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A Tseng extragradient method for solving variational inequality problems in Banach spaces

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Abstract

This paper presents an inertial Tseng extragradient method for approximating a solution of the variational inequality problem. The proposed method uses a single projection onto a half space which can be easily evaluated. The method considered in this paper does not require the knowledge of the Lipschitz constant as it uses variable stepsizes from step to step which are updated over each iteration by a simple calculation. We prove a strong convergence theorem of the sequence generated by this method to a solution of the variational inequality problem in the framework of a 2-uniformly convex Banach space which is also uniformly smooth. Furthermore, we report some numerical experiments to illustrate the performance of this method. Our result extends and unifies corresponding results in this direction in the literature.

Keywords Variational inequality · Pseudomonotone operator · Strong convergence · Banach space · Extragradient algorithm · Step-size rule

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1 Introduction

In this paper, we consider the Variational Inequality Problem (for short, VIP) defined as: Find $x \in C$ such that

$$\langle F(x), y - x\rangle \geq 0, \quad \forall\, y \in C, \tag{1.1}$$

where C is a nonempty, closed and convex subset of a real Banach space E with dual space E^*. We denote by $Sol(F, C)$, the solution set of problem (1.1). Variational inequalities play an important role in studying a wide class of unilateral, obstacle and equilibrium problems arising in several branches of pure and applied sciences in a unified and general framework (see [1, 6] and the references therein). For this reason, there have been extensive studies of this problem by many authors and therefore, several iterative algorithms have been developed for solving variational inequalities and related optimization problems in Hilbert, Banach, Hadamard and p-uniformly convex metric spaces, see [5, 16, 17, 27].

The development of iterative methods for solving the VIP goes back to the following fixed point reformulation: for any $\lambda > 0$, a point x^* is a solution of VIP (1.1), that is $x^* \in Sol(F, C)$ if and only if $x^* = P_C(x^* - \lambda F(x^*))$, where P_C is a metric projection (see, e.g., [15]). In the case where F is a L-Lipschitz continuous operator, another fixed point method is: for any $\lambda_k \in \left(0, \frac{1}{L}\right)$, a point x^* solves VIP (1.1), if and only if

$$x^* = P_C(x^* - \lambda F(P_C(x^* - \lambda F(x^*))), \tag{1.2}$$

see for example [18, Lemma 2.2]. Korpelevich [22] introduced the well known Extragradient Method (EGM), see also (Antipin [3]) by converting the fixed point (1.2) into an iterative method. The EGM is given by

$$\begin{cases} x_1 \in C, \\ y_k = P_C(x_k - \lambda_k F(x_k)), \\ x_{k+1} = P_C(x_k - \lambda_k F(y_k)), \quad k \geq 1. \end{cases} \tag{1.3}$$

Under the assumptions of monotonicity and Lipschitz continuity of F, Korpelevich showed that the sequence $\{x_k\}$ given by (1.3) converges weakly to a solution of the VIP in a finite dimensional space. This result of Korpelevich was further extended by Nadezhkina and Takahashi [26] to the framework of real Hilbert space. We note that an advantage of the method given by (1.2) is that it allows for the strong monotonicity assumption on F to be weakened to just monotonicity or even pseudomonotonicity. However, as seen in the algorithm, obtaining the next iterate x_{k+1} from the previous iterate x_k requires the calculation of two projections onto the feasible set C per each iteration. This has been observed to have an adverse effect on the computation efficiency of the method when the structure of the set C is not simple. In order to overcome this weakness of the method, Censor et al. [11] (see also [9, 10, 12]) introduced the Subgradient Extragradient Method (SEGM) where the second projection onto C is replaced by a projection onto a halfspace which can be explicitly calculated. The SEGM is presented as follows:

$$\begin{cases} x_1 \in C, \\ y_k = P_C(x_k - \lambda_k F(x_k)), \\ C_k = \{w \in H : \langle x_k - \lambda_k F(x_k) - y_k, w - y_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k}(x_k - F(y_k)), \quad k \geq 1. \end{cases} \tag{1.4}$$

Also, Cholamjiak et al. [29] introduced a new algorithm which combines the inertial contraction projection method and the Mann-type method [24] for solving monotone variational inequality problems in real Hilbert spaces. They proved strong convergence of the proposed method under some standard assumptions. Precisely, they proposed the following:

Algorithm 1 Algorithm 3.1 [40].

Initialization: Let $\gamma \in (0, 2)$, $\lambda \in (0, \frac{1}{L})$, $\theta > 0$ and $x_0, x_1 \in H$ be chosen arbitrarily. Let $\alpha_k \in (a, 1 - \beta_k)$ for some $a > 0$, where $\beta_k \in (0, 1)$ is a sequence satisfying

$$\lim_{k \to \infty} \beta_k = 0, \qquad \sum_{k=1}^{\infty} \beta_k = \infty.$$

Iterative step: Calculate x_{k+1} as follows:

Step 1: Given x_{k-1} and x_k for each $k \geq 1$, choose θ_k such that $0 \leq \theta_k \leq \bar{\theta}_k$, where

$$\bar{\theta}_k = \begin{cases} \min\left\{\theta, \dfrac{\tau_k}{||x_k - x_{k-1}||}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \theta, & \text{otherwise}. \end{cases} \tag{1.5}$$

Step 2: Compute

$$\begin{cases} w_k = x_k + \theta_k(x_k - x_{k-1})), \\ y_k = P_C(w_k - \lambda F(w_k)). \end{cases} \tag{1.6}$$

If $y_k = w_k$ or $F(y_k) = 0$ then stop, y_k is the solution. Otherwise go to **Step 3**.

Step 3: Calculate

$$z_k = w_k - \gamma \eta_k d_k,$$

where

$$d_k := w_k - y_k - \lambda(F(y_k) - (F w_k)), \qquad \eta_k := \frac{\langle w_k - y_k, d_k \rangle}{||d_k||^2}.$$

Step 4 Calculate

$$x_{k+1} = (1 - \delta_k - \beta_k)x_k + \delta_k z_k.$$

Set $k := k + 1$ and return to **Step 1**.

Following this direction, Chidume and Nnakwe [13] extended the study of the SEGM to the framework of a 2-uniformly convex and uniformly smooth Banach space. The following method was proposed for the VIP:

$$\begin{cases} x_1 \in C, \\ y_k = \Pi_C(x_k - \lambda_k F(x_k)), \\ C_k = \{w \in H : \langle x_k - \lambda_k F(x_k) - y_k, w - y_k \rangle \leq 0\}, \\ x_{k+1} = \Pi_{C_k}(x_k - F(y_k)), \quad k \geq 1, \end{cases} \tag{1.7}$$

where Π_C is the generalized projection of the Banach space E onto C and J is the normalized duality mapping from E to 2^{E^*}. They obtained and proved a weak convergence theorem using the proposed method.

