EXISTENCE OF SOLUTION FOR A VOLTERRA TYPE INTEGRAL EQUATION USING DARBO-TYPE F-CONTRACTION

Article *in* Advances in Mathematics Scientific Journal · June 2021
DOI: 10.37418/amsj.10.6.2

Francis Akutsah
University of KwaZulu-Natal

Akindele Adebayo Mebawondu
African Institute for Mathematical Sciences South Africa

Ojen Narain
University of KwaZulu-Natal

ADV MATH SCI JOURNAL
Advances in Mathematics: Scientific Journal **10** (2021), no.6, 2687–2710
ISSN: 1857-8365 (printed); 1857-8438 (electronic)
https://doi.org/10.37418/amsj.10.6.2

EXISTENCE OF SOLUTION FOR A VOLTERRA TYPE INTEGRAL EQUATION USING DARBO-TYPE *F*-CONTRACTION

Francis Akutsah, Akindele Adebayo Mebawondu, and Ojen Kumar Narain[1]

ABSTRACT. In this paper, we provide some generalizations of the Darbo's fixed point theorem and further develop the notion of F-contraction introduced by Wardowski in ([22], D. Wardowski, *Fixed points of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory and Appl., 94, (2012)). To achieve this, we introduce the notion of Darbo-type F-contraction, cyclic (α, β)-admissible operator and we also establish some fixed point and common fixed point results for this class of mappings in the framework of Banach spaces. In addition, we apply our fixed point results to establish the existence of solution to a Volterra type integral equation.

1. INTRODUCTION AND PRELIMINARIES

The theory of fixed points plays an important role in nonlinear functional analysis and it is known to be very useful in establishing the existence and uniqueness theorems of nonlinear differential and integral equations. Banach [2] in 1922 proved the well celebrated Banach contraction principle in the frame

[1]*corresponding author*

2020 *Mathematics Subject Classification.* 47H09, 47H10, 49J20, 49J40.
Key words and phrases. Darbo type F-contraction, cyclic (α, β)-admissible operator, β-admissible,fixed point and Banach space.
Submitted: 17.05.2021; *Accepted:* 29.05.2021; *Published:* 02.06.2021.

work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and in other classical and important spaces (see [1, 12–15, 17, 20, 21] and the references therein). In 2012, Wardowski [22] introduced a class of mappings called the F-contractions. This class of mappings is defined as follows:

Definition 1.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

- (F_1) *F is strictly increasing;*
- (F_2) *for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+, \lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*
- (F_3) *there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

He also established the following result:

Theorem 1.1. *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.*

Remark 1.1. *[22] If we suppose that $F(t) = \ln t$, then an F-contraction mapping becomes the Banach contraction mapping.*

It is also worth mentioning that if $F(t) = \ln\ t\ \forall\ t \in \mathbb{R}^+$, it is easy to see that conditions $(F_1) - (F_3)$ are satisfied. On the other hand, if we take $F(t) = -\frac{1}{t},\ \forall\ t \in \mathbb{R}^+$. It is easy to see that F satisfies conditions (F_1) and (F_2). We therefore denote the family of functions that satisfy condition (F_1) and (F_2) by $\mathcal{F}$ which is lager than the class of functions satisfying $(F_1) - (F_3)$.

In 2016, Chandok et al. [5] introduced another class of mappings, called the TAC-contractive and established some fixed point results in the frame work of complete metric spaces.

Definition 1.2. *Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. Then T is called a cyclic (α, β)-admissible mapping, if*

(1) $\alpha(x) \geq 1$ *for some $x \in X$ implies that $\beta(Tx) \geq 1$,*
(2) $\beta(x) \geq 1$ *for some $x \in X$ implies that $\alpha(Tx) \geq 1$.*

Definition 1.3. *Let (X, d) be a metric space and let $\alpha, \beta : X \to [0, \infty)$ be two mappings. We say that T is a TAC-contractive mapping, if for all $x, y \in X$,*

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where ψ is a continuous and nondecreasing function with $\psi(t) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0, \infty)^2 \to \mathbb{R}$ is continuous, $f(a, t) \leq a$ and $f(a, t) = a \Rightarrow a = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Theorem 1.2. *Let (X, d) be a complete metric space and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Suppose that T is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold:*

(1) *T is continuous,*
(2) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$, for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

In addition, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in F(T)$ (where $F(T)$ denotes the set of fixed points of T), then T has a unique fixed point.

Definition 1.4. *A mapping T of a convex set X is said to be affine if it satisfies the inequality*

$$T(\alpha x + (1 - \alpha)y) \leq \alpha Tx + (1 - \alpha)Ty,$$

where, $x, y \in X$ and $0 < \alpha < 1$.

In 1930, Kuratowski [11] introduced the notion of measure of noncompactness. This concept has been used by researchers around the world to establish the fixed point results for single and multivalued mappings in different abstract spaces. The notion of noncompactness gives the degree of noncompactness for bounded sets. It worth mentioning that the concept of noncompactness with some algebraic concept is useful in establishing the existence of solutions to some nonlinear problems under some favorable conditions. For example, it is well-known in the literature that the notion of noncompactness is the very tool used in establishing the Darbo's fixed point theorem for noncompact operators.

The Darbo's fixed point theorem generalizes the well-known Schauder's fixed point theorem. In what follows, we give the axiomatic way of defining the measure of noncompactness. Suppose that X is a Banach space and Y any nonempty subset of X. We use the standard notion αY and $Y + Z$, with $Z \subset X$ to denote the algebraic operations on sets. More so, the symbol $\overline{Y}, coY$ and $\overline{co}Y$ denote the closure, convex hull and convex hull closure of Y respectively. Also, we denote $\mathcal{M}_X$ by the family of all nonempty bounded subsets of the space X and by $\mathcal{N}_X$ the family of all relatively compact subsets of X.

Definition 1.5. *Let $\mathcal{M}_X$ be the set of all bounded subset of a Banach space X. A mapping $\phi : \mathcal{M}_X \to \mathbb{R}^+$ is said to be a measure of noncompactness in X if the following conditions are satisfied:*

(1) *The family* $\ker \phi = \{A \in \mathcal{M}_X : \phi(A) = 0\}$ *is a nonempty set and* $ker\phi \subset \mathcal{N}_X$;
(2) $A \subset B \Rightarrow \phi(A) \leq \phi(B)$;
(3) $\phi(A) = \phi(\overline{A})$;
(4) $\phi(coA) = \phi(A)$;
(5) *For any* $\alpha \in (0, 1)$, $\phi(\alpha A + (1 - \alpha)B) \leq \alpha\phi(A) + (1 - \alpha)\phi(B)$;
(6) *If A_n is a sequence of closed set from $\mathcal{M}_X$ such that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$ and the $\lim_{n\to\infty} \phi(A_n) = 0$, then $A_\infty = \cap_{n=1}^{\infty} A_n$ is nonempty.*

Remark 1.2. *We note that $A_\infty = \cap_{n=1}^{\infty} A_n$ is an element of the family of $ker\phi$, as such $\phi(A_\infty) \leq \phi(A_n)$ for all $n \in \mathbb{N}$, using this fact, we have that $\phi(A_\infty) = 0$. It therefore follows that $A_\infty \in ker\phi$.*

Definition 1.6. *An operator $T : X \to Y$ is said to be compact if $T(Z)$ is relatively compact in a Banach space Y for any bounded subset Z in a Banach space X.*

Theorem 1.3. *Let Y be a nonempty, bounded, closed and convex subset of a Banach space X and T be a continuous compact self operator on Y. Then T has a fixed point in the set Y.*

The next result is the Darbo's fixed point theorem which is a generalization of Theorem 1.3 using the notion of measure of noncompactness.

