Then as in Example 4.1 above, we can find $y_n \in C$ and $v_n \in Q$ respectively, such that $y_n = \frac{5}{16} w_n$ and $v_n = \frac{Aw_n + 5s}{1 - 49s}$. Take $u = 0, B = I, \gamma = 1, f = \frac{1}{10}x, \alpha_n = \frac{1}{2n^2+2}, \beta_n = \frac{1}{3n+1}$ and $\theta_n = \frac{n-1}{3n+1}$. We vary the initial values of x_0 and x_1 and then plot the graph of errors against number of iterations (Fig. 1).

Case 1: $x_0 = 0.5$ and $x_1 = 0.2$, **Case2 :** $x_0 = -0.5$ and $x_1 = 1$,
Case 3: $x_0 = -1$ and $x_1 = 0.5$ **Case4 :** $x_0 = 2$ and $x_1 = 2$.

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Declarations

Conflict of interest The authors declare that they have no competing interests.

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References

1. Abass, H.A., K.O. Aremu, L.O. Jolaoso, O.T. Mewomo. 2020. An Inertial forward-backward splitting method for approximating solutions of certain optimization of certain optimization problems, *Journal of Nonlinear Functional Analysis*. https://doi.org/10.23952/jnfa.2020.6 (**Article ID 6**).
2. Alakoya, T.O., L.O. Jolaoso, and O.T. Mewomo. 2020. Modified inertial subgradient extragradient method with self adaptive stepsize for solving monotone variational inequality and fixed point problem. *Optimization*: 1–30.
3. Ansari, Q.H., and F. Flores-Bazan. 2006. Recession methods for generalized vector equilibrium problems, *Journal of Mathematical Analysis and Applications*, **321**: 132–146.
4. Ansari, Q.H., and A. Rehan. 2014. Split feasibility and fixed point problems. In *Nonlinear Analysis: Approximation Theory, Optimization and Application*, ed. Q.H. Ansari, 281–322. New York: Springer.
5. Bryne, C. 2002. Iterative oblique projection onto convex subsets and the split feasibility problem. *Inverse Problems* 18 (2): 441–453.

