Optimal Estimate of Hyperbolic Interface Problems on Quadratic Element

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Outline

Abstract

Approximate solution of a linear hyperbolic interface problem on quadratic finite element with time discretization based on modified centered difference scheme is proposed. With the assumption that the unknown is of low regularity across the interface, the stability of the scheme is established and convergence rate of optimal order in $L^2(\Omega)$ norm is proved. The theoretical result is confirmed with an example.

Introduction

Hyperbolic partial differential equations are encountered in various physical problems such as vibrating string, vibrating membrane, shallow water waves, etc (Rao, 2007; Leissa and Qatu, 2011; Debnath, 2012). Such differential equations become interface problems when the solution domain contains materials with different properties (Brekhovskikh, 1980; Deka and Sinha, 2012; Deka and Ahmed, 2013). Due to the nature of such problems, obtaining solutions with high accuracy may be difficult (Babuška, 1970; Chen and Zou, 1998). The study of interface problems by finite element method has gained the attention of researchers within the last three decades. For recent works on elliptic interface problems, see (Karátson and Korotov, 2009; Li et al., 2010; Payne et al., 2012; Lehrenfeld and Reusken, 2017) and Deka and Ahmed (2012); Faragó et al. (2012); Mu et al. (2013); Yang (2015); Song and Yang (2017); Adewole (2017); Gupta et al. (2017, 2018); Adewole; Adewole and Payne (2018) for parabolic interface problems.

Deka and Sinha (2012) considered the convergence of finite element solution of linear hyperbolic interface problem. With the assumption that the interface can be fitted exactly using interface elements with curved edges, the authors established convergence rates of optimal order for both semi and full discretizations. Time discretization was based on symmetric difference approximation around the nodal points. Deka and Ahmed (2013) also proved convergence rates of optimal order for finite element solution of an homogenous hyperbolic interface problem. Their time discretization was again based on symmetric difference approximation around the nodal points. Approximation properties of interpolation and projection operators were used in their analysis. Linear finite element with time discretization based on implicit scheme was presented for wave equation with discontinuous coefficient in (Deka, 2017).

Adewole (2018) investigated the error contributed by semi discretization to the finite element solution of linear hyperbolic interface problems. With low regularity assumption on the solution across the interface and with the assumption that the interface could be fitted exactly, almost optimal convergence rates in $L^2(\Omega)$ and $H^1(\Omega)$ norms were established. In this work, we consider the finite element discretization where the interface is approximated by polynomials of degree two which are joined end to end and propose a modified centered difference scheme for the time discretization. Under certain regularity assumptions on the input data, we show that optimal order of convergence in the $L^2(\Omega)$ norm is obtainable for full discretization.

Problem Definition and Discretization

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ be another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$, see Figure 3.1. We consider the hyperbolic interface problem

$$u_{tt} - \nabla \cdot (a(x,t)\nabla u) + b(x,t)u = f(x,t) \quad \text{in} \quad \Omega \times (0,T] \tag{3.1}$$

with initial and boundary conditions

$$\left\{ \begin{array}{rcll} u(x,0) & = & u_0(x) & \text{in } \Omega \\ u_t(x,0) & = & u_1(x) & \text{in } \Omega \\ u(x,t) & = & 0 & \text{on } \partial\Omega \times [0,T] \end{array} \right. \tag{3.2}$$

and interface conditions

$$\left\{ \begin{array}{rcl} [u]_\Gamma & = & 0 \\ \left[a(x,t)\dfrac{\partial u}{\partial n} \right]_\Gamma & = & g(x,t) \end{array} \right. \tag{3.3}$$

where $0 < T < \infty$, the symbol $[u]$ is a jump of a quantity u across the interface Γ and n is the unit outward normal to the boundary $\partial\Omega_1$. The interface conditions are defined as the difference of the limiting values from each side of the interface. The input functions $a(x,t)$, $b(x,t)$ and $f(x,t)$ are assumed continuous on each domain but discontinuous across the interface for $t \in [0,T]$.

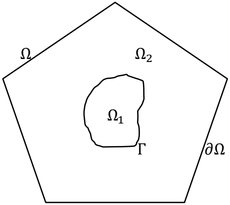

Figure: A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ

Notation and Preliminaries

In this work, we use the standard notations for Sobolev spaces and norms. For a given Banach space B, we define

$$W^{m,p}(0,T;B) = \begin{cases} u(t) \in B \text{ for a.e. } t \in (0,T) \quad \text{and} \ \sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt < 0 & \text{for } 1 \leq p < \infty \\ u(t) \in B \text{ for a.e. } t \in (0,T) \quad \text{and} \ \sum_{i=0}^{m} \text{ess} \sup_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B < 0 & \text{for } p = \infty \end{cases}$$

equipped with the norms

$$\|u\|_{W^{m,p}(0,T;B)} = \begin{cases} \left[\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt \right]^{1/p} & 1 \leq p < \infty \\ \sum_{i=0}^{m} \text{ess} \sup_{0 \leq t \leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B & p = \infty \end{cases}$$

We write $L^2(0,T;B) = W^{0,2}(0,T;B)$ and $H^m(0,T;B) = W^{m,2}(0,T;B)$. We use the definition and notation in (Adams, 1975) when m is negative or fractional.

