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On split generalized mixed equilibrium and fixed point problems of an infinite family of quasi-nonexpansive multi-valued mappings in real Hilbert spaces

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In this paper, we study split generalized mixed equilibrium problem and fixed point problem in real Hilbert spaces with a view to analyze an iterative method for approximating a common solution of split generalized mixed equilibrium problem and fixed point problem of an infinite family of a quasi-nonexpansive multi-valued mappings. The iterative algorithm introduced in this paper is designed in such a way that it does not require the knowledge of the operator norm. We state and prove a strong convergence result of the aforementioned problems and also give application of our main result to split variational inequality problem. Our result complements and extends some related results in literature.

Keywords: Split generalized mixed equilibrium problem; asymptotically nonspreading multi-valued mappings; quasi-nonexpansive mapping; iterative scheme; fixed point problem.

AMS Subject Classification: 47H06, 47H09, 47J05, 47J25

1. Introduction

Let H be a real Hilbert space with inner product and norm as $\langle .,. \rangle$ and $\|.\|$, respectively. Let C be a nonempty, closed and convex subset of H. A mapping $T : C \to C$

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is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C. \tag{1.1}$$

A point $p \in C$ is called a fixed point of T if $Tp = p$. However, if T is a multivalued mapping, that is, $T : C \to 2^C$, the fixed point of T is defined as $p \in Tp$. We denote by $F(T)$ the set of all fixed points of T.

A mapping $T : C \to C$ is called a quasi-nonexpansive mapping if

$$\|Tx - Tp\| \leq \|x - p\|, \quad \forall\, x \in C \text{ and } p \in F(T). \tag{1.2}$$

Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty, closed and bounded subsets, nonempty and compact subsets, nonempty and proximinal subset of C, respectively. The Hausdorff metric on $CB(C)$ is defined by

$$\mathcal{H}(A, B) := \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

for all $A, B \in CB(C)$ where $d(x, B) = \inf_{b\in B} \|x - b\|$.

$$\mathcal{H}(Tx, Ty) \leq L\|x - y\|, \quad x, y \in C. \tag{1.3}$$

In (1.3), if $L \in (0, 1)$, then T is called a contraction while T is called nonexpansive if $L = 1$.
In addition, T is said to be

(i) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, p) \leq \|x - y\|, \quad \forall\, x \in C, \quad p \in F(T).$$

Equilibrium problem was introduced by Blum and Oettli [4] and this problem has had a great impact and influence in the development of several branches of pure and applied sciences. Many authors have studied equilibrium problem and its generalization see ([4, 5, 9, 10, 12, 17] and papers therein). Let $F : C \times C \to \mathbb{R}$ be a nonlinear bifunction, then the Equilibrium Problem (EP) is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall\, x \in C. \tag{1.4}$$

The Generalized Mixed Equilibrium Problem (GMEP) includes fixed-point problems, variational inequality problems, optimization problems, Nash equilibria and the equilibrium problem as special cases.

Let $F : C \times C \to \mathbb{R}$ be a nonlinear bifuntion and $B : C \to H$ be a mapping. Let $\psi : C \to \mathbb{R}$ be a real-valued function, then the GMEP is to find $x^* \in C$ such that

$$F(x^*, x) + \langle Bx^*, y - x^*\rangle + \psi(y) - \psi(x^*) \geq 0, \quad \forall\, x \in C. \tag{1.5}$$

For solving GMEP (1.5), the bifunction F is said to satisfy the following conditions:

(L1) $F(x, x) = 0$ for all $x \in C$;
(L2) F is monotone, i.e $F(x, y) + F(y, x) \geq 0$, for all $x, y \in C$;

(L3) for each $x, y \in C, \lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$;
(L4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

Let H_1 and H_2 be real Hilbert spaces, C and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $F : C \times C \to \mathbb{R}, G : Q \times Q \to \mathbb{R}$ be bifunctions, $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}, \psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be functions and $B_1 : C \to H_1, B_2 : Q \to H_2$ be nonlinear mappings. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the Split Generalized Mixed Equilibrium Problem (SGMEP) is to find $x^* \in C$ such that

$$F(x^*, x) + \langle B_1x^*, x - x^*\rangle + \psi_1(x) - \psi_1(x^*) \geq 0, \quad \forall\, x \in C \tag{1.6}$$

and $y^* = Ax^* \in Q$ solves

$$G(y^*, y) + \langle B_2y^*, y - y^*\rangle + \psi_2(y) - \psi_2(y^*) \geq 0, \quad \forall\, y \in Q. \tag{1.7}$$

We denote the solution set of (1.6)–(1.7) by $\Gamma := \{x^* \in \mathrm{GMEP}(F, B_1, \psi_1) : Ax^* \in \mathrm{GMEP}(G, B_2, \psi_2)\}$.

Recently, Singthong and Suantai [19] introduced an iterative algorithm for finding a common element of the set of solutions of an equilibrium problem and fixed points set of a nonspreading-type mappings in Hilbert space. They stated and proved the following strong convergence theorem.

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(L1)-(L2)$. Let $T : C \to C$ be a k-strictly pseudo nonspreading mapping with a nonempty fixed point set and $Fix(T) \cap EP(F) \neq \emptyset$. Let $\beta \in [k, 1)$ and $T_\beta := \beta I + (1-\beta)T$. Let $\{\alpha_n\}_{n=1}^\infty \subset [0, 1)$ and $\{r_n\}_{n=1}^\infty \subset (0, \infty)$ satisfying the conditions*:

$$\lim \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty \quad \textit{and} \quad \liminf_{n\to\infty} r_n > 0.$$

Let $u \in C$ and $\{x_n\}_{n=1}^\infty, \{u_n\}_{n=1}^\infty$, $\{z_n\}_{n=1}^\infty$ be sequences in C generated from an arbitrary $x_1 \in C$ by

$$\begin{cases} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - z_n\rangle \geq 0, & \forall\, y \in C; \\ x_{n+1} = \alpha_n u + (1-\alpha_n)u_n, & n \geq 1; \\ z_n = \dfrac{1}{n}\displaystyle\sum_{m=0}^{n-1} T_\beta^m x_n, & n \geq 1. \end{cases}$$

Then $\{x_n\}_{n=1}^\infty, \{u_n\}_{n=1}^\infty$ and $\{z_n\}_{n=1}^\infty$ converge strongly to $P_{F(T)\cap EP(F)}u$, where $P_{F(T)\cap EP(F)} : H \to F(T) \cap EP(F)$ is the metric projection of H onto $F(T) \cap EP(F)$.

