Nonlinear Functional Analysis and Applications
Vol. 26, No. 3 (2021), pp. 251-273
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.03.02
http://nfaa.kyungnam.ac.kr/journal-nfaa

OUTER APPROXIMATION METHOD FOR ZEROS OF SUM OF MONOTONE OPERATORS AND FIXED POINT PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we investigate a hybrid algorithm for finding zeros of the sum of maximal monotone operators and Lipschitz continuous monotone operators which is also a common fixed point problem for finite family of relatively quasi-nonexpansive mappings and split feasibility problem in uniformly convex real Banach spaces which are also uniformly smooth. The iterative algorithm employed in this paper is design in such a way that it does not require prior knowledge of operator norm. We prove a strong convergence result for approximating the solutions of the aforementioned problems and give applications of our main result to minimization problem and convexly constrained linear inverse problem. The result present in this paper extends and complements many related results in literature.

1. Introduction

The monotone inclusion problem is to find an element $x \in H$ such that

$$0 \in B(x),$$

where $B : H \to 2^H$ is a multi-valued operator and H is a real Hilbert space. This problem is very important in many areas such as convex optimization and monotone variational inequality problems. It is worth mentioning that every monotone operator on Hilbert spaces can be regularized into single-valued, nonexpansive, Lipschitz continuous monotone operator by means of Yosida approximation notion. The inclusion problem can also be defined in terms of sum of two monotone operators M and B, where one of these operators is α-inverse strongly monotone which is $\frac{1}{\alpha}$-Lipschitz continuous.

Let E be a real Banach space with $||.||$, dual space E^* and $\langle f, x\rangle$ the value of $f \in E^*$ at $x \in E$. Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $M : E \to E^*$ be a Lipschitz continuous monotone operator.

In this paper, we consider the following inclusion problem: find $x \in E$ such that

$$0 \in (M + B)x. \tag{1.1}$$

We denote by $(M + B)^{-1}(0)$ the solution set of (1.1).

Based on a series of studies in the past years, the splitting method has been known to be a popular method for solving (1.1). The splitting methods for linear equations were introduced by Peaceman and Rashford [23]. Extensions to nonlinear equations in Hilbert spaces were carried out by Lions and Mercier [18]. Since then, many authors have considered approximating solutions of variational inclusion (1.1) using this method, (see [2, 3, 12, 27] and the references contained in).

Recently, Zhang and Jiang [37] proved the following strong convergence theorem for approximating solutions for a common zero point of the sum of two monotone operators which is also a fixed point of a family of countable quasi-nonexpansive mapping in the framework of Hilbert spaces as follows:

Theorem 1.1. [37] *Let C be a nonempty, closed and convex subset of a real Hilbert space H, $A : C \to H$ be an α-inverse strongly monotone operator and B be a maximal monotone operator on H such that $Dom(B)$ is included in*

[0]Received August 27, 2020. Revised January 26, 2021. Accepted February 19, 2021.
[0]2010 Mathematics Subject Classification: 47H06, 47H09, 47J05, 47J25.
[0]Keywords: Maximal monotone operators, relatively quasi-nonexpansive mapping, hybrid iterative scheme, split feasibility problem, fixed point problem.
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C. Let $\{S_n\} : C \to C$ be a family of countable quasi-nonexpansive mappings which are uniformly closed. Assume that

$$\Gamma := F(S_n) \cap (A+B)^{-1}(0) \neq \emptyset.$$

Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1). Let $\{x_n\}$ be a sequence of C generated by

$$\begin{cases} x_1 \in C_1 = C, \text{ chosen arbitrarily;} \\ z_n = J_{r_n}(x_n - r_n A x_n); \\ y_n = \alpha_n z_n + (1-\alpha_n) S_n z_n; \\ C_{n+1} = \{z \in C_n : ||z_n - z|| \leq ||y_n - z|| \leq ||x_n - z||\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1; \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$, $r_n \leq 2\alpha$ and $\limsup_{n\to\infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$.

Very recently, Shehu [27] considered splitting method for finding zeros of the sum of maximal monotone operator and Lipschitz continuous monotone operator in Banach spaces. He proved weak and strong convergence results and give some applications of his main result.

The Split Feasibility Problem (SFP) introduced by Censor and Elfving [10] is to find an element

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{1.2}$$

where C and Q are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively, and $A : E_1 \to E_2$ is a bounded linear operator. The SFP arises from phase retrievals and in medical image reconstruction to mention a few. For more details on SFP, we refer readers to (see[11, 21, 22, 32, 19] and other references therein).

In 2018, Ma et. al. [19] introduced an iterative algorithm to solve the SFP (1.2) and fixed point problem of quasi-ϕ-nonexpansive mappings in Banach spaces. They proved a strong convergence result to a common solution of the aforementioned problems and apply their result to convexly constrained inverse problem and split null point problem.

Remark 1.2.

(1) We observe that the inclusion problem considered in [37] is quite different from the one in (1.1) in the sense that one of the operators is a Lipschitz continuous monotone operator.
(2) The iterative algorithm employed in this article does not require prior knowledge of operator norm as the ones employed in [19] requires prior knowledge of operator norm which gives difficulties in computation.

(3) We extend the result of [37] from Hilbert spaces to a more general Banach spaces.
(4) The split feasibility problem considered in this paper finds its applications in signal processing, image reconstruction and medical care.

Motivated by the works of Shehu [27], Zhang and Jiang [37] and Ma et al. [19], we introduced a shrinking iterative algorithm for finding zeros of the sum of maximal monotone operators and Lipschitz continuous monotone operators, which is also a common fixed point of a finite family of relatively quasi-nonexpansive mappings and split feasibility problem in Banach spaces. We prove a strong convergence result for approximating solutions of the aforementioned problems and give applications of our main result to minimization problem and convexly constrained linear inverse problem. The result present in this paper extends the result of Ma et al. [19], Zhang and Jiang [37] and other related results in literature.

2. Preliminaries

We give some definitions and important results which will be useful in establishing our main results. In the sequel, we denote strong and weak convergence by "$\rightarrow$" and "$\rightharpoonup$", respectively.