Theorem 1.4. *Let Y be a nonempty, bounded, closed and convex subset of a Banach space X and T be a continuous self operator on Y. Suppose that there exists* *a constant* $k \in (0, 1)$ *such that*

$$\phi(TZ) \leq k\phi(Z) \tag{1.2}$$

for any nonempty $Z \subset Y$, *where* ϕ *is a measure of noncompactness define in* X. *Then* T *has a fixed point in the set* Y.

Hajji in [8] established a common fixed point result for commuting operator. His result generalizes and extends the Darbo's fixed point result.

Theorem 1.5. *Let* C *be a nonempty, bounded, closed and convex subset of a Banach space* E *and suppose that* $S, T : C \to C$ *are continuous operators such that*

(1) $ST = TS$;
(2) T *is affine;*
(3) *for any* $X \subset C$, *and* $0 < k < 1$, *we have*

$$\phi(ST(X)) \leq k\phi(X). \tag{1.3}$$

Then the set $\{x \in C : Tx = Sx = x\}$ *is nonempty and compact.*

Remark 1.3. *It is easy to see that if* $Tx = Ix = x$, *we obtain the Darbo's fixed point theorem.*

Motivated by the above facts it is our intention to further develop the notion of F-contraction, cyclic (α, β)-admissible mapping and generalize the Darbo's fixed point thoerem by introducing the notion of Darbo-type F-contraction and cyclic (α, β) admissible operator. We also establish some fixed point results for this class of mappings. In addition, we apply our fixed point results to establish the existence of solution to a Volterra type integral equation.

2. Main Result

In this section, we introduce the notion of Darbo-type F-contraction and cyclic (α, β)-admissible operator. In addition, we also establish some fixed point results for this class of mappings.

Definition 2.1. *Let* E *be a Banach space and let* $T : E \to E$ *be a given operator. We say that* T *is a Darbo-type-I-* F*-contraction if there exist functions* $\beta, \alpha : E \to [0, \infty)$, $F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\tau + F(\alpha(x)\beta(Tx)\phi(TY)) \leq F(\phi(Y)) \tag{2.1}$$

for any bounded subset Y and $x \in Y$ with ϕ an arbitrary measure of noncompactness and that $\phi(Y), \alpha(x)\beta(Tx)\phi(TY) > 0$.

Remark 2.1.

(1) *If $\alpha(x) = \beta(Tx) = 1$, we obtain*

$$\tau + F(\phi(TY)) \leq F(\phi(Y)), \tag{2.2}$$

which is the well-known F contraction associated with the measure of noncompactness.

(2) *If $F(x) = \ln(x)$, we have that*

$$\alpha(x)\beta(Tx)\phi(TY) \leq e^{-\tau}\phi(Y). \tag{2.3}$$

In addition, if $\alpha(x)\beta(Tx) = 1$, we obtain

$$\phi(TY) \leq e^{-\tau}\phi(Y), \tag{2.4}$$

where $e^{-\tau} \in (0, 1)$.

Definition 2.2. *Let E be a Banach space and let $T : E \to E$ be a given operator. We say that T is a Darbo-type-II- F-contraction if there exist functions $\beta, \alpha : E \to [0, \infty), F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\alpha(x)\beta(Tx) > 0 \Rightarrow \tau + F(\phi(TY)) \leq F(\phi(Y)) \tag{2.5}$$

for any bounded subset Y and $x \in Y$ with ϕ an arbitrary measure of noncompactness and that $\phi(Y), \phi(TY) > 0$.

Theorem 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible and Darbo-type-I-F-contraction such that there exist closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ with $TX_0 \subseteq X_0, \alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, where ϕ is an arbitrary measure of noncompactness. Then T has a fixed point in set C.*

Proof. We define the sequence of the set $\{X_n\}$ and element $\{x_n\}$ as follows:

$$X_n = \overline{co}(Tx_{n-1}) \text{ and } x_n = Tx_{n-1} \; \forall \; n \in \mathbb{N}.$$

From our hypothesis, since $TX_0 \subseteq X_0$, we have

$$\begin{aligned} X_1 &= \overline{co}(Tx_0) \subseteq X_0, \\ X_2 &= \overline{co}(Tx_1) \subseteq \overline{co}(Tx_0) = X_1, \\ X_3 &= \overline{co}(Tx_2) \subseteq \overline{co}(Tx_1) = X_2, \end{aligned}$$

continuing the process, we have that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots$$

and that

$$TX_n \subseteq TX_{n-1} \subseteq \overline{co}(TX_{n-1}) = X_n.$$

In what follows, we consider two cases in establishing our result based on the value of the measure of noncompactness.

Case 1. Suppose that there exists an integer $N > 0$ such that $\phi(X_N) = 0$, then X_N is a relatively compact set and also $TX_N \subseteq X_N$. Since T is a continuous self operator on C, then by Theorem 1.3 we conclude that T has a fixed point.

Case 2. Suppose that there exists and integer $n \in \mathbb{N}$ such that $\phi(X_n) > 0$ for all $n \in \mathbb{N}$. From our hypothesis, there exists $x_0 \in X_0$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$. Since T is cyclic (α, β) admissible mapping and $\alpha(x_0) \geq 1$, we have $\beta(x_1) = \beta(Tx_0) \geq 1$ and this implies that $\alpha(x_2) = \alpha(Tx_1) \geq 1$, continuing this process, we have that

$$\alpha(x_{2k}) \geq 1 \text{ and } \beta(x_{2k+1}) \geq 1 \ \forall\ k \in k \in \mathbb{N} \cup \{0\}. \tag{2.6}$$

Using similar argument, we have that

$$\beta(x_{2k}) \geq 1 \text{ and } \alpha(x_{2k+1}) \geq 1 \ \forall\ k \in k \in \mathbb{N} \cup \{0\}. \tag{2.7}$$