Also, Shehu *et al.* [18] introduced an iterative algorithm which does not require the knowledge of the operator norm and proves a strong convergence result for approximating a solution of split equality fixed point problem for quasi-nonexpansive mappings in a real Hilbert space.

Inspired by the works of Singthong and Suantai [19], Shehu *et al.* [18] and other related works in the literature, we introduce an iterative algorithm that does not require any prior knowledge of the operator norm to approximate a common solution of split generalized mixed equilibrium problem and fixed point problem for an infinite family of a multivalued quasi-nonexpansive mapping in a real Hilbert space. The result presented in this paper extends and complements the result of [13] and other recent results in literature.

2. Preliminaries

In this section, we state some well-known results which will be used in the sequel. Throughout this paper, we denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively.

Lemma 2.1 ([15]). *Let H be a real Hilbert space and $S : H \to H$ be a quasi-nonexpansive mapping. Set $S_\alpha = \alpha I + (1-\alpha)S$ for $\alpha \in [0,1)$. Then the following hold for all $x \in H$ and $p \in F(S)$:*

(I) $\|S_\alpha x - p\|^2 \leq \|x-p\|^2 - \alpha(1-\alpha)\|Sx - x\|^2$.
(II) $F(S_\alpha) = F(S)$.

Lemma 2.2. *Let H be a real Hilbert space. Then the following identities hold:*

(I) $2\langle x, y\rangle = \|x\|^2 + \|y\|^2 - \|x-y\|^2 = \|x+y\|^2 - \|x\|^2 - \|y\|^2, \forall\, x, y \in H$.
(II) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \forall\, x, y \in H$.

Lemma 2.3 ([23]). *Let C be a nonempty, closed and convex subset of a Hilbert space H. Let $B : C \to H$ be a continuous and monotone mapping, $\psi : C \to \mathbb{R}$ be a lower semicontinuous and convex function and $F : C \times C \to \mathbb{R}$ be a bifunction that satisfy $(L1) - (L4)$. For $r > 0$ and $x \in H$, then there exists $u \in C$ such that*

$$F(u,y) + \langle Bu, y-u\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0, \quad \forall\, y \in C. \quad (2.1)$$

Define a mapping $T_r^F : C \to C$ as follows:

$$T_r^F(x) = \Big\{u \in C : F(u,y) + \langle Bu, y-u\rangle + \psi(y) - \psi(u) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0, \forall\, y \in C\Big\}. \quad (2.2)$$

Then, the following assumptions hold:

(1) *T_r^F is single-valued,*
(2) *T_r^F is firmly nonexpansive, i.e., for any $x, y \in H$;*

$$\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x-y\rangle;$$

(3) $F(T_r^F) = \mathrm{GMEP}(F, B, \psi)$.
(4) $\mathrm{GMEP}(F, B, \psi)$ *is closed and convex.*

Lemma 2.4 ([16]). *Let C be a nonempty, closed and convex subset of a real uniformly Banach space E. Suppose $T : C \to E$ is a quasi-nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in C that converges weakly to x if $\{(I - T)x_n\}_{n\in\mathbb{N}}$ converges strongly to 0, then $x \in F(T)$.*

Lemma 2.5 ([6]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$. Then, for any given sequence $\{x_i\}_{i=1}^{\infty} \subset B_r(0)$ and for any given sequence $\{\lambda_i\}_{i=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \lambda_i = 1$, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to \mathbb{R}, \quad g(0) = 0,$$

such that for any positive integers i, j with $i < j$, the following inequality holds:

$$\left\|\sum_{i=1}^{\infty} \lambda_i x_i\right\|^2 = \sum_{i=1}^{\infty} \lambda_i\|x\|^2 - \lambda_i\lambda_j g(\|x_i - x_j\|).$$

Lemma 2.6 ([7]). *Let H be a real Hilbert space and $\{x_i\}_{i\in\mathbb{N}}$ be a bounded sequence in H. For $\delta_i \in (0, 1)$ such that $\sum_{i=1}^{\infty} \delta_i = 1$, the following identity holds*:

$$\left\|\sum_{i=1}^{\infty} \delta_i x_i\right\|^2 = \sum_{i=1}^{\infty} \delta_i\|x_i\|^2 - \sum_{1\leq i<j<\infty} \|x_i - x_j\|^2.$$

Lemma 2.7 ([22]). *Assume $\{a_n\}$ is a sequence of nonnegative real sequence such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \sigma_n\delta_n, \quad n > 0,$$

where $\{\sigma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that

(i) $\sum_{n=1}^{\infty} \sigma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\sigma_n\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main Result

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(L1) - (L4)$ and let G be upper semicontinuous in the first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $T_i : C \to K(C)$, for $i = 1, 2, 3, \ldots$ be a countable family of quasi-nonexpansive multi-valued mapping for*

$T_i p = \{p\}$ and $S : C \to C$ be a quasi nonexpansive mapping, respectively, such that $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ be a sequence in $(0, 1-a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{\|T_{r_n}^G - I)Aw_n\|^2}{\|A^*(T_{r_n}^G - I)Aw_n\|^2} - \varepsilon\right),$$

for $T_{r_n}^G Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then, the sequences $\{w_n\}, \{u_n\}$ and $\{x_n\}$ are generated iteratively for an arbitrary $x_0 \in C$ and a fixed point $u \in C$

$$\begin{cases} w_n = (1 - \alpha_n - t_n)x_n + \alpha_n S x_n + t_n u; \\ u_n = T_{r_n}^F (w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n); \\ x_{n+1} = \beta_{n,0} u_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \quad n \geq 1; \end{cases} \tag{3.1}$$

where $z_n^i \in T_i u_n$ and $r_n \subset (0, \infty)$ satisfy the following conditions:

(i) $\beta_{n,0}, \beta_{n,i} \in (0,1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=0}^{\infty} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} t_n = \infty$ *and* $\alpha_n + t_n < 1$;
(iv) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.