Throughout this paper, we assume C to be a nonempty, closed and convex subset of a real Banach space with norm $||\cdot||$, $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2,\ \forall x \in E\},$$

Consider the Lyapunov functional $\phi : E \times E \to [0, \infty)$ defined [4, 5] by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2,\ \forall x, y \in E.$$

Alber [4] introduced a generalized projection operator $\Pi_C : E \to C$ which is an analogue of the metric projection defined as follows:

$$\Pi_C(x) = argmin_{y\in C}\phi(y, x), x \in E.$$

That is, $\Pi_C(x) = \overline{x}$, where $\overline{x}$ is the unique solution to the minimization problem $\phi(\overline{x}, x) = \inf_{y\in C} \phi(y, x)$. In real Hilbert space, we observe that $\Pi_C(x) \equiv P_C(x)$ and $\phi(x, y) = ||x-y||^2$. It is obvious from the definition of the functional ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2. \tag{2.1}$$

Apart from inequality (2.1), the Lyapunov functional ϕ also satisfy the following inequalities:

(A_1) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$;
(A_2) $2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w)$;

(A_3) $\phi(x, y) \leq ||x||||Jx - Jy|| + ||y||||x - y||$.

Note: If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$, see [31].

We are also concerned with the functional $V : E \times E^* \to \mathbb{R}$ which is defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2 \tag{2.2}$$

for all $x \in E$ and $x^* \in E^*$. Observe that, $V(x, x^*) = \phi(x, J^{-1}x^*)$, if E is a reflexive, strictly convex and smooth Banach space and

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^* \rangle \tag{2.3}$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [26].

Let C be a closed and convex subset of E and $T : C \to C$ be a mapping. Then, $x \in C$ is called a fixed point of T, if $x = Tx$. We denote the set of fixed points of T by $F(T)$. A point $p \in C$ is called an asymptotic fixed point of T, if C contains a sequence $\{x_n\}$ such that $x_n \rightharpoonup p$ and $||x_n - Tx_n|| \to 0$ as $n \to \infty$. We denote by $\widehat{F(T)}$ the set of asymptotic fixed points of T. A mapping $T : C \to C$ is said to be relatively nonexpansive (see [20]) if the following conditions are satisfied:

$(L1)$ $F(T) \neq \emptyset$;
$(L2)$ $\phi(p, Tx) \leq \phi(p, x), \ \forall \ x \in C, \ p \in F(T)$;
$(L3)$ $F(T) = \widehat{Fix(T)}$.

If T satisfies $(L1)$ and $(L2)$, then T is said to be relatively quasi-nonexpansive. It is easy to see that the class of relative quasi- nonexpansive mappings contains the class of relatively nonexpansive mappings. Many authors have considered the relative quasi-nonexpansive mappings, (see [29, 33]).

Definition 2.1. Let $X \subset E$ be a nonempty subset. Then a mapping $A : X \to E^*$ is called

(i) γ-strongly monotone with modulus $\gamma > 0$ on X if

$$\langle Ax - Ay, x - y \rangle \geq \gamma ||x - y||^2, \ \forall \ x, y \in X;$$

(ii) monotone on X if

$$\langle Ax - Ay, x - y \rangle \geq 0, \ \forall \ x, y \in X;$$

(iii) Lipschitz continuous on X if there exists a constant $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y||, \ \forall \ x, y \in X.$$

Below is an example of a monotone operator in quantum mechanics.

Example 2.2. [27] Let the operator

$$Au := -b^2\Delta u + (f(x)+c)u(x) + u(x)\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}dy,$$

where $\Delta := \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^3$, b and c are constants, $f(x) = f_0(x) + f_1(x)$, where $f_0(x) \in L^\infty(\mathbb{R}^3)$ and $f_1(x) \in L^2(\mathbb{R}^3)$. Let $A := L + B$, where the operator L which is the schrödinger operator is the linear part of A and B defined by the last term. It is known that B is a monotone operator on $L^2(\mathbb{R}^3)$, (see p. 23 of [6]) which also implies that $A : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is also a monotone operator.

Definition 2.3. A multi-valued operator $B : E \to 2^{E^*}$ with domain $Dom(B) = \{x \in E : Bx \neq 0\}$ and the range $R(B) = \{Bx : x \in D(B)\}$ is said to be monotone if for $x, y \in D(B), a \in Bx, b \in By$, the following inequality holds:

$$\langle x - y, a - b\rangle \geq 0.$$

A monotone operator B is said to be maximal if its graph $Gra(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator.

If E is a strictly convex, reflexive and smooth Banach space and $B : E \to 2^{E^*}$ is a maximal monotone operator. Then, for any positive real number λ, we can define a nonexpansive single-valued operator $J_\lambda^B : E \to E$ by

$$J_\lambda^B(x) := (J + \lambda B)^{-1}J(x),\ x \in E.$$

This operator is called the resolvent of B for $\lambda > 0$. It is well known that $B^{-1}(0) = F(J_\lambda^B)$ for all $\lambda > 0$ and $B^{-1}(0)$ is a closed and convex subset of E.

For a real Banach space E, the modulus of convexity of E is the function $\delta_E : [0, 2] \to [0, 1]$ defined as

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon\}. \tag{2.4}$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\dfrac{||x + y||}{2} < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Also, E is p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p\epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(||x + ty|| - ||x - ty||) - 1 : ||x|| = ||y|| = 1\}. \tag{2.5}$$

E is said to be uniformly smooth if $\lim_{t\to 0}\dfrac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$. Then E is q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $p^{-1} + q^{-1} = 1$. It is also known that every q-uniformly smooth Banach space is uniformly smooth. It is also widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E.

The following are some important and useful properties of duality mapping J, for further details see [31]:

- For every $x \in E$, Jx is nonempty, closed, convex and bounded subset of E^*.
- If E is smooth or E^* is strictly convex, then J is single-valued. Also, If E is reflexive, then J is onto.
- If E is strictly convex, then J is strictly monotone, that is

$$\langle x - y, Jx - Jy\rangle > 0, x \neq y \quad \forall\, x, y \in E.$$

- If E is smooth, strictly convex and reflexive and $J^* : E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mappings on E and E^* respectively.
- If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^* = J^{-1}$ is also uniformly norm-to-norm continuous on bounded subsets of E^*.

We now state the following results which will be useful to prove our main result.

Lemma 2.4. [8] *Let $\frac{1}{p} + \frac{1}{q} = 1$, for $p, q > 1$. Then, the space E is q-uniformly smooth if and only if its dual space E^* is p-uniformly convex.*

Lemma 2.5. [34] *Let E be a 2-uniformly smooth Banach space with the best smoothness constant $k > 0$. Then, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, Jx\rangle + 2||ky||^2,\ \forall\, x, y \in E.$$

Lemma 2.6. [34] *Given a number $r > 0$, a real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x - y||);$$

for all $x, y \in E$ with $||x|| \leq r$ and $||y|| \leq r$ and $\lambda \in [0, 1]$.