It follow from (2.6) and (2.7) that $\alpha(x_n) \geq 1, \beta(x_n) \geq 1, \alpha(x_{n+1}) = \alpha(Tx_n) \geq 1$ and $\beta(x_{n+1}) = \beta(Tx_n) \geq 1$. We therefore have that

$$\begin{aligned} \tau + F(\phi(X_{n+1})) &\leq \tau + F(\alpha(x_n)\beta(x_{n+1})\phi(X_{n+1})) \\ &= \tau + F(\alpha(x_n)\beta(Tx_n)\phi(\overline{co}(TX_n))) \\ &= \tau + F(\alpha(x_n)\beta(Tx_n)\phi(TX_n)) \\ &\leq F(\phi(X_n)). \end{aligned} \tag{2.8}$$

This implies that

$$F(\phi(X_{n+1})) \leq F(\phi(X_n)) - \tau,$$

inductively, we have that

$$F(\phi(X_{n+1})) \leq F(\phi(X_0)) - n\tau.$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$, we have that

$$\lim_{n\to\infty} F(\phi(X_{n+1})) = -\infty,$$

using (F_2), we have that

$$\lim_{n\to\infty} \phi(X_{n+1}) = 0.$$

Now, using the fact that $\{X_n\}$ is nested and from (6) of Definition 1.5, we have that the set $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$, closed and convex subset of the set X_0. In addition, $\phi(X_\infty) \leq \phi(X_n)$ for all $n \in \mathbb{N}$, which implies that $\phi(X_\infty) = 0$. Thus, we have that X_∞ is an element of $ker\phi$, which follows that X_∞ is compact. More so, we have that $X_\infty \subset X_n$ and $T(X_n) \subset X_n$ for all $n \in \mathbb{N}$. Therefore, we have that $T : X_\infty \to X_\infty$ is well defined , and for any bounded $Y \subset X_\infty$, we have that $T(Y) \subset X_\infty$ and $\overline{T(Y)}$ is a compact subsets of X_∞, implies that T is a compact operator. Therefore using Theorem 1.3, we have that T has a fixed point. □

Theorem 2.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible and Darbo-type-II-F-contraction such that there exist closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ with $TX_0 \subseteq X_0, \alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$, where ϕ is an arbitrary measure of noncompactness. Then T has a fixed point in set C.*

Proof. The prove follows similar approach as in Theorem 2.1, as such we omit it. □

Remark 2.2. *Using Remark 2.1 and applying similar approach as in Theorem 2.1, we obtain similar results as in Theorem 2.1 for* (2.2) *and* (2.3).

Using Theorem 2.1, we establish the following results to the classical metric fixed point theory.

Corollary 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible mapping satisfying the following:*

(1) *for all $X \in \mathcal{M}_C, \tau > 0$ and $x, y, \in X$, we have that*

$$\tau + F(\alpha(x)\beta(Tx)\|Tx - Ty\|) \leq F(\|x - y\|), \tag{2.9}$$

such that $\alpha(x)\beta(Tx)\|Tx - Ty\| > 0$ and $\|x - y\| > 0$,

(2) *there exist closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ such that $TX_0 \subseteq X_0, \alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.*

Then T has a fixed point in set C.

Proof. Let $\phi : \mathcal{M}_E \to \mathbb{R}^+$ be define as

$$\phi(X) = \text{diam } X,$$

where diam $X = \sup\{\|x - y\| : x, y \in X\}$ stand for the diameter of X. It is well-known that ϕ is a measure of noncompactness in the space E. Using similar approach as in Theorem 2.1 it is easy to see that $\alpha(x)\beta(Tx) \geq 1$. Now applying the definition of ϕ to (2.9), we have that

$$\tau + F(\alpha(x)\beta(Tx) \sup_{x,y\in X} \|Tx - Ty\|) \leq F(\sup_{x,y\in X} \|x - y\|),$$

which implies that

$$\tau + F(\alpha(x)\beta(Tx)\phi(TX)) \leq F(\phi(X)),$$

so from Theorem 2.1, we get the desired result. □

Corollary 2.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible mapping satisfying the following:*

(1) *for all $\tau > 0$ and $x, y, \in X$, we have that*

$$\|Tx - Ty\| > 0 \Rightarrow \tau + F(\|Tx - Ty\|) \leq F(\|x - y\|)$$

(2) *there exists closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ such that $TX_0 \subseteq X_0, \alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.*

Then T has a fixed point in set C.

Corollary 2.3. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible mapping satisfying the following:*

(1) *for all $\tau > 0$ and $x, y, \in X$, we have that*

$$\alpha(x)\beta(Tx) > 0 \Rightarrow \tau + F(\|Tx - Ty\|) \leq F(\|x - y\|),$$

such that $\|Tx - Ty\| > 0$ and $\|x - y\| > 0$,

(2) *there exists closed and convex $X_0 \subseteq C$ and $x_0 \in X_0$ such that $TX_0 \subseteq X_0, \alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.*

Then T has a fixed point in set C.

Proposition 2.1. *Suppose that $\alpha(x), \beta(Tx) \geq 1$ for all $x \in E$, then the set of all fixed point of T in Theorem 2.1 is a compact set.*

Proof. By definition $F(T) = \{x \in C : Tx = x\}$ is the set of fixed point of T and $\phi(F(T)) \neq \emptyset$, then by definition and using the fact that $TF(T) = F(T)$, we have that

$$F(\phi(F(T))) = F(\phi(TF(T))) < \tau + F(\alpha(x)\beta(Tx)\phi(TF(T))) \leq F(\phi(F(T))),$$

which is a contradiction.. We therefore have it that $F(T)$ is a relatively compact set. Now, for any convergent sequence $\{x_n\} \subset F(T)$ such that $x_n \to x^*$, it is easy to see that $x^* \in C$ due to the fact that C is closed. In addition, using the continuity of T, we have that

$$Tx^* = T\lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x^* \Rightarrow Tx^* = x^*,$$

it follows that $x^* \in F(T)$, as such $F(T)$ is a compact set. □

Definition 2.3. *Let $T : E \to E$ be mapping and $\alpha, \beta : 2^E \to [0,\infty)$ be two functions. We say that T is a cyclic (α, β) admissible operator if*

(1) $\alpha(Y) \geq 1$ *for every $Y \in 2^E$ implies* $\beta(\overline{co}(TY)) \geq 1$,
(2) $\beta(Y) \geq 1$ *for every $Y \in 2^E$ implies* $\alpha(\overline{co}(TY)) \geq 1$.

Remark 2.3. *If $\alpha(Y) = \beta(Y)$, we have that $\beta(Y) \geq 1$ for every $Y \in 2^E$ implies $\beta(\overline{co}(TY)) \geq 1$, as such we obtain β-admissible as defined in [7].*

Definition 2.4. *Let E be a Banach space and let $T : E \to E$ be a given operator. We say that T is a Darbo-type-III-F-contraction if there exist functions $\beta, \alpha : 2^E \to [0,\infty)$, $F \in \mathcal{F}$ and $\tau > 0$ such that*

$$\tau + F(\alpha(Y)\beta(Y)\phi(TY)) \leq F(\phi(Y)) \tag{2.10}$$

for any bounded subset $Y \subset E$ with ϕ an arbitrary measure of noncompactness and that $\phi(Y), \alpha(Y)\beta(Y)\phi(TY) > 0$.