Then the sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converge strongly to an element in Ω.

Proof. Let $p \in \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \Gamma$, then from (3.1), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|T_{r_n}^F (w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n - p\|^2 \\ &\leq \|w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n - p\|^2 \\ &= \|w_n - p\|^2 + \gamma_n^2 \|A^*(T_{r_n}^G - I)Aw_n\|^2 \\ &\quad + 2\gamma_n \langle w_n - p, A^*(T_{r_n}^G - I)Aw_n\rangle. \end{aligned} \tag{3.2}$$

From Lemma 2.2, we have that

$$\begin{aligned} &2\gamma_n \langle w_n - p, A^*(T_{r_n}^G - I)Aw_n\rangle \\ &\quad = 2\gamma_n \langle A(w_n - p) + (T_{r_n}^G - I)Aw_n - (T_{r_n}^G - I)Aw_n, (T_{r_n}^G - I)Aw_n\rangle \\ &\quad = 2\gamma_n [\langle T_{r_n}^G Aw_n - Ap, (T_{r_n}^G - I)Aw_n\rangle - \|(T_{r_n}^G - I)Aw_n\|^2 \\ &\quad \leq 2\gamma_n \left[\frac{1}{2}\|(T_{r_n}^G - I)Aw_n\|^2 - \|(T_{r_n}^G - I)Aw_n\|^2\right] \\ &\quad = -\gamma_n \|(T_{r_n}^G - I)Aw_n\|^2. \end{aligned} \tag{3.3}$$

Therefore, from (3.2), (3.3) and condition $\gamma_n \in \big(\varepsilon, \frac{\|(T^G_{r_n}-I)Aw_n\|^2}{\|A^*(T^G_{r_n}-I)Aw_n\|^2} - \varepsilon\big)$, we have that

$$\begin{aligned}\|u_n - p\|^2 &\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(T^G_{r_n} - I)Aw_n\|^2 - \gamma_n\|(T^G_{r_n} - I)Aw_n\|^2 \\ &= \|w_n - p\|^2 + \gamma_n\big[\gamma_n\|A^*(T^G_{r_n} - I)Aw_n\|^2 - \|(T^G_{r_n} - I)Aw_n\|^2\big] \\ &= \|w_n - p\|^2 + \gamma_n^2\|A^*(T^G_{r_n} - I)Aw_n\|^2 - \gamma_n(\gamma_n + \epsilon)\|A^*(T^G_{r_n} - I)Aw_n\|^2 \\ &\leq \|w_n - p\|^2 - \gamma_n\epsilon\|A^*(T^G_{r_n} - I)Aw_n\|^2 \\ &\leq \|w_n - p\|^2. \end{aligned} \tag{3.4}$$

This implies that $\|u_n - p\| \leq \|w_n - p\|$.

Since T_i is an quasi-nonexpansive multi-valued mapping, then we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_{n,0}(u_n - p) + \sum_{i=1}^{\infty}(z_n^i - p)\|^2 \\ &\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|z_n^i - p\|^2 - \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\ &\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}d(z_n^i, p)^2 \\ &\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\mathcal{H}(T_i^n u_n, T_i^n p)^2 \\ &\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|u_n - p\|^2 \\ &= \|u_n - p\|^2 \\ &\leq \|w_n - p\|^2. \end{aligned} \tag{3.5}$$

From (3.1), the convexity of $\|.\|^2$, and the fact that $S : C \to C$ is a quasi-nonexpansive mapping, we have

$$\begin{aligned}\|w_n - p\|^2 &= \|(1 - \alpha_n - t_n)x_n + \alpha_n Sx_n + t_n u - p\|^2 \\ &= \|(1 - \alpha_n - t_n)(x_n - p) + \alpha_n(Sx_n - p) + t_n(u - p)\|^2 \\ &\leq (1 - \alpha_n - t_n)\|x_n - p\|^2 + \alpha_n\|Sx_n - p\|^2 + t_n\|u - p\|^2 \\ &\leq (1 - \alpha_n - t_n)\|x_n - p\|^2 + \alpha_n\|x_n - p\|^2 + t_n\|u - p\|^2 \\ &= (1 - t_n)\|x_n - p\|^2 + t_n\|u - p\|^2. \end{aligned} \tag{3.6}$$

Hence from (3.6), we have that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-t_n)\|x_n-p\|^2 + t_n\|u-p\|^2\\ &\leq \max[\|x_n-p\|^2, \|u-p\|^2]\\ &\vdots\\ &\leq \max[\|x_1-p\|^2, \|u-p\|^2].\end{aligned}$$

Therefore $\{x_n\}$ is bounded and consequently, we deduce that $\{u_n\}$ and $\{w_n\}$ are bounded.

Also from (3.2), (3.6), Lemma 2.5 and the fact that T_i is a quasi-nonexpansive multi-valued mapping, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \left\|\beta_{n,0}(u_n-p) + \sum_{i=1}^{\infty}(z_n^i-p)\right\|^2\\ &\leq \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|z_n^i-p\|^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &\leq \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}(d(T_iu_n,p))^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &\leq \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}(\mathcal{H}(T_iu_n,T_ip))^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &\leq \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|u_n-p\|^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &\leq \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|u_n-p\|^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &= \|u_n-p\|^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &\leq \|w_n-p\|^2 - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &= \|(1-\alpha_n)(x_n-p) + \alpha_n(Sx_n-p) + t_n(u-x_n)\|^2\\ &\quad - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\\ &= \|(1-\alpha_n)(x_n-p) + \alpha_n(Sx_n-p)\|^2 + t_n^2\|x_n-u\|^2\\ &\quad + 2t_n\langle u-x_n, (1-\alpha_n)(x_n-p) + \alpha_n(Sx_n-p)\rangle\\ &\quad - \beta_{n,0}\beta_{n,i}g(\|u_n-z_n^i\|)\end{aligned}$$

$$\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 + t_n^2\|x_n - u\|^2$$
$$+ 2t_n\langle u - x_n, (1-\alpha_n)(x_n - p) + \alpha_n(Sx_n - p)\rangle$$
$$- \beta_{n,0}\beta_{n,i}g(\|u_n - z_n^i\|). \tag{3.7}$$

We now consider two cases to establish strong convergence of $\{x_n\}$ to p.