On the other hand, despite the noted improvement in the development of these methods, the discussed methods still preserve the weakness of the extragradient method in the form of a projection onto a feasible set. Secondly, the parameter α_k is selected in a way which shows dependence on the Lipschitz constant of the underlying operator. This is also a drawback for the methods as most times the Lipschitz constant is not easy to calculate even when they are known. Recently, the Tseng Extragradient Method (TEGM) has received great attention of many authors. Compared to the EGM and the SEGM, the TEGM method requires to compute only one projection onto the feasible set. To be precise, the TEGM is given by

$$\begin{cases} x_1 \in H, \\ y_k = P_C(x_k - \lambda_k F(x_k)), \\ x_{k+1} = y_k - \lambda_k(F(y_k) - F(x_k)), \quad \forall\, k \geq 1. \end{cases} \tag{1.8}$$

Recently, Thong et al. [36] considered the problem of approximating the VIP in a real Hilbert space. By combining the inertial method, TEGM and viscosity approximation method, they proved a strong convergence theorem for solving the VIP for a monotone and Lipschitz continuous operator. We observe that obtaining strong convergence of methods for solving the VIP requires a strong monotonicity assumption of the operator or that the method is combined with any of Halpern or viscosity approximation method (see [36]).

In this paper, motivated by the literature above, we propose a method for approximating the solution of the VIP for a pseudomonotone operator. The method combines the inertial, TEGM and an Halpern method. Strong convergence theorem was proved in the framework of a 2-uniformly convex Banach space which is also uniformly smooth. We highlight the following as the advantage of our work over previous works in this direction:

(i) the simplicity of calculating projection onto the feasible set C makes our method efficient for computation;
(ii) compared to the work of Thong et al. [36], our method does not require the knowledge of the Lipschitz constant of F;
(iii) our method does not require a linesearch which has also been shown to slow down convergence rates of method (see [21, 34]) and many others that used the same approach;
(iv) our study is conducted in the framework of the Banach space.

The rest of the paper is organized as follows: In Section 2, we recall some basic definitions and important results which are required in the proof of our main results. We present our proposed method in Section 3 and then give its convergence analysis. In Section 4, we report some numerical illustrations to show the efficiency of our method compared to some of the methods discussed in Section 1. Finally, we give a concluding remark in Section 5.

2 Preliminaries

In this section, we recall some preliminary definitions and useful lemmas. Let C be a nonempty, closed and convex subset of a real Banach space E with norm $||\cdot||$. We denote by E^* and $\langle\cdot,\cdot\rangle$ respectively, the dual space of E and the duality pairing between the elements of E and E^*. We write $x_k \rightharpoonup x^*$ and $x_k \to x^*$ wweak and the strong convergence of a sequence $\{x_k\}$ to a point x^* respectively.

Let E be a real Banach space, given a function $h : E \to \mathbb{R}$,

(i) h is called Gâteaux differentiable at $x \in E$, if there exists an element of E, denoted by $h'(x)$ or $\nabla h(x)$ such that

$$\lim_{t\to 0}\frac{h(x+ty)-h(x)}{t} = \langle y, h'(x)\rangle, \quad y \in E,$$

where $h'(x)$ or $\nabla h(x)$ is called Gâteaux differential or gradient of h at x. We say h is Gâteaux differentiable on E if h is Gâteaux differentiable at every $x \in E$;

(ii) h is called weakly lower semicontinuous at $x \in E$, if $x_k \rightharpoonup x$ implies $h(x) \leq \liminf\limits_{k\to\infty} h(x_k)$. We say that h is weakly lower semicontinuous on E, if h is weakly lower semicontinuous at every $x \in E$;

(iii) if h is a convex function, then it is said to be differentiable at a point $x \in E$ if the following set

$$\partial h(x) = \{w \in E^* : h(y) - h(x) \geq \langle w, y - x\rangle, \ \ y \in E\} \tag{2.1}$$

is nonempty. Each element in $\partial h(x)$ is called a subgradient of h at x or the subdifferential of h and the inequality (2.1) is said to be the subdifferential inequality of h at x.

The function h is subdifferentiable on E, if h is subdifferentiable at every point $x \in E$. It is well known that if h is Gâteaux differentiable at x, then h is subdifferentiable at x and $\partial h(x) = \{h'(x)\}$, that is, $\partial h(x)$ is just a singleton set. For more details on Gâteaux differentiable functions on Banach spaces, see [8].

Let $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2, \ \ \forall\, x \in E\}.$$

We consider the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \ \forall\, x, y \in E.$$

Following [2], Alber introduced the generalized projection operator given by

$$\Pi_C x = \inf_{y \in C} \{\phi(y, x), \quad \forall\, x \in E\}.$$

In the real Hilbert spaces, observe that $\phi(x, y) = ||x - y||^2$ and $\Pi_C \equiv P_C$, where $P_C : H \to C$ is the metric projection of H onto C. It is obvious from the definition of ϕ that

$$(||x||^2 - ||y||^2) \leq \phi(x, y) \leq (||x||^2 + ||y||^2).$$

The functional ϕ also satisfy the following important properties (see [30, 31]):

(P1) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$;
(P2) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy\rangle$;
(P3) $\phi(x, y) = ||x||||Jx - Jy|| + ||y||||x - y||$.

We also considered the functional $V : E \times E^* \to \mathbb{R}$ which is defined by $V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x||^2$ for all $x \in E$ and $x^* \in E^*$. It is easy to see that $V(x, x^*) = \phi(x, J^{-1}x^*)$. We note that, if E is a reflexive, strictly convex and smooth Banach space, then

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\langle J^{-1}x^*, x^* - y^*\rangle,$$

for all $x \in E$ and all $x^*, y^* \in E^*$, (see [32]).

Definition 2.1 Let $F : C \to E^*$ be an operator. Then F is

(a) monotone, if

$$\langle F(x), x - y\rangle \geq \langle F(y), x - y\rangle, \quad \forall\, x, y \in C;$$

(b) pseudomonotone, if

$$\langle F(x), x - y\rangle \geq 0 \Rightarrow \langle F(y), x - y\rangle \geq 0, \quad \forall\, x, y \in C.$$

Let S_E and B_E be the unit sphere and unit closed ball of a real Banach space E respectively. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \left\{ 1 - \frac{1}{2}||x + y|| : \ x, y \in B_E, \ \ ||x - y|| \geq \epsilon \right\}.$$

A Banach space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$ and 2-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\epsilon) > c\epsilon^2$ for any $\epsilon \in (0, 2]$. It is obvious that every 2-uniformly convex Banach space is uniformly convex. The Banach space E is said to be strictly convex if $||x + y|| < 2$ for every $x, y \in S_E$ with $x \neq y$. The modulus of smoothness of E is the function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_E(t) = \sup \left\{ \frac{1}{2}(||x + ty|| - ||x - ty||) - 1 : x, \in S_E, \ \ ||y|| \leq t \right\}.$$

The Banach space E is said to be uniformly smooth if $\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0$ and 2-uniformly smooth if there exists a fixed constant $\kappa > 0$ such that $\rho_E(t) < \kappa t^2$. E is said to be smooth if the limit

$$\lim_{t\to 0}\frac{||x+ty||-||x||}{t}$$

exists for all $x, y \in S_E$. It is well known that every 2-uniformly smooth Banach space is uniformly smooth. For more on the geometry of Banach spaces (see [14, 19]).