Remark 2.4.

(1) *If $\alpha(Y) = \beta(Y) = 1$, we obtain*

$$\tau + F(\phi(TY)) \leq F(\phi(Y)), \tag{2.11}$$

which usual F contraction in the sense of measure of noncompactness.

(2) *If $F(x) = \ln(x)$, we have that*

$$\alpha(Y)\beta(Y)\phi(TY) \leq e^{-\tau}\phi(Y).$$

In addition, if $\alpha(Y) = \beta(Y) = 1,$ *we obtain*

$$\phi(TY) \leq e^{-\tau}\phi(Y),$$

where $e^{-\tau} \in (0, 1).$

Definition 2.5. *Let* E *be a Banach space and let* $T : E \to E$ *be a given operator. We say that* T *is a Darbo-type-IV-*F*-contraction if there exist functions* $\beta, \alpha : 2^E \to [0, \infty), F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\alpha(Y)\beta(Y) > 0 \Rightarrow \tau + F(\phi(TY)) \leq F(\phi(Y)) \tag{2.12}$$

for any bounded subset $Y \subset E$ *with* ϕ *an arbitrary measure of noncompactness and that* $\phi(Y), \phi(TY) > 0.$

Definition 2.6. *Let* E *be a Banach space and let* $T : E \to E$ *be a given operator. We say that* T *is a Darbo-type-V-*F*-contraction if there exist functions* $\beta : 2^E \to [0, \infty), F \in \mathcal{F}$ *and* $\tau > 0$ *such that*

$$\tau + F(\beta(Y)\phi(TY)) \leq F(\phi(Y)) \tag{2.13}$$

for any bounded subset $Y \subset E$ *with* ϕ *an arbitrary measure of noncompactness and that* $\phi(Y), \beta(Y)\phi(TY) > 0.$

Theorem 2.3. *Let* C *be a nonempty, bounded, closed and convex subset of a Banach space* E *and suppose that* $T : C \to C$ *is a continuous, cyclic* (α, β) *admissible operator and Darbo type-III-*F*-contraction such that there exists closed and convex* $X_0 \subseteq C$ *such that* $TX_0 \subseteq X_0, \alpha(X_0) \geq 1$ *and* $\beta(X_0) \geq 1,$ *where* ϕ *is an arbitrary measure of noncompactness. Then* T *has a fixed point in set* $C.$

Proof. We define the sequence of the set $\{X_n\}$ as follows:

$$X_n = \overline{co}(TX_{n-1}) \;\forall\; n \in \mathbb{N}.$$

From our hypothesis, since $TX_0 \subseteq X_0,$ we have

$$\begin{aligned} X_1 &= \overline{co}(Tx_0) \subseteq X_0, \\ X_2 &= \overline{co}(Tx_1) \subseteq \overline{co}(Tx_0) = X_1, \\ X_3 &= \overline{co}(Tx_2) \subseteq \overline{co}(Tx_1) = X_2, \end{aligned}$$

continuing the process, we have that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq X_{n+1} \supseteq \cdots$$

and that

$$TX_n \subseteq TX_{n-1} \subseteq \overline{co}(TX_{n-1}) = X_n.$$

In what follows, we consider two cases in establishing our result based on the value of the measure of noncompactness.

Case 1. Suppose that there exists an integer $N > 0$ such that $\phi(X_N) = 0$, then X_N is a relatively compact set and also $TX_N \subseteq X_N$. Since T is a continuous self operator on C, then by Theorem 1.3 we conclude that T has a fixed point.

Case 2. Suppose that there exists and integer $n \in \mathbb{N}$ such that $\phi(X_n) > 0$ for all $n \in \mathbb{N}$. From our hypothesis, there exists $X_0 \in C$ such that $\alpha(X_0) \geq 1$ and $\beta(X_0) \geq 1$. Since T is cyclic (α, β) admissible operator and $\alpha(X_0) \geq 1$, we have $\beta(X_1) = \beta(\overline{co}(TX_0)) = \beta(\phi(TX_0)) \geq 1$ and this implies that $\alpha(X_2) = \alpha(\overline{co}(TX_1)) = \alpha(\phi(TX_1)) \geq 1$, continuing this process, we have that

$$\alpha(X_{2k}) \text{ and } \beta(X_{2k+1}) \ \forall \ k \in k \in \mathbb{N} \cup \{0\}. \tag{2.14}$$

Using similar argument, we have that

$$\beta(X_{2k}) \text{ and } \alpha(X_{2k+1}) \ \forall \ k \in k \in \mathbb{N} \cup \{0\}. \tag{2.15}$$

It follow from (2.14) and (2.15) that $\alpha(X_n) \geq 1, \beta(X_n) \geq 1, \alpha(X_{n+1}) = \alpha(\phi(TX_n)) \geq 1$ and $\beta(X_{n+1}) = \beta(\phi(TX_n)) \geq 1$. It follows that

$$\begin{aligned} \tau + F(\phi(X_{n+1})) &\leq \tau + F(\alpha(X_n)\beta(X_n)\phi(X_{n+1})) \\ &= \tau + F(\alpha(X_n)\beta(X_n)\phi(\overline{co}(TX_n))) \\ &= \tau + F(\alpha(X_n)\beta(X_n)\phi(TX_n)) \\ &\leq F(\phi(X_n)). \end{aligned} \tag{2.16}$$

This implies that

$$F(\phi(X_{n+1})) \leq F(\phi(X_n)) - \tau,$$

inductively, we have that

$$F(\phi(X_{n+1})) \leq F(\phi(X_0)) - n\tau.$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$, we have that

$$\lim_{n\to\infty} F(\phi(X_{n+1})) = -\infty,$$

using (F_2), we have that

$$\lim_{n\to\infty} \phi(X_{n+1}) = 0.$$

Now, using the fact that $\{X_n\}$ is nested and from the (6) of Definition 1.5, we have that the set $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$, closed and convex subset of the set X_0. Thus, we have that X_∞ is an element of ker ϕ, which follows that X_∞ is compact. More so, we have that $X_\infty \subset X_n$ and $TX_n \subset X_n$ for all $n \in \mathbb{N}$. Therefore, we have that $T : X_\infty \to X_\infty$ is well defined , and for any bounded $Y \subset X_\infty$, we have that $T(Y) \subset X_\infty$ and $\overline{T(Y)}$ is a compact subsets of X_∞, implies that T is a compact operator. Therefore using Theorem 1.3, we have that T has a fixed point. □

Theorem 2.4. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible operator and Darbo type-IV-F-contraction such that there exists closed and convex $X_0 \subseteq C$ such that $TX_0 \subseteq X_0, \alpha(X_0) \geq 1$ and $\beta(X_0) \geq 1$, where ϕ is an arbitrary measure of noncompactness. Then T has a fixed point in set C.*

Proof. The prove follows similar approach as in Theorem 2.3, as such we omit it. □

Theorem 2.5. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible operator and Darbo type-V-F-contraction such that there exists closed and convex $X_0 \subseteq C$ such that $TX_0 \subseteq X_0$ and $\beta(X_0) \geq 1$, where ϕ is an arbitrary measure of noncompactness. Then T has a fixed point in set C.*

Proof. The prove follows similar approach as in Theorem 2.3, as such we omit it. □

Using Theorem 2.3, we establish the following results to the classical metric fixed point theory.