Lemma 2.7. [5] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \ \forall \ x \in C, y \in E.$

(ii) *If $x \in E$ and $z \in C$, then $z = \Pi_C x$ if and only if*

$$\langle z - y, Jx - Jz \rangle \geq 0, \ \forall \ y \in C.$$

(iii) *For $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$.*

Lemma 2.8. [17] *Let E be a uniformly convex and smooth Banach space, and let $\{x_n\}, \{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$ and either of $\{x_n\}$ or $\{y_n\}$ is bounded. Then, $||x_n - y_n|| \to 0$.*

Lemma 2.9. [7] *Let E be a real uniformly convex, smooth Banach space. Then, the following identities hold:*

(i) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \ \forall \ x, y \in E;$

(ii) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy \rangle, \ \forall \ x, y \in E.$

Lemma 2.10. [31] *Let E be a smooth, strictly convex, and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conclusions hold:*

(i) $z = P_C x_1,$

(ii) $\langle z - y, J(x_1 - z) \rangle \geq 0, \ \forall \ y \in C.$

Lemma 2.11. [9] *Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $M : E \to E^*$ be a Lipschitz continuous monotone operator. Then the operator $M + B$ is a maximal monotone operator.*

Lemma 2.12. [27] *Let $B : E \to 2^{E^*}$ be a maximal monotone operator and $M : E \to E^*$ be an operator. Define an operator*

$$T_\lambda x := J_\lambda^B \circ J^{-1}(J - \lambda M), \ x \in E, \ \lambda > 0.$$

Then $F(T_\lambda) = (M + B)^{-1}(0)$.

3. Main result

We suppose that E is p-uniformly convex and uniformly smooth, which implies that it dual space E^* is q-uniformly smooth and uniformly convex. Throughout this section, we assume that E_1 is a 2-uniformly convex real Banach space which is also 2-uniformly smooth and E_2 is a smooth, strictly convex and reflective Banach space, E_1^* is a 2-uniformly smooth real Banach space which is also uniformly convex. Furthermore, we suppose that J_1 and J_2 represent the normalized duality mapping of E_1 and E_2 respectively and $J_1 = (J_1^*)^{-1}$, where J_1^* is the normalized duality mapping of E_1^*.

Theorem 3.1. *Let E_1 be 2-uniformly convex and 2-uniformly smooth real Banach spaces with the best smoothness constant $0 < k \leq \frac{1}{\sqrt{2}}, E_2$ be a smooth, strictly convex and reflective Banach space. Let $\{S_i\}_{i=1}^N : E_1 \to E_1$ be a finite family of closed relatively quasi-nonexpansive mapping, $A : E_1 \to E_2$ be a bounded linear operator with adjoint A^* and Q be a nonempty, closed and convex subset of E_2. Suppose that $B : E_1 \to 2^{E_1^*}$ is a maximal monotone operator and $M : E_1 \to E_1^*$ is monotone and L-Lipschitz continuous. Let*

$$\Gamma := \{\overline{x} \in \cap_{i=1}^N F(S_i) \cap (B + M)^{-1}(0) : A\overline{x} \in Q\} \neq \emptyset.$$

Let $x_1 \in E_1$ and $C_1 = E_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n + \gamma_n A^* J_2(P_Q - I)Ax_n); \\ y_n = J_{\lambda_n}^B \circ J_1^{-1}(J_1 w_n - \lambda_n M w_n); \\ u_n = J_1^{-1}(J_1 y_n - \lambda_n(My_n - Mx_n)); \\ t_n = J_1^{-1}[(1 - \alpha_n)J_1 u_n + \alpha_n J_1 S_i u_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, t_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1;\ n \geq 1; \end{cases} \tag{3.1}$$

where P_Q is the metric projection of E_2 onto Q and $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}. Suppose $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $(0, 1)$ such that

$$\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0,$$

and the step size γ_n is chosen in such a way that for $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||(P_Q - I)Ax_n||^2}{||A^* k^2 J_2(P_Q - I)Ax_n||^2} - \varepsilon\right),$$

for all $P_Q Ax_n \neq Ax_n, \gamma_n = \gamma$ otherwise (γ being any nonnegative real number) with $\{\lambda_n\}_{n=1}^\infty$ satisfying the following condition:

$$0 < d \leq \lambda_n \leq e < \frac{1}{\sqrt{2\mu}\rho L},$$

where μ is the 2-uniform convexity constant of E_1, ρ is the 2-uniform smoothness constant of E_1^*, and L is the Lipschitz constant of M. Then, $\{x_n\}$ converges strongly to a point $\overline{x} = \Pi_\Gamma x_1$.

Proof. We divide our proof into several steps:

Step 1: We prove using Theorem 3.1 that C_n is closed and convex for each $n \geq 1$. We obtain from Theorem 3.1 that $C_1 = E_1$, therefore C_1 is closed and convex. Now assume that C_n is closed and convex, then we have

$$\phi(v, t_n) \leq \phi(v, x_n).$$

It means that

$$||v||^2 - 2\langle v, J_1 t_n\rangle + ||t_n||^2 \leq ||v||^2 - 2\langle v, J_1 x_n\rangle + ||x_n||^2.$$

Hence we obtain that

$$2\langle v, J_1 x_n - J_1 t_n\rangle \leq ||x_n||^2 - ||t_n||^2. \tag{3.2}$$

We have from (3.2) that C_{n+1} is closed and convex subset of E_1. Therefore, $\Pi_{C_{n+1}}$ is well-defined.

Step 2: We show that $\Gamma \subseteq C_n$ for all $n \geq 1$. Let $x^* \in \Gamma \subseteq C_n$, for some $n \geq 1$. Then we have from (3.1) and Lemma 2.6 that