Lemma 2.2 [2] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space X. If $x \in E$ and $q \in C$, then*

$$q = \Pi_C x \iff \langle y - q, Jx - Jq\rangle \leq 0, \quad \forall\, y \in C \tag{2.2}$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \ \forall\, y \in C, \ x \in E. \tag{2.3}$$

Lemma 2.3 [20] *Let E be a smooth and uniformly convex real Banach space, let $\{x_k\}$ and $\{y_k\}$ be two sequences in E. If either $\{x_k\}$ or $\{y_k\}$ is bounded and $\phi(x_k, y_k) \to 0$ as $k \to \infty$, then $||x_k - y_k|| \to 0$ as $k \to \infty$.*

Lemma 2.4 [7] *Let $\frac{1}{p} + \frac{1}{q} = 1, \quad p, q > 1$. The space E is q-uniformly smooth if and only if its dual E^* is p-uniformly convex.*

Lemma 2.5 [37] *Let E be a 2-uniformly smooth Banach space with the best smoothness constant $\kappa > 0$. Then, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, Jx\rangle + 2||\kappa y||^2, \quad \forall\, x, y \in E.$$

Lemma 2.6 [4] *Suppose E is 2-uniformly convex Banach space. Then, there exists a constant $c \geq 1$ such that*

$$\phi(x, y) \geq \frac{1}{c}||x-y||^2, \quad \forall\, x, y \in E.$$

Lemma 2.7 [33] *Let $\{a_k\}$ be a sequence of nonnegative real numbers, $\{\alpha_k\}$ be a sequence of real numbers in $(0, 1)$ such that $\sum_{k\to 1}^{\infty} \alpha_k = \infty$ and $\{b_k\}$ be a sequence of real numbers. Assume that*

$$a_{k+1} \leq (1-\alpha_k)a_k + \alpha_k b_k, \quad \forall\, k \geq 1.$$

If $\limsup_{j\to\infty} b_{k_j} \leq 0$ for every subsequence $\{a_{k_j}\}$ of $\{a_k\}$ satisfying the condition

$$\liminf_{j\to\infty}(a_{k_j+1} - a_{k_j}) \geq 0,$$

then $\lim_{k\to\infty} a_k = 0$.

Definition 2.8 (see [25, 28]). Let $F : C \to E^*$ be an operator. The Minty Variational Inequalities (MVI) consist of finding a point $x^* \in C$ such that

$$\langle F(y), y - x^* \rangle \geq 0, \quad \forall\, y \in C. \tag{2.4}$$

We denote by $M(F, C)$, the set of solution of (2.4). Some existence results for the MVI have been presented in [28]. Also, the assumption that $M(F, C) \neq \emptyset$ has been used for solving VIP (1.1) in finite dimensional spaces (see, e.g., [35]). It is easy to prove that pseudomonotonicity implies $M(F, C) \neq \emptyset$, but the converse is not true.

Lemma 2.9 [25] *Consider the VIP* (1.1). *Suppose the mapping* $h : [0, 1] \to E^*$ *defined by* $h(t) = F(tx + (1 - t)y)$ *for all* $x, y \in C$ *and* $t \in [0, 1]$ *(i.e., h is hemicontinuous), then* $M(F, C) \subset Sol(F, C)$*. Moreover, if F is pseudomonotone, then* $Sol(F, C)$ *is well defined and* $M(F, C) = Sol(F, C)$.

3 Main result

In this section, we present the convergence analysis of a Tseng extragradient-like method with generalized projection for VIP. Let C be a nonempty, closed and convex subset of a real 2-uniformly convex Banach space E which is also uniformly smooth with dual E^*. We represent by c and κ respectively the 2-uniformly convexity constant and 2-uniformly smoothness constant of E and E^*. For $i = 1, 2 \cdots m$, let $h_i : E \to \mathbb{R}$ be a family of convex, weakly lower semicontinous and Gâteaux differentiable functions such that $h_i'(\cdot)$ is K_i-Lipschitz continuous with $K = \max_{1 \leq i \leq m} K_i$. We consider a problem of finding a point in the set $Sol(F, C)$.

For this purpose, we assume the following conditions:

Assumption 3.1

(A1) *The feasible set C is nonempty, closed and convex.*

(A2) *The mapping* $F : C \to E^*$ *is pseudomonotone on* E*, L-Lipschitz continuous on* E *and weakly sequentially continuous on* C*. However, our proposed method does not require the information of L to be known.*

(A3) *The solution set* $Sol(F, C)$ *is nonempty.*

(A4) *The feasible set C is defined by*

$$C := \cap_{i=1}^m C^i$$

where

$$C^i := \{z \in E : h_i(z) \leq 0\}.$$

In addition, we assume that $\{\tau_k\}$ is a positive sequence such that $\tau_k = \circ(\beta_k)$, which implies that $\lim_{k\to\infty} \dfrac{\tau_k}{\beta_k} = 0$, where $\{\beta_k\} \subset (0, 1)$ satisfies $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\lim_{k\to\infty} \beta_k = 0$.

We study the convergence analysis of the following Tseng extragradient method.

Algorithm 2 Tseng extragradient method for VIP.

Initialization: Choose $\mu \in \left(0, \frac{1}{2c\kappa^2}\right)$ and $\theta > 0$. Select initial points $x_0,\ x_1 \in C$, $\alpha_0 > 0$ and set the counter $k := 1$. For $i = 1, 2 \cdots, m$ and given the current iterate w_k, construct the family of half spaces

$$C_k^i := \{z \in E : h_i(w_k) + \langle h_i'(w_k), z - w_k\rangle \leq 0\}$$

and set

$$C_k = \cap_{i=1}^m C_k^i.$$

Iterative step: Calculate x_{k+1} and α_{k+1} as follows:

Step 1: Given $x_{k-1},\ x_k$ and α_k, for each $k \geq 1$, choose θ_k such that $\theta_k \in [0, \bar{\theta}_k]$, where

$$\bar{\theta}_k = \begin{cases} \min\left\{\theta, \frac{\tau_k}{||x_k - x_{k-1}||}\right\}, & \text{if } x_k \neq x_{k-1}, \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2: Compute

$$\begin{cases} w_k = J^{-1}(Jx_k + \theta_k(Jx_{k-1} - Jx_k)), \\ y_k = \Pi_{C_k} J^{-1}(Jw_k - \alpha_k F(w_k)), \\ \alpha_{k+1} = \begin{cases} \min\left\{\alpha_k, \frac{\mu||y_k - w_k||}{||F(y_k) - F(w_k)||}\right\}, & \text{if } ||F(y_k) - F(w_k)|| > 0, \\ \alpha_k, & \text{otherwise,} \end{cases} \\ z_k = J^{-1}(Jy_k - \alpha_k(F(y_k) - F(w_k))), \\ x_{k+1} = J^{-1}(\beta_k Ju + (1 - \beta_k)Jz_k). \end{cases} \tag{3.2}$$

Stopping criterion: If $x_{k+1} = w_k = y_k$ for some $k \geq 1$ then stop. Otherwise set $k := k + 1$ and return to **Iterative step**.