Case 1. Assume that $\{\|x_n - p\|\}$ is monotonically nonincreasing sequence. Then $\{x_n\}$ is convergent and clearly

$$\lim_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_{n+1} - p\|. \tag{3.8}$$

Thus from (3.7), conditions (iii) and (iv) of (3.1), we have that

$$0 \leq \varepsilon^2 g(\|u_n - z_n^i\|) \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2$$
$$+ t_n^2\|x_n - u\|^2 + 2t_n\langle u - x_n, (1-\alpha_n)(x_n - p) + \alpha_n(Sx_n - p)\rangle \to 0, \quad \text{as } \infty.$$

Hence, we have that

$$\lim_{n\to\infty} g(\|u_n - z_n^i\|) = 0$$

and by property of g in Lemma 2.3, we have that $\lim_{n\to\infty}\|u_n - z_n^i\| = 0$. Since $\{u_n\}$ and $\{x_n\}$ are bounded, we have that

$$\lim_{n\to\infty} d(u_n, T_i u_n) \leq \lim_{n\to} \|u_n - z_n^i\| = 0. \tag{3.9}$$

Also, from (3.7), we have that

$$\alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2$$
$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + t_n^2 + 2t_n\langle u - x_n, (1-\alpha)(x_n - p) + \alpha_n(Sx_n - p),$$

hence from conditions (iii) and (iv) of (3.1), we have that

$$\|Sx_n - x_n\| = 0. \tag{3.10}$$

From (3.1), we have that

$$\|x_{n+1} - p\|^2 = \left\|\beta_{n,0}(u_n - p) + \sum_{i=1}^{\infty}\beta_{n,i}(z_n^i - p)\right\|^2$$
$$= \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty}\|z_n^i - p\|^2 - \sum_{i=1}^{\infty}\beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2$$
$$- \sum_{i,j=1, i\neq j}^{\infty}\beta_{n,i}\beta_{n,k}\|z_n^i - z_n^k\|^2$$

$$
\begin{aligned}
&\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty} \|z_n^i - p\|^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\
&\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty} \beta_{n,i}(d(T_i u_n, p))^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\
&\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty} \beta_{n,i}(\mathcal{H}(T_i u_n, T_i p))^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\
&\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty} \beta_{n,i}\|u_n - p\|^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\
&\leq \beta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^{\infty} \|u_n - p\|^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\
&= \|u_n - p\|^2 - \sum_{i=1}^{\infty} \beta_{n,0}\beta_{n,i}\|u_n - z_n^i\|^2 \\
&\leq \|w_n - p\|^2 + \gamma_n^2\|A^*(T_{r_n}^G - I)Aw_n\|^2 - \gamma_n\|(T_{r_n}^G - I)Aw_n\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 + t_n^2\|x_n - u\|^2 \\
&\quad + 2t_n\langle u - x_n, (1-\alpha)(x_n - p) \\
&\quad + \alpha_n(Sx_n - p)\rangle + \gamma_n^2\|A^*(T_{r_n}^G - I)Aw_n\|^2 - \gamma_n\|T_{r_n}^G - I)Aw_n\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 + t_n^2\|x_n - u\|^2 \\
&\quad + 2t_n\langle u - x_n, (1-\alpha)(x_n - p) + \alpha_n(Sx_n - p)\rangle \\
&\quad + \gamma_n[\gamma_n\|A^*(T_{r_n}^G - I)Aw_n\|^2 - \|(T_{r_n}^G - I)Aw_n\|^2]. \qquad (3.11)
\end{aligned}
$$

It then follows from condition (i) of (3.1) and the condition $\gamma_n \in \left(\varepsilon, \frac{\|T_{r_n}^G - I)Aw_n\|^2}{\|A^*(T_{r_n}^G - I)Aw_n\|^2} - \varepsilon\right)$, that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2 &- \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 + t_n^2\|x_n - u\|^2 \\
&+ 2t_n\langle u - x_n, (1-\alpha_n)(x_n - p) + \alpha_n(Sx_n - p)\rangle \\
&- \varepsilon\|A^*(T_{r_n}^G - I)Aw_n\|^2, \qquad (3.12)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\varepsilon\|A^*(T_{r_n}^G - I)Aw_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + t_n(\|x_n - u\|^2 \\
+ 2\langle u - x_n, (1-\alpha_n)(x_n - p) + \alpha_n(Sx_n - p)\rangle). \qquad (3.13)
\end{aligned}
$$

Hence,

$$
\lim_{n\to\infty} \|A^*(T_{r_n}^G - I)Aw_n\|^2 = 0. \qquad (3.14)
$$

From condition (i) of (3.1) and (3.14), we obtain that

$$\begin{aligned}\gamma_n\|(T^G_{r_n}-I)Aw_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\quad + t_n(\|x_n-u\|^2+2\langle u-x_n,(1-\alpha_n)(x_p)+\alpha_n(Sx_n-p)\rangle\\ &\quad + \gamma_n^2\|A^*(T^G_{r_n}-I)Aw_n\|^2).\end{aligned}$$