$$\begin{aligned}
\phi(x^*, t_n) &= \phi(x^*, J_1^{-1}((1-\alpha_n)J_1 u_n + \alpha_n J_1 S_i u_n)) \\
&= ||x^*||^2 - 2\langle x^*, (1-\alpha_n)J_1 u_n + \alpha_n J_1 S_i u_n\rangle \\
&\quad + ||(1-\alpha_n)J_1 u_n + \alpha_n J_1 S_i u_n||^2 \\
&\leq ||x^*||^2 - 2(1-\alpha_n)\langle x^*, J_1 u_n\rangle - 2\alpha_n\langle x^*, J_1 S_i u_n\rangle + (1-\alpha_n)||u_n||^2 \\
&\quad + \alpha_n||S_i u_n||^2 - \alpha_n(1-\alpha_n)g(||J_1 u_n - J_1 S_i u_n||) \\
&= (1-\alpha_n)\phi(x^*, u_n) + \alpha_n\phi(x^*, S_i u_n) \\
&\quad - \alpha_n(1-\alpha_n)g(||J_1 u_n - J_1 S_i u_n||) \\
&\leq (1-\alpha_n)\phi(x^*, u_n) + \alpha_n\phi(x^*, u_n)) \\
&\quad - \alpha_n(1-\alpha_n)g(||J_1 u_n - J_1 S_i u_n||) \\
&= \phi(x^*, u_n) - \alpha_n(1-\alpha_n)g(||J_1 u_n - J_1 S_i u_n||) \\
&\leq \phi(x^*, u_n) \\
&= \phi(x^*, J_1^{-1}(J_1 y_n - \lambda_n(My_n - Mw_n))) \\
&= ||x^*||^2 - 2\langle x^*, (J_1 y_n - \lambda_n(My_n - Mw_n))\rangle \\
&\quad + ||J_1^{-1}(J_1 y_n - \lambda_n(My_n - Mw_n))||^2 \\
&= ||x^*||^2 - 2\langle x^*, J_1 y_n - \lambda_n(My_n - Mw_n)\rangle \\
&\quad + ||(J_1 y_n - \lambda_n(My_n - Mw_n))||^2 \\
&= ||x^*||^2 - 2\langle x^*, J_1 y_n\rangle + 2\lambda_n\langle x^*, My_n - Mw_n\rangle \\
&\quad + ||J_1 y_n - \lambda_n(My_n - Mw_n)||^2.
\end{aligned} \tag{3.3}$$

But from Lemma 2.5, we have that

$$\begin{aligned}
||J_1 y_n - \lambda_n(My_n - Mw_n)||^2 &\leq ||J_1 y_n||^2 - 2\lambda_n\langle My_n - Mw_n, y_n\rangle \\
&\quad + 2k^2||\lambda_n(My_n - Mw_n)||^2.
\end{aligned} \tag{3.4}$$

On substituting (3.4) into (3.3), we obtain

$$\begin{aligned}\phi(x^*, u_n) &\leq ||x^*|| - 2\langle x^*, J_1 y_n\rangle + 2\lambda_n\langle x^*, My_n - Mw_n\rangle + ||J_1 y_n||^2 \\ &\quad - 2\lambda_n\langle My_n - Mw_n, y_n\rangle + 2k^2||\lambda_n(My_n - Mw_n)||^2 \\ &= ||x^*||^2 - 2\lambda_n\langle My_n - Mw_n, y_n - x^*\rangle - 2\langle x^*, J_1 y_n\rangle \\ &\quad + 2k^2||\lambda_n(My_n - Mw_n)||^2 + ||y_n||^2 \\ &= \phi(x^*, y_n) - 2\lambda_n\langle My_n - Mw_n, y_n - x^*\rangle \\ &\quad + 2k^2||\lambda_n(My_n - Mw_n)||^2. \end{aligned} \tag{3.5}$$

Observe from Lemma 2.9 (i) that

$$\begin{aligned}\phi(x^*, y_n) &= \phi(x^*, w_n) + \phi(w_n, y_n) + 2\langle x^* - w_n, J_1 w_n - J_1 y_n\rangle \\ &= \phi(x^*, w_n) + \phi(w_n, y_n) + 2\langle w_n - x^*, J_1 y_n - J_1 w_n\rangle. \end{aligned} \tag{3.6}$$

Also, using Lemma 2.9 (ii), we get

$$\phi(w_n, y_n) = 2\langle y_n - w_n, J_1 y_n - J_1 w_n\rangle - \phi(y_n, w_n). \tag{3.7}$$

On substituting (3.6) and (3.7) into (3.5), we obtain that

$$\begin{aligned}\phi(x^*, u_n) &= \phi(x^*, w_n) + \phi(w_n, y_n) + 2\langle w_n - x^*, J_1 y_n - J_1 w_n\rangle \\ &\quad - 2\lambda_n\langle My_n - Mw_n, y_n - x^*\rangle + 2k^2||\lambda_n(My_n - Mw_n)||^2 \\ &= \phi(x^*, w_n) + \phi(w_n, y_n) - 2\langle y_n - w_n, J_1 y_n - J_1 w_n\rangle \\ &\quad + 2\langle y_n - x^*, J_1 y_n - J_1 w_n\rangle - 2\lambda_n\langle My_n - Mw_n, y_n - x^*\rangle \\ &\quad + 2k^2||\lambda_n(My_n - Mw_n)||^2 \\ &\leq \phi(x^*, w_n) - \phi(y_n, w_n) + 2\langle y_n - x^*, J_1 y_n - J_1 w_n\rangle \\ &\quad - 2\lambda_n\langle My_n - Mw_n, y_n - x^*\rangle + 2k^2||\lambda_n(My_n - Mw_n)||^2 \\ &= \phi(x^*, w_n) - \phi(y_n, w_n) + 2k^2||\lambda_n(My_n - Mw_n)||^2 \\ &\quad - 2\langle J_1 w_n - J_1 y_n - \lambda_n(Mw_n - My_n), y_n - x^*\rangle. \end{aligned} \tag{3.8}$$

Using (3.1), it is clear that

$$J_1 w_n - \lambda_n M w_n \in (J_1 + \lambda_n B) y_n.$$

Also, using the fact that B is a maximal monotone, there exists $r_n \in By_n$ such that

$$J_1 w_n - \lambda_n M w_n = J_1 w_n + \lambda_n r_n.$$

Hence

$$r_n = \frac{1}{\lambda_n}(J_1 w_n - J_1 y_n - \lambda_n M w_n). \tag{3.9}$$

Since $M+B$ is maximal monotone and $My_n + r_n \in (M+B)y_n$, we obtain

$$\langle My_n + r_n, y_n - x^* \rangle \geq 0. \tag{3.10}$$

On substituting (3.9) into (3.10), we have

$$\langle J_1 w_n - J_1 y_n - \lambda_n (Mw_n - My_n), y_n - x^* \rangle \geq 0. \tag{3.11}$$

Using (3.11) in (3.8), we obtain that

$$\begin{aligned}
\phi(x^*, u_n) &\leq \phi(x^*, w_n) - \phi(y_n, w_n) + 2k^2 ||\lambda_n (My_n - Mw_n)||^2 \\
&\leq \phi(x^*, w_n) - \phi(y_n, w_n) + 2k^2 \lambda_n^2 L^2 \mu \phi(y_n, w_n) \\
&\leq \phi(x^*, w_n) - (1 - 2k^2 \lambda_n^2 L^2 \mu) \phi(y_n, w_n). \qquad (3.12)
\end{aligned}$$