Lemma 3.2 [38] *The sequence* $\{\alpha_k\}$ *generated by* (3.2) *is a monotonically decreasing sequence and*

$$\lim_{k\to\infty} \alpha_k = \alpha \geq \min\left\{\frac{\mu}{L}, \alpha_0\right\},$$

with $\alpha > 0$.

Remark 3.3 From the definition of C and C_k, it is easy to see that $C \subset C_k$. Indeed, for each $i = 1, 2, \cdots, m$ and $x \in C^i$, we have by the subdifferential inequality that

$$h_i(w_k) + \langle h_i'(w_k), x - w_k\rangle \leq h_i(x) \leq 0.$$

By the definition of C_k^i, we have that $x \in C_k^i$. Hence, $C^i \subset C_k^i$ for all i and therefore $C \subset C_k$ for all $k \geq 1$.

Remark 3.4 If $w_k = y_k$, then $w_k \in Sol(F, C)$.

Proof Assume that $w_k = y_k$ for some $k \geq 1$. Then by (3.2), we have

$$w_k = \Pi_{C_k} J^{-1}(Jw_k - \alpha_k F(w_k)).$$

Using this, we assert that $w_k \in C_k$, thus $w_k \in C_k^i$ for each $i = 1, 2, \cdots, m$. By the definition of C_k^i, we have that $h_i(w_k) + \langle h_i'(w_k), w_k - w_k\rangle \leq 0$. This implies that $h_i(w_k) \leq 0$ for each i, hence $w_k \in C$. □

From the definition of $\{y_k\}$ and the property of Π_{C_k} in Lemma 2.2, we have

$$\langle Jw_k - \alpha_k F(w_k) - Jw_k, y - w_k\rangle \leq 0, \quad \forall\, y \in C_k$$

or equivalently

$$\alpha_k \langle F(w_k), y - w_k\rangle \geq 0, \quad \forall\, y \in C_k.$$

Since $\alpha_k > 0$, we obtain that $\langle F(w_k), y - w_k\rangle \geq 0$. Hence, $w_k \in Sol(F, C_k)$. The conclusion follows from this, $w_k \in C$ and $C \subset C_k$. That is $w_k \in Sol(F, C)$.

Remark 3.5 From (3.1) of Algorithm 3.2, we have

$$\lim_{k\to\infty} \theta_k \big(\phi(x^*, x_{k-1}) - \phi(x^*, x_k)\big) = 0.$$

Proof Indeed, we have that $\theta_k ||x_k - x_{k-1}|| \leq \tau_k$ for each $k \geq 1$, which together with $\lim\limits_{k\to\infty} \dfrac{\tau_k}{\beta_k} = 0$ implies

$$\lim_{k\to\infty} \frac{\theta_k}{\beta_k} ||x_k - x_{k-1}|| \leq \lim_{k\to\infty} \frac{\tau_k}{\beta_k} = 0. \tag{3.3}$$

□

Now,

$$\begin{aligned}\phi(x^*, x_{k-1}) - \phi(x^*, x_k) &= ||x||^2 - 2\langle x^*, Jx_{k-1}\rangle + ||x_{k-1}||^2 \\ &\quad -(||x^*||^2 - 2\langle x^*, Jx_k\rangle + ||x_k||^2) \\ &= ||x_{k-1}||^2 - ||x_k||^2 + 2\langle x^*, Jx_k - Jx_{k-1}\rangle - ||x_k||^2 \\ &\leq ||x_{k-1} - x_k||(||x_k|| + ||x_{k-1}||) + 2||x^*||||Jx_{k-1} - Jx_k||.\end{aligned}$$

Since J is norm-to-norm continuous on subsets of E^*, we obtain from (3.3) that

$$\lim_{k\to\infty} \beta_k \cdot \frac{\theta_k}{\beta_k} ||Jx_k - Jx_{k-1}|| = \lim_{k\to\infty} \beta_k \cdot \frac{\theta_k}{\beta_k} ||x_k - x_{k-1}|| = 0, \tag{3.4}$$

hence,

$$\lim_{k\to\infty} \beta_k \left(\frac{\theta_k}{\beta_k} ||x_{k-1} - x_k||(||x_k|| + ||x_{k-1}||) + 2\frac{\theta_k}{\beta_k} ||x^*||||Jx_{k-1} - Jx_k|| \right) = 0.$$

Thus,

$$\lim_{k\to\infty} \theta_k \big(\phi(x^*, x_{k-1}) - \phi(x^*, x_k)\big) = 0.$$

Lemma 3.6 *Assume that Assumption 3.1 holds and* $\{w_k\}$ *is a sequence generated by Algorithm 3.2. Let* $\{w_{k_j}\}$ *be a subsequence of* $\{w_k\}$ *converging weakly to* $\bar{x} \in E$ *and* $\lim\limits_{j\to\infty} ||w_{k_j} - y_{k_j}|| = 0$, *then* $\bar{x} \in Sol(F, C)$.

Proof Using the definition of $\{y_k\}$ and Lemma 2.2, we have

$$\langle Jw_{k_j} - \alpha_{k_j} F(w_{k_j}) - Jy_k, y - y_{k_j}\rangle \leq 0, \quad \forall\, y \in C_k,$$

equivalently

$$\frac{1}{\alpha_{k_j}}\langle Jw_{k_j} - Jy_{k_j}, y - y_{k_j}\rangle \leq \langle F(w_{k_j}), y - y_{k_j}\rangle, \quad \forall\, y \in C_k.$$

It follows that

$$\frac{1}{\alpha_{k_j}}\langle Jw_{k_j} - Jy_{k_j}, y - y_{k_j}\rangle + \langle F(w_{k_j}), y_{k_j} - w_{k_j}\rangle \leq \langle F(w_{k_j}), y - w_{k_j}\rangle, \quad \forall\, y \in C_k. \tag{3.5}$$

Since $||w_{k_j} - y_{k_j}|| \to 0$ as $j \to \infty$ and J is norm-to-norm continuous on subsets of E, we obtain $||Jw_{k_j} - Jy_{k_j}|| \to 0$. Taking limit as $j \to \infty$ in (3.5), we obtain

$$\liminf_{j\to\infty}\langle F(w_{k_j}), y - w_{k_j}\rangle \geq 0, \quad \forall\, y \in C_k. \tag{3.6}$$

Using this, $w_{k_j} \in C$ and $C \subset C_{k_j}$, we have that

$$\liminf_{j\to\infty}\langle F(w_{k_j}), y - w_{k_j}\rangle \geq 0, \quad \forall\, y \in C. \tag{3.7}$$

Next, we show that $\bar{x} \in C$. Indeed, it follows from $y_{k_j} \in C_{k_j}$ that

$$h_i(w_{k_j}) + \langle h_i'(w_{k_j}), y_{k_j} - w_{k_j}\rangle \leq 0.$$

By using Cauchy Schwartz inequality, we have

$$\begin{aligned} h_i(w_{k_j}) &\leq \langle h_i'(w_{k_j}), w_{k_j} - y_{k_j}\rangle \\ &\leq ||h_i'(w_{k_j})|| \cdot ||w_{k_j} - y_{k_j}||. \end{aligned}$$