Corollary 2.4. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, cyclic (α, β) admissible operator satisfying the following:*

(1) *for all $X \in \mathcal{M}_E, \tau > 0$ and $x, y, \in X$, we have that*

$$\tau + F(\alpha(X)\beta(X)\|Tx - Ty\|) \leq F(\|x - y\|),$$

where $\alpha(X)\beta(X)\|Tx - Ty\| > 0$ and $\|x - y\| > 0$,

(2) *there exists closed and convex $X_0 \subseteq C$ such that $TX_0 \subseteq X_0, \alpha(X_0) \geq 1$ and $\beta(X_0) \geq 1$.*

Then T has a fixed point in set C.

Proof. let $\phi : \mathcal{M}_E \to \mathbb{R}^+$ be defined as

$$\phi(X) = \text{diam } X,$$

where diam $X = \sup\{\|x - y\| : x, y \in X\}$ stand for the diameter of X. It is well-known that ϕ is a measure of noncompactness in the space E. Therefore, we have that

$$\tau + F(\alpha(X)\beta(X) \sup_{x,y \in X} \|Tx - Ty\|) \leq F(\sup_{x,y \in X} \|x - y\|),$$

we have

$$\tau + F(\alpha(X)\beta(X)\phi(TX)) \leq F(\phi(X)),$$

so from Theorem 2.3, we obtain the desired result. □

Corollary 2.5. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $T : C \to C$ is a continuous, β admissible operator satisfying the following:*

(1) *for all $X \in \mathcal{M}_E, \tau > 0$ and $x, y, \in X$, we have that*

$$\tau + F(\beta(X)\|Tx - Ty\|) \leq F(\|x - y\|),$$

where $\beta(X)\|Tx - Ty\| > 0$ and $\|x - y\| > 0$,

(2) *there exists closed and convex $X_0 \subseteq C$ such that $TX_0 \subseteq X_0$ and $\alpha(X_0) \geq 1$.*

Then T has a fixed point in set C.

Theorem 2.6. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $S, T : C \to C$ are continuous operators such that*

(1) $ST = TS$;

(2) *for any $X \subset C, T\overline{co}(X) = \overline{co}(T(X))$ and*

$$\tau + F(\phi(S(X))) \leq F(\phi(T(X))), \tag{2.17}$$

where $\tau > 0, F \in \mathcal{F}$ and ϕ an arbitrary measure of noncompactness such that $\phi(S(X)) > 0$ and $phi(T(X)) > 0$.

Then,

(1) *the set $F(S) = \{x \in C : S(x) = x\}$ is nonempty and compact.*

(2) *T has a fixed point and the set $F(T) = \{x \in C : Tx = x\}$ is closed and invariant by S.*

(3) *if T is affine, then S and T have a common fixed point so the set $\{x \in C : Tx = Sx = x\}$ is compact.*

Proof.

(1) We define the sequence of the set $\{X_n\}$ as $X_n = \overline{co}(S(x_{n-1}))$. Using our hypothesis, we have

$$\begin{aligned} TX_1 &= T\overline{co}(S(X_0)) \subset \overline{co}(S(T(X_0))) \subset \overline{co}(S(X_0)) = X_1, \\ TX_2 &= T\overline{co}(S(X_1)) \subset \overline{co}(S(T(X_1))) \subset \overline{co}(S(X_1)) = X_2, \\ TX_3 &= T\overline{co}(S(X_2)) \subset \overline{co}(S(T(X_2))) \subset \overline{co}(S(X_2)) = X_3, \end{aligned}$$

continuing the process, we have that

$$TX_n = T\overline{co}(S(X_{n-1})) \subset \overline{co}(S(T(X_{n-1}))) \subset \overline{co}(S(X_{n-1})) = X_n.$$

In what follows, we consider two cases in establishing our result based on the value of the measure of noncompactness.

Case 1. Suppose that there exists an integer $N > 0$ such that $\phi(X_N) = 0$, then X_N is a relatively compact set. Since S is a continuous self operator on C, then by Theorem 1.3 we conclude that S has a fixed point.

Case 2. Suppose that there exists and integer $n \in \mathbb{N}$ such that $\phi(X_n) > 0$ for all $n \in \mathbb{N}$. We therefore have that

$$\begin{aligned} \tau + F(\phi(X_{n+1})) &= \tau + F(\phi(\overline{co}(S(X_n)))) \\ &= \tau + F(\phi(S(X_n))) \\ &\le F(\phi(T(X_n))). \\ &\le F(\phi(X_n)). \end{aligned} \tag{2.18}$$

This implies that

$$F(\phi(X_{n+1})) \le F(\phi(X_n)) - \tau,$$

inductively, we have that

$$F(\phi(X_{n+1})) \le F(\phi(X_0)) - n\tau.$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$, we have that

$$\lim_{n\to\infty} F(\phi(X_{n+1})) = -\infty,$$

using (F_2), we have that

$$\lim_{n\to\infty} \phi(X_{n+1}) = 0.$$

Now, using the fact that $\{X_n\}$ is nested and from (6) of Definition 1.5, we have that the set $X_\infty = \cap_{n=1}^{\infty} X_n \neq \emptyset$, closed and convex subset of the set X_0. In addition, $\phi(X_\infty) \leq \phi(X_n)$ for all $n \in \mathbb{N}$, which implies that $\phi(X_\infty) = 0$. Thus, we have that X_∞ is an element of ker ϕ, which follows that X_∞ is compact. More so, we have that $X_\infty \subset X_n$ and $S(X_n) \subset S(X_{n-1}) \subset \overline{co}(S(X_{n-1})) = X_n$ for all $n \in \mathbb{N}$. Therefore, we have that $S : X_\infty \to X_\infty$ is well defined , and for any bounded $Y \subset X_\infty$, we have that $S(Y) \subset X_\infty$ and $\overline{S(Y)}$ is a compact subsets of X_∞, implies that S is a compact operator. Therefore using Theorem 1.3, we have that S has a fixed point. Hence the set $F(S) = \{x \in C : Sx = x\}$ is closed. Also using the fact that $ST = TS$, we have that

$$T(Sx) = S(Tx) = Tx.$$

Hence $T(F(S)) \subset F(S)$ and since

$$\begin{aligned} F(\phi(F(S))) &= F(\phi(S(F(S)))) \\ &< \tau + F(\phi(S(F(S)))) \\ &\leq F(\phi(T(F(S)))) \\ &\leq F(\phi(F(S))), \end{aligned}$$

which is a contradiction. We therefore have it that $F(S)$ is a relatively compact set. Now, for any convergent sequence $\{x_n\} \subset F(S)$ such that $x_n \to x^*$, it is easy to see that $x^* \in C$ due to the fact that C is closed. In addition, using the continuity of S, we have that

$$Sx^* = S\lim_{n\to\infty} x_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} x_{n+1} = x^* \Rightarrow Sx^* = x^*,$$

it follows that $x^* \in F(S)$, as such $F(S)$ is a compact set.