By applying the condition on λ_n, (3.6), we have that

$$\begin{aligned}
\phi(x^*, u_n) &\leq \phi(x^*, y_n) \\
&\leq \phi(x^*, w_n) \\
&= \phi(x^*, J_1^{-1}(J_1 x_n + \gamma_n A^* J_2 (P_Q - I) A x_n)) \\
&= ||x^*||^2 - 2\langle x^*, J_1 x_n + \gamma_n A^* J_2 (P_Q - I) A x_n \rangle \\
&\quad + ||J_1 x_n + \gamma_n A^* J_2 (P_Q - I) A x_n||^2 \\
&\leq ||x^*||^2 - 2\langle x^*, J_1 x_n \rangle - 2\langle x^*, \gamma_n A^* J_2 (P_Q - I) A x_n \rangle \\
&\quad + ||x_n||^2 + 2\gamma_n \langle A x_n, J_2 (P_Q - I) A x_n \rangle + 2||k\gamma_n A^* J_2 (P_Q - I) A x_n||^2 \\
&\leq \phi(x^*, x_n) - 2\gamma_n \langle A x^* - A x_n, J_2 (P_Q - I) A x_n \rangle \\
&\quad + 2k^2 \gamma_n^2 ||A^* J_2 (P_Q - I) A x_n||^2. \qquad (3.13)
\end{aligned}$$

But, from Lemma 2.10, we have that

$$\begin{aligned}
&\langle A x^* - A x_n, J_2 (P_Q - I) A x_n \rangle \\
&\quad = \langle A x^* - P_Q A x_n + P_Q A x_n - A x_n, J_2 (P_Q - I) A x_n \rangle \\
&\quad = \langle A x^* - P_Q A x_n, J_2 (P_Q - I) A x_n \rangle \\
&\qquad + \langle (P_Q - I) A x_n, J_2 (P_Q - I) A x_n \rangle \\
&\quad = \langle A x^* - P_Q A x_n, J_2 (P_Q - I) A x_n \rangle + ||(P_Q - I) A x_n||^2 \\
&\quad \geq ||(P_Q - I) A x_n||^2. \qquad (3.14)
\end{aligned}$$

On substituting (3.14) into (3.13), we obtain that

$$\begin{aligned}
&\phi(x^*, w_n) \\
&\leq \phi(x^*, x_n) - 2\gamma_n ||(P_Q - I) A x_n||^2 + 2k^2 \gamma_n^2 ||A^* J_2 (P_Q - I) A x_n||^2 \\
&\leq \phi(x^*, x_n) - 2\gamma_n \big[||(P_Q - I) A x_n||^2 - k^2 \gamma_n ||A^* J_2 (P_Q - I) A x_n||^2 \big]. \qquad (3.15)
\end{aligned}$$

By applying the condition on γ_n in Theorem 3.1 we have that

$$\phi(x^*, u_n) \leq \phi(x^*, w_n) \leq \phi(x^*, x_n), \tag{3.16}$$

and hence

$$\phi(x^*, t_n) \leq \phi(x^*, x_n). \tag{3.17}$$

We therefore conclude that $x^* \in C_{n+1}$. This implies that $\Gamma \subseteq C_n$ for all $n \geq 1$. Hence, (3.1) is well-defined.

Step 3: We show that $\{x_n\}$ is a Cauchy sequence. Let $x^* \in \Gamma$, by using the definition of C_n, we have that $x_n = \Pi_{C_n} x_1$ for all $n \geq 1$. It follows from Lemma 2.7, we have that

$$\begin{aligned} \phi(x_n, x_1) &= \phi(\Pi_{C_n} x_1, x_1) \\ &\leq \phi(x^*, x_1) - \phi(x^*, \Pi_{C_n} x_1) \\ &\leq \phi(x^*, x_1), \ \forall\ n \geq 1. \end{aligned}$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded. More so, since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subseteq C_n$, we have that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \ \forall\ n \geq 1. \tag{3.18}$$

Therefore, $\{\phi(x_n, x_1)\}$ is non-decreasing. So, the limit also exists. From Lemma 2.7, we obtain that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1), \end{aligned} \tag{3.19}$$

thus, we have that

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.20}$$

Applying Lemma 2.8, we obtain that

$$\lim_{n \to \infty} ||x_{n+1} - x_n|| = 0. \tag{3.21}$$

Suppose $x_n = \Pi_{C_n} x_1 \subseteq C_m$, for some positive integers m, n with $m \leq n$, then applying Lemma 2.7 and using the same approach as in (3.19), we obtain that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \\ &\leq \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.22}$$

Since $\lim_{n\to\infty} \phi(x_n, x_1)$ exists, it follows from (3.22) and Lemma 2.8 that $\lim_{n\to\infty} ||x_n - x_m|| = 0$. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence.

Step 4: Let $\{x_n\}$ be a sequence generated by (3.1). Then we have the followings.

(i) $\lim_{n\to\infty} ||(P_Q - I)Ax_n|| = 0.$

(ii) $\lim_{n\to\infty} ||S_i u_n - u_n|| = 0.$

(iii) $\lim_{n\to\infty} ||y_n - w_n|| = 0.$

Since $x_{n+1} = \Pi_{C_{n+1}} \in C_{n+1} \subseteq C_n$, by the definition of C_{n+1}, (3.18) and (3.20), we have that

$$\phi(x_{n+1}, t_n) \le \phi(x_{n+1}, x_n) \to 0, \ (n \to \infty). \tag{3.23}$$

It follows from Lemma 2.8 that

$$\lim_{n\to\infty} ||x_{n+1} - t_n|| = 0. \tag{3.24}$$

Also, from (3.21) and (3.24), we have that

$$\lim_{n\to\infty} ||t_n - x_n|| = 0. \tag{3.25}$$

From (3.3), (3.12) and (3.15), we have that

$$\begin{aligned}\phi(x^*, t_n) \le \phi(x^*, x_n) &- \alpha_n(1-\alpha_n) g(||J_1 u_n - S_i u_n||) \\ &- 2\gamma_n \big[||(P_Q - I)Ax_n||^2 - k^2 \gamma_n ||A^* J_2 (P_Q - I)Ax_n||^2\big] \\ &- (1 - 2k^2 \lambda_n^2 L^2 \mu) \phi(y_n, w_n).\end{aligned} \tag{3.26}$$