Since h_i' is Lipschitz continuous and $\{w_{k_j}\}$ is bounded, we have that $\{h_i'(w_{k_j})\}$ is bounded. Thus, there exists $K_i > 0$ such that $||h_i'(w_{k_j})||$ for each i. Therefore, we obtain

$$h_i(w_{k_j}) \leq K \cdot ||w_{k_j} - y_{k_j}||,$$

where $K = \max_{1\leq i\leq m}\{K_i\}$. Hence, by the weakly continuity of h_i, we have

$$h_i(\bar{x}) \leq \liminf_{j\to\infty} h_i(w_{k_j}) \leq \lim_{j\to\infty} K \cdot ||w_{k_j} - y_{k_j}|| = 0.$$

Therefore, $\bar{x} \in C$. □

Now choose a sequence $\{\epsilon_j\}$ of positive numbers such that $\{\epsilon_j\}$ is decreasing and $\epsilon_j \to 0$ as $j \to \infty$. For each $j \geq 1$, denote by N_{k_j} the smallest positive integer such that

$$\langle F(w_{k_j}), y - w_{k_j}\rangle + \epsilon_j \geq 0, \quad \forall\, j \in N_{k_j}.$$

Observe that $\{N_{k_j}\}$ is nondecreasing since $\{\epsilon_j\}$ is decreasing. Thus for each $j \geq 1$, since $\{w_{k_j}\} \subset C$, we have $F(w_{k_j}) \neq 0$ and by setting $A_{N_{k_j}} := \dfrac{F(w_{N_{k_j}})}{||F(w_{N_{k_j}})||^2}$, we get $\langle F(w_{k_j}), A_{N_{k_j}}\rangle = 1$, for each $j \geq 1$. It follows from (3.7) for each $j \geq 1$, that

$$\langle F(w_{N_{k_j}}), w + \epsilon_j A_{N_{k_j}} - w_{N_{k_j}}\rangle \geq 0.$$

Now, since F is pseudomonotone, we obtain

$$\langle F(y+\epsilon_j A_{N_{k_j}}), y+\epsilon_j A_{N_{k_j}} - w_{N_{k_j}}\rangle \geq 0.$$

Therefore,

$$\begin{aligned}\langle F(y), y - w_{N_{k_j}}\rangle &\geq \langle F(y) - F(y+\epsilon_j A_{N_{k_j}}), y+\epsilon_j A_{N_{k_j}} - w_{N_{k_j}}\rangle \\ &\quad -\epsilon_j\langle F(y), A_{N_{k_j}}\rangle \end{aligned} \tag{3.8}$$

To obtain the conclusion of the lemma, we show that $\lim\limits_{j\to\infty} \epsilon_j A_{N_{k_j}} = 0$. To see this, since $w_{k_j} \rightharpoonup \bar{x} \in E$ and F is weakly sequentially continuous on C, it follows that $\{F(w_{N_{k_j}})\}$ converges weakly to $F(\bar{x})$. We suppose $F(\bar{x}) \neq 0$, otherwise $\bar{x} \in Sol(F, C)$. Since norm $||\cdot||$ is sequentially weakly lower semicontinuous, we have

$$||F(\bar{x})|| \leq \liminf_{j\to\infty} ||F(w_{k_j})||.$$

Since $w_{N_{k_j}} \subset w_{k_j}$ and $\epsilon_j \to 0$ as $j \to \infty$, we get

$$0 \leq \limsup_{j\to\infty} ||\epsilon_j A_{N_{k_j}}|| = \limsup_{j\to\infty} \frac{\epsilon_j}{||F(w_{k_j})||} \leq \frac{\limsup \epsilon_j}{\liminf\limits_{j\to\infty} ||F(w_{k_j})||} = 0,$$

thus, $\lim\limits_{j\to\infty} \epsilon_j A_{N_{k_j}} = 0$. Therefore,

$$\liminf_{j\to\infty} \langle F(y), y - w_{N_{k_j}}\rangle \geq 0.$$

Hence, for all $y \in C$, we have

$$\langle F(y), y-\bar{x}\rangle = \lim_{j\to\infty} \langle F(y), y - w_{N_{k_j}}\rangle = \liminf_{j\to\infty} \langle F(y), y - w_{N_{k_j}}\rangle \geq 0.$$

Therefore by Lemma 2.9, we obtain $\bar{x} \in Sol(F, C)$.

Lemma 3.7 *The sequence $\{x_k\}$ defined iteratively by Algorithm 3.2 is bounded and satisfies the inequality*

$$\phi(x^*, z_k) \leq \phi(x^*, w_k) - \left(1 - \frac{2c\mu k^2 \alpha_k^2}{\alpha_{k+1}^2}\right)\phi(y_k, w_k). \tag{3.9}$$

Proof Let $x^* \in Sol(F, C)$, then from (3.2), we have

$$\begin{aligned}\phi(x^*, z_k) &= \phi(x^*, J^{-1}(Jy_k - \alpha_k(F(y_k) - F(w_k)))) \\ &= ||x^*||^2 - 2\langle x^*, Jy_k - \alpha_k(F(y_k) - F(w_k))\rangle \\ &\quad + ||Jy_k - \alpha_k(F()y)_k - F(w_k))||^2 \\ &= ||x^*||^2 - 2\langle x^* Jy_k\rangle + 2\alpha_k\langle x^*, F(y_k) - F(w_k)\rangle \\ &\quad + ||Jy_k - \alpha_k(F(y_k) - F(w_k))||^2. \end{aligned} \tag{3.10}$$

Using Lemma 2.3, we get that E^* is 2-uniformly smooth and so by Lemma 2.4, we get

$$\begin{aligned}||Jy_k - \alpha_k(F(y_k) - F(w_k))||^2 &\leq ||Jy_k||^2 - 2\alpha_k\langle y_k, F(y_k) - F(w_k)\rangle \\ &\quad + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2\end{aligned}$$

Substituting this in (3.10) and applying (P1), we obtain

$$
\begin{aligned}
\phi(x^*, z_k) \le\ & ||x^*||^2 - 2\langle x^*, Jy_k\rangle + 2\alpha_k\langle x^*, F(y_k) - F(w_k)\rangle \\
& + ||Jy_k||^2 - 2\alpha_k\langle y_k, F(y_k) - F(w_k)\rangle \\
& + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2 \\
=\ & \phi(x^*, y_k) + 2\alpha_k\langle x^* - y_k, F(y_k) - F(w_k)\rangle + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2 \\
=\ & \phi(x^*, w_k) + \phi(w_k, y_k) + 2\langle x^* - w_k, Jw_k - Jy_k\rangle \\
& + 2\alpha_k\langle x^* - y_k, F(y_k) - F(w_k)\rangle \\
& + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2.
\end{aligned}
\tag{3.11}
$$

From (P2), we have $\phi(w_k, y_k) = -\phi(y_k, w_k) + 2\langle y_k - w_k, Jy_k - Jw_k\rangle$. Using this in (3.11), we obtain

$$
\begin{aligned}
\phi(x^*, z_k) \le\ & \phi(x^*, w_k) - \phi(y_k, w_k) - 2\alpha_k\langle y_k - x^*, F(w_k)\rangle \\
& + 2\alpha_k\langle x^* - y_k, F(y_k) - F(w_k)\rangle \\
& + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2 \\
=\ & \phi(x^*, w_k) - \phi(y_k, w_k) - 2\alpha_k\langle y_k - x^*, F(y_k)\rangle \\
& + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2.
\end{aligned}
\tag{3.12}
$$