Hence,

$$\lim_{n\to\infty}\|(T^G_{r_n}-I)Aw_n\|^2=0. \tag{3.15}$$

Also,

$$\begin{aligned}\|u_n-p\|^2 &= \|T^F_{r_n}(w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p\|^2\\ &\le \langle u_n-p,w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p\rangle\\ &= \frac{1}{2}[\|u_n-p\|^2+\|w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p\|^2\\ &\quad -\|u_n-p-(w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p)\|^2]\\ &\le \Bigg[\frac{1}{2}\|u_n-p\|^2+\|w_n-p\|^2+\gamma_n(\gamma_n\|A^*(T^G_{r_n}-I)Aw_n\|^2\\ &\quad -\|(T^G_{r_n}-I)Aw_n\|^2)-\|u_n-p-(w_n+\gamma_nA^*(T^G_{r_n}-I)Aw_n-p)\|^2\Bigg]\\ &\le \frac{1}{2}[\|u_n-p\|^2+\|w_n-p\|^2-(\|u_n-w_n\|^2+\gamma_n^2\|A^*(T^G_{r_n}-I)Aw_n\|\\ &\quad -2\gamma_n\langle u-w_n,A^*(T^G_{r_n}-I)Aw_n\rangle)]\\ &\le \frac{1}{2}[\|u_n-p\|^2+\|w_n-p\|^2-\|u_n-w_n\|^2+\gamma_n^2\|A^*(T^G_{r_n}-I)Aw_n\|\\ &\quad +2\gamma_n\langle u-w_n,A^*(T^G_{r_n}-I)Aw_n\rangle)].\end{aligned} \tag{3.16}$$

That is,

$$\|u_n-p\|^2\le\|w_n-p\|^2-\|u_n-w_n\|^2+2\gamma_n\|u_n-w_n\|\;\|A^*(T^G_{r_n}-I)Aw_n\|. \tag{3.17}$$

It follows from condition (i) of (3.1) and (3.17) that

$$\|x_{n+1}-p\|^2\le\|w_n-p\|^2-\|u_n-w_n\|^2+2\gamma_n\|u_n-w_n\|\;\|A^*(T^G_{r_n}-I)Aw_n\|, \tag{3.18}$$

this implies that

$$\begin{aligned}\|u_n-w_n\|^2 &\le \|w_n-p\|^2-\|x_{n+1}-p\|^2+2\gamma_n\|u_n-w_n\|\;\|A^*(T^G_{r_n}-I)Aw_n\|\\ &= \|(1-\alpha_n-t_n)x_n+\alpha_nSx_n+t_nu-p\|^2-\|x_{n+1}-p\|^2\\ &\quad +2\gamma_n\|u_n-w_n\|\;\|A^*(T^G_{r_n}-I)Aw_n\|\end{aligned}$$

$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 - \alpha_n(1-\alpha_n)\|Sx_n - x_n\|^2 + t_n(\|x_n - u\|^2 + 2\langle u - x_n, (1-\alpha_n)(x_n - p) + \alpha_n(Sx_n - p)\rangle) + 2\gamma_n\|u_n - w_n\| \, \|A^*(T_{r_n}^G - I)Aw_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.19}$$

Also, from (3.10) and condition (iv) of (3.1) , we have that

$$\|w_n - x_n\| = \|(1-\alpha_n - t_n)(x_n - x_n) + \alpha_n(Sx_n - x_n) + t_n(u - x_n)\| \to 0, \quad \text{as } n \to \infty. \tag{3.20}$$

From (3.19) and (3.20), we have that

$$\|u_n - x_n\| \leq \|x_n - w_n\| + \|w_n - u_n\| \to 0, \quad \text{as } n \to \ \infty. \tag{3.21}$$

From (3.1) and (3.9), we have that

$$\begin{aligned} \|x_{n+1} - u_n\| &= \left\| \beta_{n,0}u_n + \sum_{i=1}^{\infty} \beta_{n,i} z_n^i - u - u_n \right\| \\ &= \left\| \beta_{n,0}(u_n - u_n) + \sum_{i=1}^{\infty} (z_n^i - u_n) \right\| \\ &\leq \sum_{i=1}^{\infty} \beta_{n,i} \|z_n^i - u_n\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.22}$$

Hence, from (3.21) and (3.22) we have that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - u_n\| + \|u_n - x_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.23}$$

It follows from (3.9), (3.21) and the demiclosedness principle that $\{u_n\}$ that converges weakly to $p \in \cap_{i=1}^{\infty} F(T_i) \cap F(S)$ and consequently $\{x_n\}$ and $\{w_n\}$ converges weakly to p.

Next, we show that $p \in \mathrm{GMEP}(F, B_1, \psi_1)$. Since $u_n = T_{r_n}^F(w_n + \gamma_n A^*(T_{r_n}^G - I)Aw_n)$, we have

$$F(u_n, u) + \langle B_1 u_n, u - u_n\rangle + \psi_1(u) - \psi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - w_n\rangle - \frac{1}{r_n}\langle u - u_n, \gamma_n A^*(T_{r_n}^G - I)Aw_n\rangle \geq 0, \quad \forall \, u \in C. \tag{3.24}$$

Hence, from the monotonicity of $\phi_1(x, u) := F(x, u) + \langle B_1 x, u - x\rangle + \psi(u) - \psi(x)$, we have

$$\frac{1}{r_n}\langle u - u_n, u_n - w_n\rangle - \frac{1}{r_n}\langle u - u_n, \gamma_n A^*(T_{r_n}^G - I)Aw_n) \geq F(u, u_n) + \langle B_1 u, u_n - u\rangle + \psi(u_n) - \psi(u), \tag{3.25}$$

which implies that

$$\frac{1}{r_{n_k}}\langle u-u_{n_k}, u_{n_k}-w_{n_k}\rangle - \frac{1}{r_{n_k}}\langle u-u_{n_k}, \gamma_n A^*(T^G_{r_{n_k}}-I)Aw_{n_k}\rangle$$
$$\geq F(u,u_{n_k}) + \langle B_1 u, u_{n_k}-u\rangle + \psi(u_{n_k}) - \psi(u). \tag{3.26}$$

Since $u_n \rightharpoonup p$, then it follows from (3.15), (3.18), (3.20), (3.21) and $(L4)$ that

$$F(u,p) + \langle B_1 u, p-u\rangle + \psi_1(p) - \psi_1(u) \leq 0, \quad \forall\, u \in C. \tag{3.27}$$