It then follows that

$$\begin{aligned}&\alpha_n(1-\alpha_n) g(||J_1 u_n - S_i u_n||) \\ &\quad \le \phi(x^*, x_n) - \phi(x^*, t_n) \\ &\quad = ||x^*||^2 - 2\langle x^*, J_1 x_n\rangle + ||x_n||^2 - ||x^*||^2 + 2\langle x^*, J_1 t_n\rangle - ||t_n||^2 \\ &\quad = 2\langle x^*, J_1 t_n - J_1 x_n\rangle + ||x_n||^2 - ||t_n||^2 \\ &\quad \le 2||x^*|| \ ||J_1 t_n - J_1 x_n|| + ||x_n - t_n|| \ (||x_n|| + ||t_n||).\end{aligned} \tag{3.27}$$

Since E_1 is 2-uniformly convex and uniformly smooth Banach space, J_1 is uniformly continuous from norm-to-norm. Then, we obtain from (3.25) that

$$\lim_{n\to\infty} ||J_1 t_n - J_1 x_n|| = 0. \tag{3.28}$$

By applying the condition $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and (3.28) in (3.26), we obtain that

$$\lim_{n\to\infty} g(||J_1 u_n - J_1 S_i u_n||) = 0. \tag{3.29}$$

Using the property of g in Lemma 2.6, we have that

$$\lim_{n\to\infty} ||J_1 u_n - J_1 S_i u_n|| = 0. \tag{3.30}$$

Since J_1^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} ||u_n - S_i u_n|| = 0. \tag{3.31}$$

Also, from (3.26) and following the same approach in (3.27), we have that

$$\phi(y_n, w_n) = 0. \tag{3.32}$$

Applying Lemma 2.8 in (3.32), we have

$$\lim_{n\to\infty} ||y_n - w_n|| = 0. \tag{3.33}$$

From (3.26), condition on γ_n in (3.1) and following the approach in (3.27), we get that

$$\lim_{n\to\infty} ||(P_Q - I)Ax_n|| = 0. \tag{3.34}$$

From (3.1), we have

$$J_1(t_n) - J_1(u_n) = \alpha_n(J_1 S_i u_n - J_1(u_n)),$$

it implies that

$$||J_1 t_n - J_1 u_n|| = \alpha_n ||J_1 S_i u_n - J_1 u_n||.$$

Thus, from (3.30), we have that

$$\lim_{n\to\infty} ||J_1 t_n - J_1 u_n|| = 0. \tag{3.35}$$

Since E_1 is 2-uniformly convex and uniformly smooth real Banach space and J_1^{-1} is uniformly norm-to-norm weakly continuous on bounded subset of E_1^*, we obtain that

$$\lim_{n\to\infty} ||t_n - u_n|| = 0. \tag{3.36}$$

From (3.24) and (3.36), we get that

$$\lim_{n\to\infty} ||x_{n+1} - u_n|| = 0. \tag{3.37}$$

Also, from (3.25) and (3.36), we obtain that

$$\lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{3.38}$$

More so, from (3.37) and (3.38), we get

$$||u_{n+1} - u_n|| \le ||u_{n+1} - x_{n+1}|| + ||x_{n+1} - u_n|| \to 0 \text{ as } n \to \infty. \tag{3.39}$$

We also have that

$$\begin{aligned} ||u_n - S_{i+l} u_n|| \le{}& ||u_n - u_{n+l}|| + ||u_{n+l} - S_{i+l} u_{n+l}|| \\ &+ ||S_{i+l} u_{n+l} - S_{i+l} u_n||, \end{aligned}$$

for all $l = 1, 2, ..., N$. Using the assumption of S_l, we know that S_l is uniformly continuous. It then follows from (3.31) and (3.39) that

$$\lim_{n\to\infty} ||u_n - S_{i+l}u_n|| = 0, \ \forall\ l = 1, 2, ..., N. \tag{3.40}$$

Thus, we have

$$\lim_{n\to\infty} ||u_n - S_l u_n|| = 0, \ \forall\ l = 1, 2, ..., N. \tag{3.41}$$

Since S_l is closed for each $l = 1, 2, ..., N$ and $\{x_n\} \rightharpoonup \overline{x}$, we have that

$$\overline{x} \in \cap_{i=1}^{N} F(S_i).$$

Step 5: We show that $\overline{x} \in (M + B)^{-1}(0)$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\overline{x} \in E_1$ such that $x_{n_k} \rightharpoonup \overline{x}$. Suppose $(v, u) \in Gra(M + B)$. This implies that

$$J_1 u - Mv \in Bv.$$

More so, we obtain from (3.1) that

$$y_{n_k} = (J_1 + \lambda_{n_k} B)^{-1} J_1 \circ J_1^{-1}(J_1 w_{n_k} - \lambda_{n_k} M w_{n_k}),$$

which implies

$$(J_1 - \lambda_{n_k} M) w_{n_k} \in (J_1 + \lambda_{n_k} B) y_{n_k},$$

and thus

$$\frac{1}{\lambda_{n_k}}(J_1 w_{n_k} - J_1 y_{n_k} - \lambda_{n_k} M w_{n_k}) \in B y_{n_k}.$$

Using the fact that B is maximal monotone, we obtain

$$\langle v - y_{n_k}, J_1 u - Mv - \frac{1}{\lambda_{n_k}}(J_1 w_{n_k} - J_1 y_{n_k} - \lambda_{n_k} M w_{n_k})\rangle \geq 0.$$

Hence, we have

$$\begin{aligned}
\langle v - y_{n_k}, J_1 u\rangle &\geq \langle v - y_{n_k}, Mv + \frac{1}{\lambda_{n_k}}(J_1 w_{n_k} - J_1 y_{n_k} - \lambda_{n_k} M w_{n_k})\rangle \\
&= \langle v - y_{n_k}, Mv - M w_{n_k}\rangle \\
&\quad + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}}(J_1 w_{n_k} - J_1 y_{n_k})\rangle \\
&= \langle v - y_{n_k}, Mv - M y_{n_k}\rangle \\
&\quad + \langle v - y_{n_k}, M y_{n_k} - M w_{n_k}\rangle \\
&\quad + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}}(J_1 w_{n_k} - J_1 y_{n_k})\rangle \\
&\geq \langle v - y_{n_k}, M y_{n_k} - M w_{n_k}\rangle \\
&\quad + \langle v - y_{n_k}, \frac{1}{\lambda_{n_k}}(J_1 w_{n_k} - J_1 y_{n_k})\rangle.
\end{aligned}$$