Note that by $x^* \in Sol(F, C)$, we have $\langle F(x^*), y_k - x^*\rangle \ge 0, \quad \forall\ y \in C$. It follows that $\langle F(y_k), y_k - x^*\rangle \ge 0$, since F is pseudomonotone. Thus we have from (3.12), that

$$
\phi(x^*, z_k) \le \phi(x^*, w_k) - \phi(y_k, w_k) + 2\kappa^2\alpha_k^2||F(y_k) - F(w_k)||^2.
$$

By applying Lemma 2.6, we obtain

$$
\begin{aligned}
\phi(x^*, z_k) \le\ & \phi(x^*, w_k) - \phi(y_k, w_k) + \frac{2c\mu\kappa^2\alpha_k^2}{\alpha_{k+1}^2}\phi(y_k, w_k) \\
=\ & \phi(x^*, w_k) - \left(1 - \frac{2c\mu\kappa^2\alpha_k^2}{\alpha_{k+1}^2}\right)\phi(y_k, w_k).
\end{aligned}
$$

□

Again from (3.2), we have

$$
\begin{aligned}
\phi(x^*, w_k) =\ & \phi(x^*, J^{-1}((1-\theta_k)Jx_k + \theta_k Jx_{k-1})) \\
\le\ & (1-\theta_k)\phi(x^*, x_k) + \theta_k\phi(x^*, x_{k-1}).
\end{aligned}
\tag{3.13}
$$

From (3.2) and (3.13), we have

$$\begin{aligned}\phi(x^*, x_{k+1}) &= \phi(x^*, J^{-1}(\beta_k Ju + (1-\beta_k)Jz_k)) \\ &\le \beta_k\phi(x^*, u) + (1-\beta_k)\phi(x^*, z_k) \qquad (3.14)\\ &\le \beta_k\phi(x^*, u) + (1-\beta_k)\phi(x^*, w_k) \\ &\le \beta_k\phi(x^*, u) + (1-\beta_k)((1-\theta_k)\phi(x^*, x_k) + \theta_k\phi(x^*, x_{k-1})) \\ &\le \max\left\{\phi(x^*, u), \max\{\phi(x^*, x_k), \phi(x^*, x_{k-1})\}\right\} \\ &\le \vdots \\ &\le \max\left\{\phi(x^*, u), \max\{\phi(x^*, x_1), \phi(x^*, x_0)\}\right\}. \qquad (3.15)\end{aligned}$$

We are now in the position to state and prove our strong convergence theorem.

Theorem 3.8 *Assume that Assumption 3.1 holds and let* $\{x_k\}$ *be the sequence given by Algorithm 3.2, then* $\{x_k\}$ *converges strongly to a point* $x^* = \Pi_{Sol(F,C)}u \in Sol(F, C)$.

Proof Let $x^* \in Sol(F, C)$. From (3.2), we obtain

$$\begin{aligned}\phi(x^*, x_{k+1}) &= \phi(x^*, J^{-1}(\beta_k Ju + (1-\beta_k)Jz_k)) \\ &= V(x^*, \beta_k Ju + (1-\beta_k)Jz_k) \\ &\le V(x^*, \beta_k Ju + (1-\beta_k)Jz_k - \beta_k(Ju - Jx^*)) \\ &\quad +2\langle J^{-1}(\beta_k Ju + (1-\beta_k)Jz_k) - p, \beta_k(Ju - Jx^*)\rangle \\ &= \beta_k V(x^*, Jx^*) + (1-\beta_k)V(x^*, Jz_k) + 2\beta_k\langle x_{k+1} - x^*, Ju - Jx^*\rangle \\ &= \beta_k\phi(x^*, x^*) + (1-\beta_k)\phi(x^*, z_k) + 2\langle x_{k+1} - x^*, Ju - Jx^*\rangle \\ &\le (1-\beta_k)\phi(p, w_k) + 2\beta_k\langle x_{k+1} - x^*, Ju - Jx^*\rangle \\ &\le (1-\beta_k)((1-\theta_k)\phi(x^*, x_k) + \theta_k\phi(x^*, x_{k-1})) + 2\beta_k\langle x_{k+1} - x^*, Ju - Jx^*\rangle \\ &\le (1-\beta_k)\phi(x^*, x_k) + \beta_k\left(\frac{\theta_k}{\beta_k}\phi(x^*, x_{k-1}) + 2\langle x_{k+1} - x^*, Ju - Jx^*\rangle\right). \qquad (3.16)\end{aligned}$$

To show that $\{||x_k - x^*||^2\} \to 0$ as $k \to \infty$. It suffices to show that $\limsup_{k\to\infty}\langle x_{k+1} - x^*, Ju - Jx^*\rangle \le 0$, and then apply Lemma 2.7 to (3.16). Suppose there exists a subsequence $\{\phi(x^*, x_{k_j})\}$ of $\{\phi(x^*, x_{k_j+1})\}$ satisfying

$$\limsup_{j\to\infty}[\phi(x^*, x_{k_j+1}) - \phi(x^*, x_{k_j})] \ge 0.$$

Consider such a subsequence, we have from (3.9) and Remark 3.6, that

$$\begin{aligned}\limsup_{j\to\infty}\left(1 - \frac{2c\mu\kappa^2\alpha_{k_j}^2}{\alpha_{k_j+1}^2}\right)\phi(y_{k_j}, w_{k_j}) &\le \limsup_{j\to\infty}\left(\beta_{k_j}\phi(x^*, u) + (1-\beta_{k_j})\phi(x^*, w_{k_j}) - \phi(x^*, x_{k_j+1})\right) \\ &= \limsup_{j\to\infty}\left(\beta_{k_j}(\phi(x^*, u) - \phi(x^*, w_{k_j}))\right) \\ &\quad + \limsup_{j\to\infty}(\phi(x^*, w_{k_j}) - \phi(x^*, x_{k_j+1})) \\ &\le \limsup_{j\to\infty}(\phi(x^*, w_{k_j}) - \phi(x^*, x_{k_j+1}))\end{aligned}$$

$$
\begin{aligned}
&\le \limsup \big((1-\theta_{k_j})\phi(x^*, x_{k_j})+\theta_{k_j}\phi(x^*, x_{k_j-1})-\phi(x^*, x_{k_j+1})\big)\\
&= \limsup_{j\to\infty} \big(\phi(x^*, x_{k_j}) - \phi(x^*, x_{k_j+1})\big)\\
&\quad + \limsup_{j\to\infty} \theta_{k_j} \big(\phi(x^*, x_{k_j-1}) - \phi(x^*, x_{k_j})\big)\\
&= -\liminf_{j\to\infty} \big(\phi(x^*, x_{k_j+1}) - \phi(x^*, x_{k_j})\big)\\
&\quad + \limsup_{j\to\infty} \theta_{k_j} \big(\phi(x^*, x_{k_j-1}) - \phi(x^*, x_{k_j})\big)\\
&\le 0.
\end{aligned} \tag{3.17}
$$

Since $\left(1 - \dfrac{2c\mu\kappa^2\alpha_{k_j}^2}{\alpha_{k_j+1}^2}\right) > 0$, we obtain by (3.17), that

$$\lim_{j\to\infty} \phi(y_{k_j}, w_{k_j}) = 0. \tag{3.18}$$

It follows from Lemma 2.3 and the boundedness $\{y_{k_j}\}$, that $||y_{k_j} - w_{k_j}|| \to 0$ as $j \to \infty$.