6. Byrne, C. 2004. A unified treatment of some iterative algorithms in signal processing and image reconstruction. *Inverse Problems* 20: 103–120.
7. Blum, E., and W. Oettli. 1994. From optimization and variational inequalities to equilibrium problems. *The Mathematics Student* 63: 123–145.
8. Censor, Y., and T. Elfving. 1994. A multi projection algorithm using Bregman projections in a product space. *Numerical Algorithms* 8 (2): 221–239.
9. Censor, Y., T. Elfving, N. Kopf, and T. Bortfeld. 2005. The multiple-sets split feasibility problem and its applications for inverse problems. *Inverse Problems* 21: 2071–2084.
10. Censor, Y., T. Bortfeld, B. Martin, and A. Trofimov. 2006. A unified approach for inversion problems in intensity-modulated radiation therapy. *Physics in Medicine and Biology* 51: 2353–2365.
11. Chen, G., X. Huang, and X. Yang. 2005. *Vector Optimization: Set-Valued and Variational Analysis. Lecture Notes in Economics and Mathematical Systems*, vol. 541. Berlin: Springer.
12. Chidume, C.E. 2009. *Geometric properties of Banach spaces and nonlinear spaces and nonlinear iterations Lecture Notes in Mathematics*. New York: Springer (ISBN 978- 84882-189-7).
13. Combettes, P.L. 2004. Solving monotone inclusions via compositions of nonexpansive averaged operators. *Optimization* 53: 475–504.
14. Ezeora, J. N., H. A. Abass. C. Izuchukwu. 2020. Strong convergence theorem of an inertial algorithm to a common solution of minimization and fixed point problems. *Matematicki Vesnik*
15. Giannessi, F. (ed.). 2000. *Vector Variational Inequalities and Vector Equilibrium*, vol. 38. Dordrecht: Kluwer Academic Publishers.
16. Gong, X.H., and H.M. Yue. 2008. Existence of efficient solutions and strong solutions for vector equilibrium problems. *Journal of Nanchang University* 32: 1–5.
17. Kazmi, K.R. 2000. On vector equilibrium problem. *Proceedings of the Indian Academy of Sciences (Mathematical Sciences)* 110: 213–223.
18. Kazmi, K.R., and A. Raouf. 2005. A class of operator equilibrium problems. *Journal of Mathematical Analysis and Applications* 308: 554–566.
19. Kazmi, K.R., S.H. Rizvi, and M. Farid. 2015. A viscosity Cesaro mean approximation method for split generalized vector equilibrium problem and fixed point problem. *Journal of the Egyptian Mathematical Society* 23 (2): 362–370.
20. Kazmi, K.R., and M. Farid. 2013. Some iterative schemes for generalized vector equilibrium problems and relatively nonexpansive mappings in Banach spaces. *Mathematical Sciences* 7: 19.
21. Kazmi, K.R., and S.H. Rizvi. 2013. Iterative approximation of a common solution of a split generalized equilibrium problem and a fixed point problem for nonexpansive semigroup. *Mathematical Sciences*. https://doi.org/10.1186/2251-7456-7-1.
22. Jolaoso, L.O., A. Taiwo, T.O. Alakoya, and O.T. Mewomo. 2019. A strong convergence theorem for solving variational inequalities using an inertial viscosity subgradient extragradient algorithm with self adaptive stepsize. *Demonstratio Mathematica* 52: 183–203.
23. Jolaoso, L.O., A. Taiwo, A. Alakoya, and O.T. Mewomo. 2019. A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem. *Computational and Applied Mathematics*. https://doi.org/10.1007/s40314-019-1014-2.
24. Jung, J.S. 2006. Iterative algorithms with some control conditions for quadratic optimizations. *Pan-American Mathematical Journal* 16: 13–25.
25. Jung, J.S. 2011. A general iterative scheme for k- strictly pseudo-contractive mappings and optimization problems. *Applied Mathematics and Computation* 217: 5581–5588.
26. Li, J., N.J. Huang, and J.K. Kim. 2003. On implicit vector equilibrium problems. *Journal of Mathematical Analysis and Applications* 283: 501–502.
27. Li, Q.Y., and S.H. Wang. 2003. Viscosity approximation methods for strong vector equilibrium problems and fixed point problems. *Journal of Nanchang University* 26: 1–5.
28. Luc, D.T. 1989. *Theory of Vector Opimization. Lecture Notes in economics and Mathematical Systems*, vol. 319. Berlin: Springer.
29. Marino, G., and H.K. Xu. 2006. A general iterative method for nonexpansive mappings in Hilbert spaces. *Journal of Mathematical Analysis and Applications* 318: 43–52. https://doi.org/10.1016/j.jmaa.2005.05.028.
30. Oyewole, O.K., C. Izuchukwu, C.C. Okeke, and O.T. Mewomo. 2020. Inertial approximation method for split variational inclusion problem in Banach spaces. *International Journal of Nonlinear Analysis and Applications* 11 (2): 285–304.

31. Peng, J.W., and J.C. Yao. 2009. Two extragradient methods for generalized mixed equilibrium problems, nonexpansive mappings and monotone mappings. *Computers and Mathematics with Applications* 58: 1287–1301.
32. Polyak, B.T. 1964. Some methods of speeding up the convergence of iteration methods. *USSR Computational Mathematics and Mathematical Physics* 4(5): 1–17.
33. Rizvi, S.H. 2018. A strong convergence theorem for split mixed equilibrium and fixed point problems for nonexpansive mappings. *Journal of Fixed Point Theory and Applications* 20 (1): 8. https://doi.org/10.1007/s11784-018-0487-8.
34. Rouhani, B.D., K.R. Kazmi, and M. Farid. 2017. Common solutions to some systems of variational inequalities and fixed point problems. *Fixed Point Theory* 18 (1): 167–190.
35. Shan, S.Q., and N.J. Huang. 2012. An iterative method for generalized method for generalized mixed vector equilibrium problems and fixed point of nonexpansive mappings and variational inequalities. *Taiwanese Journal of Mathematics* 16 (5): 1681–1705.
36. Suzuki, T. 2005. Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals. *Journal of Mathematical Analysis and Applications* 305: 227–239.
37. Tan, N.X. 2004. On the existence of solutions of quasivariational inclusion problems. *Journal of Optimization Theory and Applications* 123: 619–638.
38. Witthayarata, U., Y.J. Cho, and P. Kumam. 2012. Approximation algorithm for fixed points of nonlinear operators and solutions of mixed equilibrium problems and variational inclusion problems with applications. *Journal of Nonlinear Science and Applications* 5: 475–494.
39. Yao, Y., Y.J. Cho, and Y.C. Liou. 2011. Iterative algorithms for variational inclusions, mixed equilibrium and fixed point problems with application to optimization problems. *Central European Journal of Mathematics* 9: 640–656.

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