(2) Using a similar approach as in (1), it is easy to see that T has a fixed point and that $F(T) = \{x \in C : Tx = x\}$ is closed. More so, Using the fact that $ST = TS$, we also have that Sx is the fixed point of T, thus $F(T)$ is invariant by S.

(3) Using the fact that T is affine, we have that $F(T)$ is convex. In addition, we have that $S(F(T)) \subset F(T), T(F(T)) \subset F(T)$ and for any $X \subset F(T)$, we have that

$$\tau + F(\phi(S(X))) \leq F(\phi(T(X))).$$

It is easy to see from the results of (1) that S has a fixed point in $F(T)$, therefore, we have that S and T have a common fixed point. Using the fact that S is continuous and by hypothesis (2) the set of common fixed point of S and T is compact.

□

Theorem 2.7. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and suppose that $S, T : C \to C$ are continuous operators such that*

(1) $ST = TS$;
(2) *for any $X \subset C, T\overline{co}(X) = \overline{co}(T(X)), T$ is affine and*

$$\tau + F(\phi(ST(X))) \leq F(\phi(T(X))), \tag{2.19}$$

where $\tau > 0, F \in \mathcal{F}$ and ϕ an arbitrary measure of noncompactness such that $\phi(S(X)) > 0$ and $phi(T(X)) > 0$.

Then S and T have a common fixed point.

Proof. Suppose that $U(x) = ST(x)$. It is easy to see that $U : C \to C$, $TU = UT$ and U is continuous. Using our hypothesis, we have that

$$\tau + F(\phi(U(X))) = \tau + F(\phi(ST(X))) \leq F(\phi(T(X))) \tag{2.20}$$

and by Theorem 2.6, we have that U and T have a common fixed point. More so, the set $K = \{x \in C : Ux = Tx = x\}$ is not empty and compact. Now, observe that for all $x \in K$, we have that

$$x = Ux = ST(x) = Sx.$$

Hence S and T have a common fixed point. □

3. Applications

In this section, we present an application of our fixed point result, that is Theorem 2.1 to the following Volterra type integral equation

$$x(t) = f(t, x(t)) + \int_0^{\eta(t)} G(t, s, x(s))ds,\ t \in \mathbb{R}^+, \tag{3.1}$$

where $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ $G : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions. It is well-known that the space of all bounded and continuous real-valued function $BC(\mathbb{R}^+)$ defined on $\mathbb{R}^+$ with the norm $\|x\| = \sup\{|x(t)| : t > 0\}$ is a Banach space. The measure of noncompactness ϕ on the family of all nonempty bounded subset of $BC(\mathbb{R}^+)$ say $\mathcal{M}BC(\mathbb{R}^+)$ is defined as follows:

$$\phi(X) = \omega_0(X) + \limsup_{t\to\infty} \text{diam } X(t), \tag{3.2}$$

where diam $X(t) = \sup\{|x(t) - y(t) : x, y \in X\}$ and $X(t) = \{x(t) : x \in X\}$. The modulus of continuity for any $x \in X$ and $\epsilon > 0$ is given by

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s) : t, s \in [0, T], |t - s| \leq \epsilon\}, \tag{3.3}$$

where

$$\omega^T(X, \epsilon) = \sup\{\omega^T(x, \epsilon) : x \in X\},$$
$$\omega_0^T(X) = \lim_{\epsilon\to 0} \omega^T(X, \epsilon),$$
$$\omega_0(X) = \lim_{T\to\infty} \omega_0^T(X).$$

To establish the existence of a solution of the functional integral equation (3.1), we can consider the operator T defined by

$$Tx(t) = f(t, x(t)) + \int_0^{\eta(t)} G(t, s, x(s))ds,\ t \in \mathbb{R}^+, x \in BC(\mathbb{R}^+). \tag{3.4}$$

The problem of existence of a solution (3.1) is equivalent to the problem of existence of a fixed point of (3.4).

Theorem 3.1. *Let T be an operator define by* (3.4) *on $BC(\mathbb{R}^+)$ and suppose the following conditions hold:*

(1) *the function $t \to |f(t, 0)|$ is bounded and a member of $BC(\mathbb{R}^+)$ that is*

$$M_1 = \sup\{|f(t, 0)| : t \in \mathbb{R}^+\} < \infty; \tag{3.5}$$

(2) *there exist* $\tau > 0$, $\alpha, \beta : BC(\mathbb{R}^+) \to \mathbb{R}^+$, *such that*

$$|f(t,u) - f(t,v)| \leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}}|u - v|, \tag{3.6}$$

where $\alpha(x)\beta(Tx) \geq 1$;

(3) *the function* $G : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is a continuous function and there exists positive solution say* r_0 *of the inequality*

$$M_1 + \frac{1}{\alpha(x)\beta(Tx)e^{\tau}}|r_0| + M_2 < r_0, \tag{3.7}$$

where M_2 *is a positive constant defined by the inequality*

$$M_2 = \sup\left\{\left|\int_0^{\eta(t)} G(t,s,x(s))ds\right|,\ t \in \mathbb{R}^+, x \in BC(\mathbb{R}^+)\right\} \tag{3.8}$$

and

$$\lim_{t\to\infty} \int_0^{\eta(t)} |G(t,s,x(s)) - G(t,sy(s))|ds = 0, \tag{3.9}$$

uniformly with respect to $x, y \in BC(\mathbb{R}^+)$.

Then T *has a fixed point in* $BC(\mathbb{R}^+)$.