Finally, we show that $||x_{k_j+1} - x_{k_j}|| \to 0$ as $j \to \infty$. First, from (3.2), we have

$$
\begin{aligned}
||Jz_{k_j} - Jy_{k_j}|| &= ||Jy_{k_j} - \alpha_{k_j}(F(y_{k_j}) - F(w_{k_j})) - Jy_{k_j}||\\
&\le \alpha_{k_j}||F(y_{k_j}) - F(w_{k_j})||\\
&\le \frac{\mu\alpha_{k_j}}{\alpha_{k_j+1}}||y_{k_j} - w_{k_j}||,
\end{aligned}
$$

which implies that $||Jz_{k_j} - Jy_{k_j}|| \to 0$ as $j \to \infty$. Since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E^*, we have that

$$\lim_{j\to\infty} ||z_{k_j} - y_{k_j}|| = 0. \tag{3.19}$$

Again from (3.2), we have

$$||Jx_{k_j+1} - Jz_{k_j}|| = \beta_{k_j}||u - z_{k_j}|| + (1-\beta_{k_j})||Jz_{k_j} - Jz_{k_j}|| \to 0.$$

This implies that

$$||x_{k_j+1} - z_{k_j}|| = 0, \tag{3.20}$$

since J^{-1} is norm-to-norm uniformly continuous on bounded subsets of E^*. Therefore, by applying triangle inequalities, we obtain

$$
\begin{cases}
\lim\limits_{j\to\infty} ||y_{k_j} - w_{k_j}|| = \lim\limits_{j\to\infty} (||y_{k_j} - w_{k_j}|| + ||w_{k_j} - x_{k_j}||) = 0,\\
\lim\limits_{j\to\infty} ||x_{k_j+1} - y_{k_j}|| = \lim\limits_{j\to\infty} (||x_{k_j+1} - z_{k_j}|| + ||z_{k_j} - y_{k_j}||) = 0,\\
\lim\limits_{j\to\infty} ||x_{k_j+1} - x_{k_j}|| = \lim\limits_{j\to\infty} (||x_{k_j+1} - z_{k_j}|| + ||z_{k_j} - x_{k_j}||) = 0.
\end{cases} \tag{3.21}
$$

Since the sequence $\{x_{k_j}\}$ is bounded, it follows that there exists a subsequence $\{x_{k_{j_i}}\}$ of $\{x_{k_j}\}$ that converges weakly to $\bar{x} \in E$, such that

$$\limsup_{j\to\infty}\langle Ju - Jx^*, x_{k_j+1} - x^*\rangle = \lim_{i\to\infty} = \langle Ju - Jx^*, x_{k_{j_i}+1} - x^*\rangle.$$

Table 1 Computation result for Example 4.1

m		Algorithm 2	Algorithm 1
5	No of iter	39	314
	CPU (time)	0.0211	0.0622
8	no of iter	44	979
	CPU (time)	0.0281	0.1593
10	No of iter	46	1216
	CPU (time)	0.0758	0.2592

It is easy to see by (3.21) that $x_{k_{j_i+1}} \rightharpoonup \bar{x}$, hence we have by $x^* = \Pi_{Sol(F,C)}u$ and Lemma 2.2, that

$$\begin{aligned}\limsup_{j\to\infty}\langle Ju - Jx^*, x_{k_j+1} - x^*\rangle &= \lim_{i\to\infty} = \langle Ju - Jx^*, x_{k_{j_i+1}} - x^*\rangle \\ &= \langle Ju - Jx^*, \bar{x} - x^*\rangle \le 0. \qquad (3.22)\end{aligned}$$

Also, from (3.21), we obtain $y_{k_j} \rightharpoonup \bar{x}$, thus by Lemma 3.7, $\bar{x} \in Sol(F, C)$. Hence by (3.16), (3.22), condition (iii), and Lemma 2.7, we have that

$$\lim_{k\to\infty} \phi(x^*, x_k) = 0.$$

It follows from this and Lemma 2.3 that $||x_k - x^*|| \to 0$ as $k \to \infty$. Therefore $\{x_k\}$ converges strongly to $x^* \in Sol(F, C)$. □

4 Numerical example

Example 4.1 The following example has been considered by many authors in the literature (see [23, 35]). The operator F is defined by $F = Mx + q$, where $M = B^T B + P + Q$ with $P, Q \in \mathbb{R}^{m\times m}$ are randomly generated matrices such that P is skew-symmetric, Q is a diagonal matrix of nonnegative (i.e., Q is positive definite) entries and $q = 0$. We define the feasible set by $C = \{x \in E : ||x|| = 5\}$. It is easy to check that the zero vector is feasible and therefore the unique solution of the corresponding variational inequality. We present the numerical results using the different values of m. In this example, we choose $\beta_k = \dfrac{1}{10(k+1)}$, $u = \mu = 0.5$, $\theta = 0.5$, $\alpha_0 = 0.25$. We make comparison with Algorithm 2 with the same parameter and suitable one where their algorithm differ from ours and as necessary. We terminate the iterations at $||x_{k+1} - x_k|| \le 10^{-3}$. The results are reported in Table 1 and Fig. 1.