Now for fixed $u \in C$, let $u_t = tu + (1-t)p$ for all $t \in (0,1)$. This implies that $u_t \in C$. Thus from $(L1)$ and $(L4)$, we obtain

$$\begin{aligned} 0 &= F(u_t,u_t) + \langle B_1 u_t, u_t-u_t\rangle + \psi_1(u_t) - \psi_1(u_t) \\ &\leq t[F(u_t,u) + \langle B_1 u_t, u-u_t\rangle + \psi_1(u) - \psi_1(u_t)] \\ &\quad + (1-t)[F(u_t,p) + \langle B_1 u_t, p-u_t\rangle + \psi_1(p) - \psi_1(u_t)] \\ &\leq t[F(u_t,u) + \langle B_1 u_t, u-u_t\rangle + \psi_1(u) - \psi_1(u_t)]. \end{aligned} \tag{3.28}$$

Therefore

$$F(u_t,u) + \langle B_1 u_t, u-u_t\rangle + \psi_1(u) - \psi_1(u_t) \geq 0. \tag{3.29}$$

Furthermore, from (L4), we have that

$$F(p,u) + \langle B_1 p, u-p\rangle + \psi_1(u) - \psi_1(p) \geq 0, \tag{3.30}$$

which implies that $p \in \mathrm{GMEP}(F,B_1,\psi_1)$. Next we show that $Ap \in \mathrm{GMEP}(G,B_2,\psi_2)$. since $\{w_n\}$ is bounded and $w_n \rightharpoonup p$ and since A is a bounded linear operator, $Aw_{n_k} \to Ap$.

Set $v_{n_k} = Aw_{n_k} - T^G_{r_{n_k}}Aw_{n_k}$. Then we have that $Aw_{n_k} - v_{n_k} = T^G_{r_{n_k}}Aw_{n_k}$, and from (3.15), we have that

$$\lim_{n\to\infty} v_{n_k} = 0. \tag{3.31}$$

Therefore, from the definition of $T^G_{r_{n_k}}$, we observe that

$$G(Aw_{n_k}-v_{n_k},u) + \langle B_2 w_{n_k}-v_{n_k}, u-w_{n_k}+v_{n_k}\rangle + \psi_2(u) - \psi_2(u)(w_{n_k}-v_{n_k})$$
$$+ \frac{1}{r_{n_k}}\langle u-(w_{n_k}-v_{n_k}), (w-n_k-v_{n_k})-w_{n_k}\rangle \geq 0, \quad \forall\, u \in C. \tag{3.32}$$

Since G is upper semicontinuous in the first argument, then G is defined as

$$\phi_2(x,y) := G(x,u) + \langle B_2 x, u-x\rangle + \psi_2(u) - \psi_2(x). \tag{3.33}$$

Thus, taking lim sup of the inequality (3.32) as $k \to \infty$ and using the assumption $(L3)$, we have

$$G(Ap,u) + \langle B_2 Ap, u-Ap\rangle + \psi_2(u) - \psi_2(Ap) \geq 0, \quad \forall\, u \in C; \tag{3.34}$$

which implies that $Ap \in \mathrm{GMEP}(G,B_2,\psi_2)$. Hence $p \in \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \Gamma$. We now show that $\{x_n\}$ converges strongly to p.

From (3.1), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \left\|\beta_{n,0}u_n + \sum_{i=1}^{\infty}\beta_{n,i}z_n^i - p\right\|^2 \\
&= \left\|\beta_{n,0}(u_n-p) + \sum_{i=1}^{\infty}\beta_{n,i}(z_n^i - p)\right\|^2 \\
&\le \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|z_n^i-p\|^2 - \sum_{i=1}^{\infty}\beta_{n,0}\beta_{n,i}\|u_n-z_n^i\|^2 \\
&\quad - \sum_{i,j=1,i\neq j}\beta_{n,i}\beta_{n,j}\|z_n^i-z_n^j\|^2 \\
&\le \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}d(T_iu_n,p)^2 - \sum_{i=1}^{\infty}\beta_{n,0}\beta_{n,i}\|u_n-z_n^i\|^2 \\
&\le \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\mathcal{H}(T_iu_n,T_ip) \\
&\le \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|u_n-p\|^2 \\
&\le \beta_{n,0}\|u_n-p\|^2 + \sum_{i=1}^{\infty}\beta_{n,i}\|u_n-p\|^2 \\
&= \|u_n-p\|^2 \\
&\le \|w_n-p\|^2 \\
&= \|(1-\alpha_n-t_n)x_n + \alpha_nSx_n + t_nu - p\|^2 \\
&= \|(1-\alpha_n-t_n)(x_n-p) + \alpha_n(Sx_n-p) + t_n(u-p)\|^2 \\
&\le \|(1-\alpha_n-t_n)(x_n-p) + \alpha_n(Sx_n-p)\|^2 + 2t_n\langle x_{n+1}-p, u-p\rangle \\
&\le [(1-\alpha_n-t_n)\|x_n-p\| + \alpha_n\|Sx_n-p\|]^2 + 2t_n\langle x_{n+1}-p, u-p\rangle \\
&\le (1-t_n)\|x_n-p\|^2 + t_n(2\langle x_{n+1}-p, u-p\rangle).
\end{aligned}
$$

Therefore, using Lemma 2.7, we have that $\|x_n-p\| \to 0$, as $n\to\infty$.

Case 2. Assume that $\{\|x_n-p\|\}$ is a monotonically increasing sequence. Set $\Upsilon_n = \|x_n-p\|^2$ and let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping defined for all $n\ge n_0$ (for some large enough n_0) by

$$\tau(n) := \max\{k\in\mathbb{N} : k\le n, \Upsilon_k \le \Upsilon_{k+1}\}. \tag{3.35}$$

Obviously, $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$\Upsilon_\tau(n) \leq \Upsilon_{\tau(n)+1}, \quad \text{for } n \geq n_0.$$