Applying (3.33) and using the fact that M is Lipschitz continuous, we obtain that

$$\lim_{n\to\infty} ||M y_{n_k} - M w_{n_k}|| = 0.$$

More so, we obtain that $\langle v - \overline{x}, J_1 u\rangle \geq 0$. By the maximal monotonicity of $M + B$, we obtain that $0 \in (M + B)\overline{x}$. Therefore, we conclude that

$$\overline{x} \in (M + B)^{-1}(0).$$

Step 6: We show that $A\overline{x} \in Q$. Using Lemma 2.7, we have that

$$\begin{aligned}
||(I - P_Q)A\overline{x}||^2 &= \langle J_2(A\overline{x} - P_Q(A\overline{x})), A\overline{x} - P_Q(A\overline{x})\rangle \\
&= \langle J_2(A\overline{x} - P_Q(A\overline{x})), A\overline{x} - Ax_n + Ax_n \\
&\quad - P_Q(Ax_n) + P_Q(Ax_n) - P_Q(A\overline{x})\rangle \\
&= \langle J_2(A\overline{x} - P_Q(A\overline{x})), A\overline{x} - Ax_n\rangle \\
&\quad + \langle J_2(A\overline{x} - P_Q(A\overline{x})), Ax_n - P_Q(Ax_n)\rangle \\
&\quad + \langle J_2(A\overline{x} - P_Q(A\overline{x})), P_Q(Ax_n) - P_Q(A\overline{x})\rangle \\
&\leq \langle J_2(A\overline{x} - P_Q(A\overline{x})), A\overline{x} - Ax_n\rangle \\
&\quad + \langle J_2(A\overline{x} - P_Q(A\overline{x})), Ax_n - P_Q(Ax_n)\rangle. \qquad (3.42)
\end{aligned}$$

Using the fact that A is a bounded linear operator and (3.34), we have that

$$\lim_{n\to\infty} ||Ax_n - A\overline{x}|| = 0,$$

this implies that $||(I - P_Q)A\overline{x}|| = 0$, and hence

$$A\overline{x} \in Q,$$

Fom the Step 5 and Step 6, we conclude that $\overline{x} \in \Gamma$.

Step 7: We prove that $\{x_n\}$ converges strongly to $\overline{x}$. Let $\overline{x} = \Pi_\Gamma x_1$ and $\overline{x} \in \Gamma$. Then, from $x_n = \Pi_{C_n} x_1$ and $\overline{x} \in \Gamma \subseteq C_n$, we have

$$\phi(x_n, x_1) \leq \phi(\overline{x}, x_1), \tag{3.43}$$

which implies that

$$\begin{aligned}\phi(\overline{x}, x_1) &\leq \liminf_{n\to\infty} \phi(x_n, x_1) \\ &\leq \phi(\overline{x}, x_1).\end{aligned} \tag{3.44}$$

From the definition of $\overline{x} = \Pi_\Gamma x_1$, we have that $x^* = \overline{x}$. Hence

$$\liminf_{n\to\infty} x_n = \overline{x} = \Pi_C x_1.$$

We therefore conclude that $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$, where $\overline{x} = \Pi_\Gamma x_1$. This completes the proof. □

Corollary 3.2. *Suppose that E_1, and C be as defined in Theorem 3.1 and $S : E \to E$ be a nonexpansive mapping. Suppose that $B : E_1 \to 2^{E_1^*}$ is a maximal monotone operator and $M : E_1 \to E_1^*$ is monotone and L-Lipschitz continuous. Let*

$$\Gamma := \{\overline{x} \in C : \overline{x} \in F(S) \cap (B + M)^{-1}(0)\} \neq \emptyset.$$

Let $x_1 \in E$ and $C = E$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = J^B_{\lambda_n} \circ J_1^{-1}(J_1 x_n - \lambda_n M x_n); \\ u_n = J_1^{-1}(J_1 y_n - \lambda_n (M y_n - M x_n)); \\ t_n = J_1^{-1}[(1 - \alpha_n) J_1 u_n + \alpha_n J_1 S u_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, t_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1;\ n \geq 1; \end{cases} \tag{3.45}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E onto C_{n+1}. Suppose $\{\alpha_n\}_{n=1}^\infty$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$, with $\{\lambda_n\}_{n=1}^\infty$ satisfying the following condition:

$$0 < d \leq \lambda_n \leq e < \frac{1}{\sqrt{2\mu}\rho L},$$

where μ is the 2-uniform convexity constant of E, ρ is the 2-uniform smoothness constant of E^*, and L is the Lipschitz constant of M. Then the sequence $\{x_n\}$ converges strongly to a point $\overline{x} = \Pi_\Gamma x_1$.

Corollary 3.3. *Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $M : H_1 \to H$ be an α-inverse strongly monotone operator with $\alpha > 0$ and $B : H_1 \to 2^{H_1}$ be a maximal monotone operator on H_1 such that $Dom(B)$ is included in H_1. Let $\{S_n\}$:*

$H_1 \to H_1$ be a family of countable quasi-nonexpansive mappings which are uniformly closed, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that

$$\Gamma := \{\overline{x} \in C : \overline{x} \in F(S_n) \cap (M+B)^{-1}(0) \text{ and } A\overline{x} \in Q\} \neq \emptyset.$$

Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1). Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_1 \in C_1 = C, \text{ chosen arbitrarily;} \\ w_n = x_n + \gamma_n A^*(P_Q - I)Ax_n; \\ z_n = J_{r_n}(w_n - r_n A w_n); \\ y_n = \alpha_n z_n + (1-\alpha_n) S_n z_n; \\ C_{n+1} = \{z \in C_n : ||z_n - z|| \leq ||y_n - z|| \leq ||x_n - z||\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1; \end{cases}$$

where P_Q is the metric projection on H_2, $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$, $r_n \leq 2\alpha$, $\limsup_{n\to\infty} \alpha_n < 1$. and and the step size γ_n is chosen in such a way that for $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||(P_Q - I)Ax_n||^2}{||A^*(P_Q - I)Ax_n||^2} - \varepsilon\right),$$

for all $P_Q A x_n \neq A x_n, \gamma_n = \gamma$ otherwise (γ being any nonnegative real number). Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_1$.

Remark 3.4. We observe that Corollary 3.3 coincide with the main result of [37]. Just that a new problem (split feasibilty problem) was added to their iterative algorithm.

4. Applications

1. Convexly Constrained Linear Inverse Problem: Consider the convexly constrained linear inverse problem (see [13]) which is defined by

$$\begin{cases} Ax = b, \\ x \in C, \end{cases} \tag{4.1}$$

where H_1 and H_2 are two real Hilbert spaces, C, Q are closed convex subset of H_1 and H_2 respectively, $A : H_1 \to H_2$ is a bounded linear operator and $b \in Q$. We denote by Ω the solution set of (4.1).