Proof. To start with, we need to show that the operator T defined on $BC(r_0) = \{x \in BC(\mathbb{R}^+) : \|x\| \leq r_0\}$. Obviously Tx is continuous for any $x \in BC(\mathbb{R}^+)$. It follows that

$$\begin{aligned} |Tx(t)| &= \left|f(t,x(t)) + \int_0^{\eta(t)} G(t,s,x(s))ds\right| \\ &= \left|f(t,x(t)) - f(t,0) + f(t,0) + \int_0^{\eta(t)} G(t,s,x(s))ds\right| \\ &\leq |f(t,x(t)) - f(t,0)| + |f(t,0)| + \left|\int_0^{\eta(t)} G(t,s,x(s))ds\right| \\ &\leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}}|x(t)| + M_1 + \left|\int_0^{\eta(t)} G(t,s,x(s))ds\right|. \end{aligned} \tag{3.10}$$

Thus, from the above equation, using (3.7) and (3.8), we have that

$$\|Tx\|_\infty \leq M_1 + M_2 + \frac{1}{\alpha(x)\beta(Tx)e^{\tau}}\|x\|_\infty \leq r_0. \tag{3.11}$$

Hence T is well-defined and we have that $T(B(r_0)) \subset B(r_0)$. In what follows, we show that the operator $T : B(r_0) \to B(r_0)$ is continuous. Let $x, y \in B(r_0)$ and for any $\epsilon > 0$, $\|x - y\|_{B(r_0)} < \frac{\epsilon}{2}$, we then have that

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| f(t, x(t)) + \int_0^{\eta(t)} G(t, s, x(s))ds - [f(t, y(t)) \right. \\
&\qquad \left. + \int_0^{\eta(t)} G(t, s, y(s))ds] \right| \\
&\leq |f(t, x(t)) - f(t, y(t))| \\
&\qquad + \int_0^{\eta(t)} |G(t, s, x(s)) - G(t, s, y(s))| ds \\
&\leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}} |x(t) - y(t)| \\
&\qquad + \int_0^{\eta(t)} |G(t, s, x(s)) - G(t, s, y(s))| ds.
\end{aligned} \tag{3.12}$$

Now using (3.9) there exists $T > 0$ such that if $t > T$, we have that

$$\int_0^{\eta(t)} |G(t, s, x(s)) - G(t, s, y(s))| ds \leq \frac{\epsilon}{2}, \tag{3.13}$$

for any $x, y \in BC(\mathbb{R}^+)$. We now consider the following cases.

Case 1: Suppose that $t > T$, using (3.13) and (3.12), we have that

$$|Tx(t) - Ty(t)| \leq \frac{\epsilon}{2\alpha(x)\beta(Tx)e^{\tau}} + \frac{\epsilon}{2} < \epsilon. \tag{3.14}$$

Case 2: Suppose that $t \in [0, T]$, we have that

$$\begin{aligned}
|Tx(t) - Ty(t)| &\leq \frac{\epsilon}{2\alpha(x)\beta(Tx)e^{\tau}} + \int_0^{\eta(t)} |G(t, s, x(s)) - G(t, s, y(s))| ds \\
&< \frac{\epsilon}{2} + \eta_T \eta(\epsilon),
\end{aligned} \tag{3.15}$$

where $\eta_T = \sup\{\eta(t) : t \in [0, T]\}$ and

$$\begin{aligned}
\eta(\epsilon) = \sup\{&G(t, s, x(s)) - G(t, s, y(s)) : t \in [0, T], s \in [0, \beta_T], \\
&x, y \in [-r_0, r_0], \|x - y\|_{BC(\mathbb{R}^+)} < \epsilon\}.
\end{aligned}$$

Therefore, since G is continuous on $[0, T] \times [0, \eta_T] \times [-r_0, r_0]$, we have that $\eta(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, from (3.14)and (3.15) we have that T is a continuous function on $B(r_0)$.

Now, we establish that T has a fixed point in $B(r_0)$. To do this, suppose that for any $T, \epsilon > 0$, Y a nonempty subset of $B(r_0)$ and for any $s_1, s_2 \in [0, T]$ with $|s_1 - s_2| \leq \epsilon$, we obtain that

$$\begin{aligned} |Tx(s_1) - Tx(s_2)| &= \Big| f(s_1, x(s_1)) + \int_0^{\eta(s_1)} G(s_1, \zeta, x(\zeta)) d\zeta - [f(s_2, x(s_2)) \\ &\quad + \int_0^{\eta(s_2)} G(s_2, \zeta, x(\zeta)) d\zeta] \Big| \\ &\leq |f(s_1, x(s_1)) - f(s_2, x(s_2))| \\ &\quad + \Big| \int_0^{\eta(s_1)} G(s_1, \zeta, x(\zeta)) d\zeta - \int_0^{\eta(s_2)} G(s_2, \zeta, x(\zeta)) d\zeta] \Big| \\ &\leq |f(s_1, x(s_1)) - f(s_2, x(s_1))| + |f(s_2, x(s_1)) - f(s_2, x(s_2))| \\ &\quad + \int_0^{\eta(s_1)} |G(s_1, \zeta, x(\zeta)) - G(s_2, \zeta, x(\zeta))| d\zeta \\ &\quad + \int_{\eta(s_2)}^{\eta(s_1)} |G(s_2, \zeta, x(\zeta))| d\zeta \\ &\leq \omega_{r_0}^T(f, \epsilon) + \frac{\omega^T(x, \epsilon)}{\alpha(x)\beta(Tx)e^\tau} + \int_0^{\eta(s_1)} \omega_{r_0}^T(G, \epsilon) d\zeta \\ &\quad + \int_{\eta(s_2)}^{\eta(s_1)} |G(s_2, \zeta, x(\zeta))| d\zeta \\ &\leq \omega_{r_0}^T(f, \epsilon) + \frac{\omega^T(x, \epsilon)}{\alpha(x)\beta(Tx)e^\tau} + \eta_T \omega_{r_0}^T(G, \epsilon) + K\omega^T(G, \epsilon). \end{aligned} \tag{3.16}$$

Since x is an arbitrary element in Y, the above inequality becomes

$$\omega^T(T(Y), \epsilon) \leq \omega_{r_0}^T(f, \epsilon) + \frac{\omega^T(Y, \epsilon)}{\alpha(x)\beta(Tx)e^\tau} + \eta_T \omega_{r_0}^T(G, \epsilon) + K\omega^T(G, \epsilon). \tag{3.17}$$

More so, using the fact that f on $[0, T] \times [-r_0, r_0]$ and G on $[0, T] \times [0, T] \times [-r_0, r_0]$ is uniformly continuous, we have that $\omega_{r_0}^T(f, \epsilon) \to 0, \omega_{r_0}^T(G, \epsilon) \to 0$. Now, taking limit as $\epsilon \to 0$ in (3.17), we have that

$$\omega_0^T(T(Y)) \leq \frac{\omega_0^T(Y)}{\alpha(x)\beta(Tx)e^\tau} \Rightarrow \alpha(x)\beta(Tx)e^\tau \omega_0^T(T(Y)) \leq \omega_0^T(Y) \tag{3.18}$$

and taking limit as $T \to \infty$, we have that

$$\omega_0(T(Y)) \leq \frac{\omega_0(Y)}{\alpha(x)\beta(Tx)e^\tau} \Rightarrow \alpha(x)\beta(Tx)e^\tau \omega_0(T(Y)) \leq \omega_0(Y). \tag{3.19}$$