Following the case argument as in Case 1, we can show that

$$\lim_{\tau(n)\to\infty} \|(T^G_{r_{\tau(n)}} - I)Aw_{\tau(n)}\| = 0.$$

By the same argument as (3.7) and (3.23) in Case 1, we conclude that $\{x_{\tau(n)}\}, \{y_{\tau(n)}\}$ and $\{w_{\tau(n)}\}$ converge weakly to $p \in F(T_i) \cap F(S) \cap \Gamma$. Now for all $n \geq n_0$,

$$\begin{aligned}
0 &\leq \|x_{\tau(n)+1} - p\|^2 - \|x_{\tau(n)} - p\|^2 \\
&\leq (1 - t_{\tau(n)})\|x_{\tau(n)} - p\|^2 + 2t_n\langle x_{\tau(n)+1} - p, u - p\rangle - \|x_{\tau(n)} - p\|^2 \\
&= t_{\tau(n)}[2t_{\tau(n)}\langle x_{\tau(n)+1} - p, u - p\rangle - \|x_{\tau(n)} - p\|^2].
\end{aligned}$$

Therefore,

$$\|x_{\tau(n)} - p\|^2 \leq t_{\tau(n)}[2t_{\tau(n)}\langle x_{\tau(n)+1} - p, u - p\rangle] \to 0, \quad \text{as } n \to \infty. \tag{3.36}$$

Hence,

$$\lim_{n\to\infty} \|x_{\tau(n)} - p\| = 0 \tag{3.37}$$

and

$$\lim_{n\to\infty} \Upsilon_{\tau(n)} = \lim_{n\to\infty} \Upsilon_{\tau(n)+1}. \tag{3.38}$$

Furthermore, for $n \geq n_0$, it is easily observed that $\Upsilon_{\tau(n)} \leq \Upsilon_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$) since $\Upsilon_j > \Upsilon_{j+1}$ for $\tau(n) + 1 \leq j \leq n$.

Consequently, for all $n \geq n_0$,

$$0 < \Upsilon_n \leq \max\{\Upsilon_\tau(n), \Upsilon_{\tau(n)+1}\} = \Upsilon_{\tau(n)+1}.$$

So $\lim_{n\to\infty} \Upsilon_n = 0$, that is $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converge strongly to $p \in F(T_i) \cap F(S) \cap \Gamma, \forall\ n > 0$. □

Remark 3.2. (i) If $B_1 = 0$ and $B_2 = 0$, then SGMEP (1.6)–(1.7) reduces to the following Split Mixed Equilibrium Problem (SMEP), find $x^* \in C$ such that

$$F(x^*, x) + \psi_1(x) - \psi - 1(x^*) \geq 0, \quad \forall\, x \in C \tag{3.39}$$

and $y^* = Ax^* \in Q$ solves

$$G(y^*, y) + \psi_2(y) - \psi(y^*) \geq 0, \quad \forall! y \in Q; \tag{3.40}$$

with solution set $\Theta_1 := \{x^* \in MEP(F, \psi_1) : Ax^* \in MP(G, \psi_2)\}$.

(ii) If $\psi_1 = \psi_2 = 0$ in SGMEP (1.6)–(1.7), then we have the following Split Generalized Equilibrium Problem, find $x^* \in C$ such that

$$F(x^*, x) + \langle B_1 x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C \tag{3.41}$$

and $y^* = Ax^* \in Q$ solves

$$G(y^*, y) + \langle B_2 y^*, y - y^* \rangle \geq 0, \quad \forall\, y \in Q; \tag{3.42}$$

with solution set $\Theta_2 := \{x^* \in GEP(F, B_1) : Ax^* \in GEP(G, B_2)\}$.

(iii) If $B_1 = B_2$ and $\psi_1 = \psi_2 = 0$, we have the following Split Equilibrium Problem studied by Kazmi and Rizvi [12] in 2013 which is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall\, x \in C \tag{3.43}$$

and $y^* = Ax^* \in Q$ solves

$$G(y^*, y) \geq 0, \quad \forall\, y \in Q, \tag{3.44}$$

with the solution set $\Theta_3 := \{x^* \in EP(F) : Ax^* \in EP(G)\}$.

(iv) If $F = G = 0$ and $\psi_1 = \psi_2 = 0$, then SGMEP (1.6)–(1.7) becomes Split Variational Inequality Problem (SVIP), which is to find $x^* \in C$ such that

$$\langle B_1 x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C \tag{3.45}$$

and $y^* = Ax^* \in Q$ solves

$$\langle B_2 y^*, y - y^* \rangle \geq 0, \quad \forall\, y \in Q. \tag{3.46}$$

We denote by SVIP(B_1, B_2) the solution set of (3.45)–(3.46).

Corollary 3.3. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets of H_1 and H_2 respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(L1) - (L4)$ and G is upper semicontinuous in the first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $T_i : C \to K(C)$, for $i = 1, 2, 3, \ldots$ be a countable family of quasi-nonexpansive multi-valued mappings for $T_i p = \{p\}$ and $S : C \to C$ be a quasi nonexpansive mapping, respectively, such that $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$, then the step size γ_n is chosen in such a way that for some $\varepsilon > 0$,*

$$\gamma_n \in \left(\varepsilon, \frac{\|T_{r_n}^G - I)Aw_n\|^2}{\|A^*(T_{r_n}^G - I)Aw_n\|^2} - \varepsilon \right),$$

for $T_{r_n}^G Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then, the sequences $\{w_n\}, \{u_n\}$ and $\{x_n\}$ are generated iteratively for an arbitrary

$x_0 \in C$ and a fixed point $u \in C$

$$\begin{cases} w_n = (1-\alpha_n)x_n + \alpha_n x_n, \\ u_n = T^F_{r_n}(w_n + \gamma_n A^*(T^G_{r_n} - I)Aw_n), \\ x_{n+1} = \beta_{n,0}u_n + \sum_{i=1}^{N} \beta_{n,i} z^i_n, \quad n \geq 1, \end{cases} \tag{3.47}$$

where $z^i_n \in T_i u_n$ and $r_n \subset (0,\infty)$ satisfy the following conditions:

(i) $\beta_{n,0}, \beta_{n,i} \in (0,1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=1}^{N} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequences $\{x_n\}, \{u_n\}$ and $\{w_n\}$ converge strongly to an element in Ω.