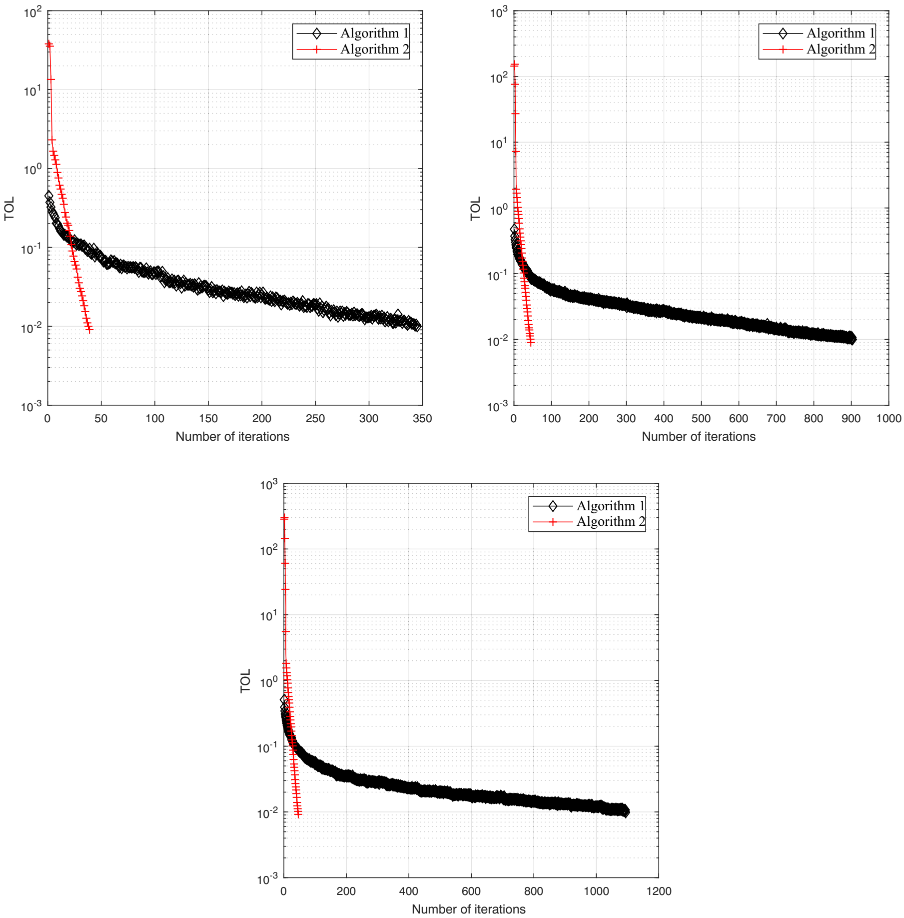

Fig. 1 Example 4.1

Example 4.2 Let $E = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^{\infty}$ of scalars in $\mathbb{R}$, that is

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \cdots, x_i \cdots), \quad x_i \in \mathbb{R} \text{ and } \sum_{i=1}^{\infty} |x_i|^2 < \infty \right\},$$

with the inner product $\langle \cdot, \cdot \rangle : \ell_2(\mathbb{R}) \times \ell_2(\mathbb{R}) \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{i=1}^{\infty} x_i y_i$ and the norm $|| \cdot || : \ell_2(\mathbb{R}) \to \mathbb{R}$ by $||x|| := \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$, where $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty}$. Let $C = \{x \in E : ||x|| \leq 5\}$, and let $F : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be defined by

$F(x) = (5 - ||x||)x$ for all $x \in \ell_2(\mathbb{R})$, it is easy to see that F is $L = \frac{11}{4}$-Lipschitz continuous. The projection onto C is easily computed as

$$P_C(x) = \begin{cases} x & \text{if} \quad ||x|| \leq 5, \\ \dfrac{5x}{||x||} & \text{otherwise.} \end{cases}$$

During this experiment, we choose $\beta_k = \dfrac{1}{(2k+1)^{0.5}}$, $u = \mu = 0.5$, $\theta = 0.5$, $\theta_k = \bar{\theta}_k$, $\tau_k = \dfrac{1}{k^{2.2}}$ and $\alpha_0 = 0.25$. For Algorithm 1, the following are also used for the parameters $\delta_k = 0.5 - \beta_k$ and $\gamma = 0.01$. We terminate the iterations if $||x_{k+1} - x_k|| \leq 10^{-5}$. We test Algorithms 1 and 2 for different cases of the initial points x_0 and x_1. The result of this experiment are displayed in Fig. 2.

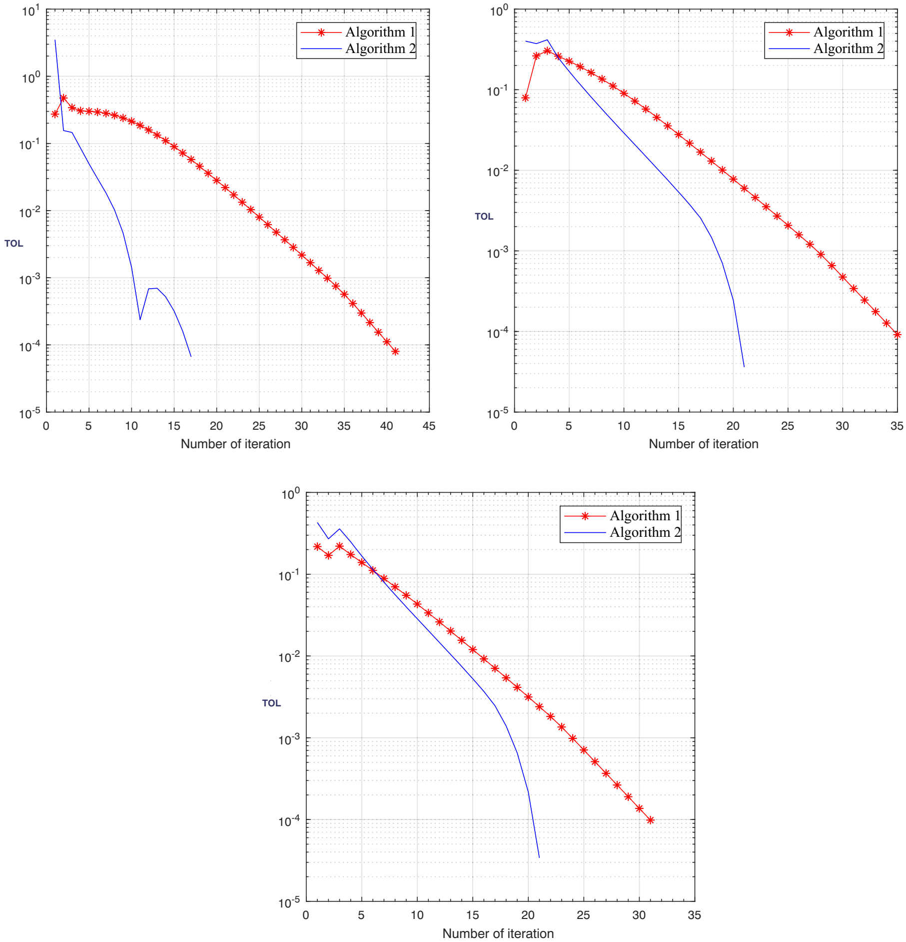

Fig. 2 Example 4.2. Top left: **Case 1**, Top right **Case 2**, Bottom: **Case 3**

Case 1 $x_0 = (1, 0, 0 \cdots, 0, \cdots)'$ and $x_1 = (4, 0, 0 \cdots, 0, \cdots)'$.
Case 2 $x_0 = (0.5, 0, 0 \cdots, 0, \cdots)'$ and $x_1 = (2, 0, 0 \cdots, 0, \cdots)'$.
Case 3 $x_0 = (-1, 0, 0 \cdots, 0, \cdots)'$ and $x_1 = (1, 0, 0 \cdots, 0, \cdots)'$.

5 Conclusion

In this paper, we have considered an iterative approximation of the solutions of pseudomonotone variational inequality problem. We proposed an inertial Tseng algorithm originally used for finding zeros of sum of monotone operators. The proposed method uses a single projection onto a half space which can be easily evaluated. The method considered in this paper does not require the knowledge of the Lipschitz constant as it uses variable stepsizes from step to step which are updated over each iteration by a simple calculation. We proved a strong convergence theorem of the sequence generated by this method to a solution of the VIP in the framework of a 2-uniformly convex Banach space which is also uniformly smooth. We also presented some numerical examples to further show the efficiency of the method.

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