It thus follows, for any $x, y \in Y$ and $t \in \mathbb{R}^+$, we have that

$$\begin{aligned}
|Tx(t) - Ty(t)| &= \left| f(t, x(t)) + \int_0^{\eta(t)} G(t, s, x(s))ds - [f(t, y(t)) \right. \\
&\quad \left. + \int_0^{\eta(t)} G(t, s, y(s))ds] \right| \\
&\leq |f(t, x(t)) - f(t, y(t))| \\
&\quad + \left| \int_0^{\eta(t)} G(t, s, x(s))ds - \int_0^{\eta(t)} G(t, s, y(s))ds \right| \\
&\leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}} |x(t) - y(t)| \\
&\quad + \left| \int_0^{\eta(t)} G(t, s, x(s)) - G(t, s, y(s))ds \right|.
\end{aligned}$$

From the above inequality, using the concept of diameter of a set, we have that

$$\text{diam}(TX)(t) \leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}} \text{diam}(Y) + \left| \int_0^{\eta(t)} G(t, s, x(s)) - G(t, s, y(s))ds \right|,$$

taking limit as $t \to \infty$ in the above inequality and using (3.9), we have that

$$\limsup_{t\to\infty} \text{diam}(TX)(t) \leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}} \limsup_{t\to\infty} \text{diam}(Y)(t) \tag{3.20}$$

$$\Rightarrow \quad \alpha(x)\beta(Tx)e^{\tau} \limsup_{t\to\infty} \text{diam}(TX)(t) \leq \limsup_{t\to\infty} \text{diam}(Y)(t),$$

using (3.2), (3.20) and (3.19), we have that

$$\phi(TY) \leq \frac{1}{\alpha(x)\beta(Tx)e^{\tau}} \phi(Y) \quad \Rightarrow \quad e^{\tau}\alpha(x)\beta(Tx)\phi(TY) \leq \phi(Y).$$

Taking $\ln$ of both sides, we have that

$$\tau + \ln(\alpha(x)\beta(Tx)\phi(TY)) \leq \ln(\phi(Y))$$

by definition, taking $F(x) = \ln(x)$, we obtain

$$\tau + F(\alpha(x)\beta(Tx)\phi(TY)) \leq F(\phi(Y)).$$

It follows that T is a Darbo-type-I-F-contraction and conditions of Theorem 2.1 are satisfied, hence T has a fixed point in $B(r_0)$, which solves the integral equation (3.1) in $BC(\mathbb{R}^+)$. □

4. Acknowledgement

The research of the second author is wholly supported by the Centre of Excellence in Mathematical and Statistical Sciences (COE-MaSS) Postdoctoral Fellowship. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

References

[1] G. V. R. Babu, P. D. Sailaja: *A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces,* Thai J. Math. **9** (2011), 1–10.

[2] S. Banach: *Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales,* Fundamenta Mathematicae, **3** (1922), 133–181.

[3] J. Banas, K. Goebe l: *Measures of noncompactness in Banach spaces.* In: Lecture Notes in Pure and Applied Mathematics, **60**, Dekker, New York (1980).

[4] J. Banas, J. Mursaleen: *Sequence Spaces and Measures of Noncompactness with Applications to Differential and Integral Equations.* Springer, New Delhi (2014).

[5] S. Chandok, K. Tas, A. H. Ansari: *Some fixed point results for TAC-type contractive mappings* J. Funct. Space, 2016, ID 1907676.

[6] K. Goebel: *A coincidence theorem,* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., **16** (1968), 733–735.

[7] R. U. Hajji, G. Dhananjay, P. Kumam ; *Generalization of Darbo's fixed point theorem for new condensing operators with application to a functional integral equation,* Demonstr. Math., **52** (2019), 166–182.

[8] A. Hajji: *A generalization of Darbo's fixed point and common solutions of equations in Banach spaces,* Fixed Point Theory Appl. 2013, 1–9.

[9] G. Jungck: *Commuting mappings and fixed points,* Amer. Math. Monthly, **83**(4) (1976), 261–263.

[10] G. Jungck, N. Hussain: *Compatible maps and invariant approximations,* J. Math. Anal. Appl., **325**(2) (2007), 1003–1012.

[11] C. Kuratowski: *Sue les espaces complet,* Fund. Math., **15** (1930), 301–309.

[12] A. A. Mebawondu, C. Izuchukwu, H. A. Abass, O. T. Mewomo: *Some results on generalized mean nonexpansive mapping in complete metric space,* Bol. Soc. Paran. Mat., accepted.

[13] A. A. Mebawondu, O. T. Mewomo: *Application of fixed point results for modified generalized F-contracton mappings to solve boundary value problems,* PanAmerican Mathematical Journal, **20** (4) (2019), 45–68.

[14] A. A. MEBAWONDU, O. T. MEWOMO: *Some fixed pont results for a modified F-contraction via a new type of (α, β)-cyclic admissible mappings in metric space,* Bulletin University. Transilvania Brasov, Series III: Mathematics, informatics, Physics, **12** (61) (2019), 77–94.

[15] A. MEIR, E. KEELER: *A theorem on contraction mappings,* J. Math. Anal. Appl., **28** (1969), 326–329.

[16] M. MURSALEEM, R. ARAB: *On existence of solution of a class of quadratic-integral equations using contraction defined by simulation functions and measures of noncompactness,* Carpathian J. Math., **34**(3) (2018), 371–378.

[17] H. PIRI, P. KUMAM: *Some fixed point theorems concerning F-contraction in complete metric spaces,* Fixed Point Theory and Appl., **210** (2014).

[18] N. A. SECELEAN: *Iterated function systems consisting of F-contractions,* Fixed Point Theory and Appl., **2013** (2013), art.no.277.

[19] M. SGROI, C. VETRO: *Multi-valued F-contractions and the solution of certain functional and integral equations,* Filomat, **27** (2013), 1259–1268.

[20] M. VATH: *Volterra and Integral Equations of Vector Functions,* Chapman and Hall/CRC Pure and Applied Mathematics, Marcel Dekker, New York (2000).

[21] F. VETRO: *F-contractions of Hardy-Rogers type and application to multistage decision processes,* Nonlinear Anal. Model. Control, **21** (2016), 531–546.

[22] D. WARDOWSKI: *Fixed points of a new type of contractive mappings in complete metric spaces,* Fixed Point Theory and Appl., **2012** (2012), art.no. 94.

SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE,
UNIVERSITY OF KWAZULU-NATAL,
DURBAN, SOUTH AFRICA.
Email address: 216040405@stu.ukzn.ac.za, akutsah@gmail.com

SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE,
UNIVERSITY OF KWAZULU-NATAL,
DURBAN, SOUTH AFRICA.
DSI-NRF CENTRE OF EXCELLENCE IN MATHEMATICAL AND STATISTICAL SCIENCES (COE-MASS)
JOHANNESBURG, SOUTH AFRICA.
Email address: dele@aims.ac.za, akindeletempo1989@gmail.com

SCHOOL OF MATHEMATICS, STATISTICS AND COMPUTER SCIENCE,
UNIVERSITY OF KWAZULU-NATAL,
DURBAN, SOUTH AFRICA.
Email address: naraino@ukzn.ac.za