Corollary 3.4. *Let H_1 and H_2 be two real Hilbert spaces, let $C \subset H_1$ and $Q \subset H_2$ be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* the adjoint of A. Let $F : C \times C \to \mathbb{R}$ and $G : Q \times Q \to \mathbb{R}$ be bifunctions satisfying conditions $(L1) - (L4)$ and G be upper semicontinuous in the first argument. Let $B_1 : C \to H_1$ and $B_2 : Q \to H_2$ be continuous and monotone mappings, $\psi_1 : C \to \mathbb{R} \cup \{+\infty\}$ and $\psi_2 : Q \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and convex functions. Let $T_i : C \to K(C)$, for $i = 1, 2, 3, \ldots$ be a countable family of quasi-nonexpansive multi-valued mappings for $T_i p = \{p\}$ and $S : C \to C$ be a nonexpansive mapping, respectively, such that $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \Gamma \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ and $\{t_n\}$ be a sequence in $(0, 1-a)$ for some $a > 0$. Let the step size γ_n be chosen in such a way that for some $\varepsilon > 0$,*

$$\gamma_n \in \left(\varepsilon, \frac{\|T^G_{r_n} - I)Aw_n\|^2}{\|A^*(T^G_{r_n} - I)Aw_n\|^2} - \varepsilon\right),$$

for $T^G_{r_n}Aw_n \neq Aw_n$ and $\gamma_n = \gamma$, otherwise (γ being any nonnegative real number). Then, the sequences $\{w_n\}, \{u_n\}$ and $\{x_n\}$ are generated iteratively for an arbitrary $x_0 \in C$ and a fixed point $u \in C$

$$\begin{cases} w_n = (1-\alpha_n - t_n)x_n + \alpha_n Sx_n + t_n u; \\ u_n = T^F_{r_n}(w_n + \gamma_n A^*(T^G_{r_n} - I)Aw_n); \\ x_{n+1} = \beta_{n,0}u_n + \sum_{i=1}^{\infty} \beta_{n,i} z^i_n, \quad n \geq 1; \end{cases} \tag{3.48}$$

where $z^i_n \in T_i u_n$ and $r_n \subset (0,\infty)$ satisfy the following conditions:

(i) $\beta_{n,0}, \beta_{n,i} \in (0,1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=1}^{\infty} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} t_n = \infty$ *and* $\alpha_n + t_n < 1$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{w_n\}$ *converge strongly to an element in* $p \in \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \Gamma$.

4. Application to SVIP

Variational inequality problem is one of the most important problems in optimization as it is used in studying differential equations, minimax problems, and has certain applications to mechanics and economic theory. Also, the SVIP is known to include certain optimization problems such as split feasibility problem, split zero problem and split minimization problem as special cases, (see [2, 3, 11, 14]). We now state a result in solving $\mathrm{SVIP}(B_1, B_2)$ as discussed in (3.45)–(3.46).

Theorem 4.1. *Let* H_1 *and* H_2 *be two real Hilbert spaces, let* $C \subset H_1$ *and* $Q \subset H_2$ *be nonempty, closed and convex subsets of* H_1 *and* H_2, *respectively. Let* $A : H_1 \to H_2$ *be a bounded linear operator and* A^* *the adjoint of* A. *Let* $B_1 : C \to H_1$ *and* $B_2 : Q \to H_2$ *be continuous and monotone mappings, and* $T_i : C \to K(C)$, *for* $i = 1, 2, 3, \ldots$ *be a countable family of quasi-nonexpansive multi-valued mappings for* $T_i p = \{p\}$, *and* $S : C \to C$ *be a nonexpansive mapping, respectively. Assume* $\Omega := \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \mathrm{SVIP}(B_1, B_2) \neq \emptyset$ *with* $\{\alpha_n\}$ *being a sequence in* $(0, 1)$ *and* $\{t_n\}$ *being a sequence in* $(0, 1 - a)$ *for some* $a > 0$. *Let the step size* γ_n *be chosen in such a way that for some* $\varepsilon > 0$,

$$\gamma_n \in \left(\varepsilon, \frac{\|P_Q(I - r_n B_2) - I)Aw_n\|^2}{\|A^*(P_Q(I - r_n B_2) - I)Aw_n\|^2} - \varepsilon\right),$$

for $P_Q(I - r_n B_2)Aw_n \neq Aw_n$ *and* $\gamma_n = \gamma$, *otherwise* (γ *being any nonnegative real number*). *Then, the sequences* $\{w_n\}, \{u_n\}$ *and* $\{x_n\}$ *are generated iteratively for an arbitrary* $x_0 \in C$ *and a fixed point* $u \in C$

$$\begin{cases} w_n = (1 - \alpha_n - t_n)x_n + \alpha_n S x_n + t_n u, \\ u_n = P_C(I - r_n B_1)(w_n + \gamma_n A^*(P_Q(I - r_n B_2))Aw_n), \\ x_{n+1} = \beta_{n,0} u_n + \displaystyle\sum_{i=1}^{\infty} \beta_{n,i} z_n^i, \quad n \geq 1, \end{cases} \tag{4.1}$$

where $z_n^i \in T_i u_n$ *and* $r_n \subset (0, \infty)$ *satisfy the following conditions*:

(i) $\beta_{n,0}, \beta_{n,i} \in (0, 1), \liminf_{n\to\infty} \beta_{n,0}\beta_{n,i} > 0$ *such that* $\sum_{i=1}^{\infty} \beta_{n,i} = 1$;
(ii) $\liminf_{n\to\infty} r_n > 0$;
(iii) $\lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} t_n = \infty$ *and* $\alpha_n + t_n < 1$;
(iv) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then the sequences $\{x_n\}, \{u_n\}$ *and* $\{w_n\}$ *converge strongly to an element in* $p \in \cap_{i=1}^{\infty} F(T_i) \cap F(S) \cap \mathrm{SVIP}(B_1, B_2)$.

5. Conclusion

In this work, we study split generalized mixed equilibrium problem and fixed point problem in real Hilbert spaces and establish strong convergence result. The results obtained in this paper generalize, extend and unify the results established in [1, 8, 12, 20]. In addition, the iterative algorithm introduced in this paper is designed in such a way that it does not require the knowledge of the operator norm. Lastly, we present an application of our main result to split variational inequality problem.

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