Landweber introduced the following iterative algorithm to approximate the solution of (4.1) (see [14]) as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P_C(x_n - \gamma A^*(Ax_n - b)), \ n \geq 1, \end{cases}$$

where A^* is the adjoint of A, $0 < \gamma < 2\alpha$ with $\alpha = \frac{1}{||A||^2}$, then $\{x_n\}$ converges weakly to a solution of (4.1).

Now, we introduce an iterative algorithm to approximate (4.1) and prove the following strong convergence result.

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces. Let C, Q be a nonempty, closed and convex subsets of a real Hilbert space H_1 and H_2 respectively. Let $M : H_1 \to H_1$ be an α-inverse strongly monotone operator with $\alpha > 0$ and $B : H_1 \to 2^{H_1}$ be a maximal monotone operator on H_1 such that $Dom(B)$ is included in H_1. Let $\{S_n\} : H_1 \to H_1$ be a family of countable quasi-nonexpansive mappings which are uniformly closed, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Assume that*

$$\Gamma := \{\overline{x} \in C : \overline{x} \in F(S_n) \cap (M + B)^{-1}(0) \cap \Omega\} \neq \emptyset.$$

Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in [0,1). Let $\{x_n\}$ be a sequence in C generated by

$$\begin{cases} x_1 \in C_1 = C, \text{ chosen arbitrarily}; \\ w_n = x_n - \gamma A^*(Ax_n - b); \\ z_n = J_{r_n}(w_n - r_n M w_n); \\ y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n; \\ C_{n+1} = \{z \in C_n : ||z_n - z|| \leq ||y_n - z|| \leq ||x_n - z||\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \ n \geq 1; \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n\to\infty} r_n > 0$, $r_n \leq 2\alpha$, $\limsup_{n\to\infty} \alpha_n < 1$. and γ is a positive constant satisfying $0 < \gamma < \frac{1}{||A||^2}$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. Then the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_1$.

Proof. This is a a consequence of Corollary 3.3 by taking $P_Q(Ax_n) = b$. $\square$

2. Minimization Problem:

Definition 4.2. Let Q be a convex subset of a vector space X and $f : Q \to \mathbb{R} \cup \{+\infty\}$ be a map. Then,

(i) f is calleed convex if for each $\lambda \in [0, 1]$ and $x, y \in Q$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y);$$

(ii) f is called proper if there exists at least one $x \in Q$ such that

$$f(x) \neq +\infty;$$

(iii) f is called lower semi-continuous at $x_0 \in Q$ if

$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$

Let E be a real Banach space, we consider the following minimization of composite objective function of the type:

$$\min_{x \in E} f(x) + g(x), \tag{4.2}$$

where $f : E \to \mathbb{R} \cup \{+\infty\}$ is a proper, convex and lower semi-continuous function and $g : E \to \mathbb{R}$ is a convex function. In this setting, we assume that g is the smooth part of the functionals, while f is assumed to be non-smooth.

Precisely, we assume that g is Gâteaux-differentiable with derivative $\bigtriangledown g$ which is Lipschitz continuous with constant L. It is easy to see from Theorem 3.13 ([24]) that

$$\langle \bigtriangledown g(x) - \bigtriangledown g(y), x - y \rangle \geq \frac{1}{L} || \bigtriangledown g(x) - \bigtriangledown g(y)||^2, \ \forall \ x, y \in E.$$

Hence, $\bigtriangledown g$ is monotone and Lipschitz continuous. It can be seen that (4.2) is equivalent to finding $x \in E$ such that

$$0 \in \partial f(x) + \bigtriangledown g(x). \tag{4.3}$$

Problem (4.3) is a special case of (1.1) with $M := \bigtriangledown g$ and $B = \partial f$.

We denote by Ω the solution set of (4.3). Also, for fixed $r > 0$ and $z \in E$, it has been shown in [27] that the resolvent of ∂f which is denoted as $J_r^{\partial f}$ is defined as

$$J_r^{\partial f}(z) = \arg\min_{y \in E} \big\{ f(y) + \frac{1}{2r} ||y||^2 - \frac{1}{r} \langle y, Jz \rangle \big\}.$$

This can be re-written using (3.1) as

$$y_n = \arg\min_{y \in E} \big\{ f(y) + \frac{1}{2\lambda_n} ||y||^2 - \frac{1}{\lambda_n} \langle y, Jw_n - \lambda_n \bigtriangledown g(w_n) \rangle \big\}.$$

Theorem 4.3. *Let E_1, E_2, A, A^*, and Q be as defined in Theorem 3.1 and suppose that*

$$\Gamma := \{ \overline{x} \in C : \overline{x} \in \cap_{i=1}^N F(S_i) \cap \Omega : A\overline{x} \in Q \} \neq \emptyset.$$

Let $x_1 \in E_1$ and $C_1 = E_1$, and $\{x_n\}$ be a sequence generated by

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n + \gamma_n A^* J_2(P_Q - I)Ax_n); \\ y_n = argmin_{y\in E}\{f(y) + \frac{1}{2\lambda_n}||y||^2 - \frac{1}{\lambda_n}\langle y, J_1 w_n - \lambda_n \bigtriangledown g(w_n)\}; \\ u_n = J_1^{-1}(J_1 y_n - \lambda_n(\bigtriangledown g(y_n) - \bigtriangledown g(x_n)); \\ t_n = J_1^{-1}[(1-\alpha_n)J_1 u_n + \alpha_n J_1 S_i u_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, t_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1;\ n \geq 1; \end{cases} \tag{4.4}$$

where P_Q is the metric projection of E_2 onto Q and $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1}. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and the step size γ_n is chosen in such a way that for $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{||(P_Q - I)Ax_n||^2}{||A^* k^2 J_2(P_Q - I)Ax_n||^2} - \varepsilon\right),$$

for all $P_Q Ax_n \neq Ax_n, \gamma_n = \gamma$ otherwise (γ being any nonnegative real number) with $\{\lambda_n\}_{n=1}^{\infty}$ satisfying the following condition:

$$0 < d \leq \lambda_n \leq e < \frac{1}{\sqrt{2\mu}\rho L},$$

where μ is the 2-uniform convexity constant of E_1, ρ is the 2-uniform smoothness constant of E_1^*, and L is the Lipschitz constant of $\bigtriangledown g$. Then, $\{x_n\}$ converges strongly to a point $\overline{x} = \Pi_\Gamma x_1$.

Acknowledgments The first and second authors acknowledge with thanks the bursary and financial support from Department of Science and Technology and National Research Foundation, Republic of South Africa, Centre of Excellence in Mathematical and Statistical Sciences (DST-NRF COE-MaSS) Post Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS. And the fourth author was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

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