For

$$f(x,t) = \begin{cases} f_1(x,t) & \text{in} \quad \Omega_1 \times (0,T] \\ f_2(x,t) & \text{in} \quad \Omega_2 \times (0,T] \end{cases}$$

with $f_1 \in H^m(\Omega_1)$ and $f_2 \in H^m(\Omega_2)$, $m = 2, 3$, we define

$$\|f\|_{H^m(\Omega)} = \|f_1\|_{H^m(\Omega_1)} + \|f_2\|_{H^m(\Omega_2)}, \quad t \in (0,T].$$

Throughout this presentation, C is a generic positive constant (which is independent of the mesh parameter h and the time step size k) and may take on different values at different occurrences.

We recall that for $u \in H^1(\Omega)$, the boundary value of u (ie $u_{|\partial\Omega}$) is defined on $H^{1/2}(\partial\Omega)$ the trace space of $H^1(\Omega)$. Similarly, the trace space on the interface Γ is $H^{1/2}(\Gamma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous and satisfies the embedding

$$\|z\|_{L^2(\partial\Omega)} \leq \|z\|_{H^{1/2}(\partial\Omega)} \leq c_0 \|z\|_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega) \tag{3.4}$$

See Atkinson and Han (2009) for more information on trace operator.

Regarding the regularity for the solution of the interface problem (3.1)−(3.3), we have the following result (Deka and Sinha, 2012):

Theorem 3.1

Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{1/2}(\Gamma))$, $u_0 \in H_0^1(\Omega)$ *and* $u_1 \in Y \cap L_0^2(\Omega)$. *Then problem* (3.1)−(3.3) *has a unique solution*

$$u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^2(0,T;Y)$$

where

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2) \ , \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2).$$

Full Discretization

$\mathfrak{T}_h$ denotes a partition of Ω into disjoint six-node triangles K (called elements) such that no vertex of any triangle lies on the interior or side of another triangle. Let $\mathfrak{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ;

$$\mathfrak{T}_h^\star = \{K \in \mathfrak{T}_h : K \cap \Gamma \neq \phi\}$$

$K \in \mathfrak{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K \in \mathfrak{T}_h^\star} K$. The triangulation $\mathfrak{T}_h$ of the domain Ω satisfies the following conditions

(i) $\bar{\Omega} = \bigcup_{K \in \mathfrak{T}_h} \bar{K}$

(ii) If $\bar{K}_1, \bar{K}_2 \in \mathfrak{T}_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.

(iii) The edge of $K \in \mathfrak{T}_h^\star$ that intersect with the interface is called the interface edge of K and is denoted by K_I, $\Gamma_h = \bigcup_{K \in \mathfrak{T}_h^\star} K_I$. Each $K \in \mathfrak{T}_h^\star$ has only one interface edge and Γ intersects K_I at minimum of three nodes. The interface edge of each interface element is a curve defined by a second degree polynomial through three nodes (see Figure 3.2). These three nodes are chosen such that two are at the vertices and one almost at the middle.

(iv) For each element $K \in \mathfrak{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles respectively. It is assumed that, for some fixed $h_0 > 0$, there exists two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \leq h \leq C_1 \bar{r}_K \quad \forall\, h \in (0, h_0)$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise polynomials of degree two through the nodes on each $K \in \mathfrak{T}_h$ and vanish on $\partial\Omega$. We ensure that each element has six nodes. This is necessary to ensure continuity across the element edges. All interface elements are constructed from the reference element (triangle in this case) through a geometric transformation. See Ciarlet (1978) for more information on the construction of isoparametric finite elements.

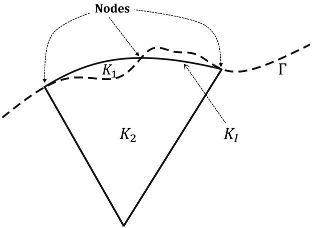

Figure: A typical interface element

The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x) \ ,$$

where each basis function ϕ_j, $(j = 1, 2, \dots, N_h)$ is a second degree polynomial satisfying

$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

For the approximation g_h of g, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathfrak{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be second degree polynomials corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h then

$$g_h(t,x) = \sum_{j=1}^{n_h} \beta_j(t)\psi_j(x) \ .$$

The finite element discretization determines the nature of the error obtained. With the assumption that the interface can be fitted exactly using interface elements with curved edges, optimal convergence rate is possible (see Sinha and Deka (2005) for example).

Convergence rate of optimal order is also obtainable when the approximation to the interface and the finite element spaces meet certain conditions (Li et al., 2010). Such conditions include $\Omega_h^\star \in S_\delta$ where S_δ is a $\delta-$neighbourhood of the interface, with $\delta = O(h^3)$. With this condition, interface elements need to divide more rapidly than non-interface elements to guarantee the optimal convergence rate. In both cases, we have, by standard finite element interpolation theory (Ciarlet, 1978; Thomée, 2006),

Lemma 3.2

For the Lagrange interpolation operator $\pi_h : C(\bar{\Omega}) \to S_h$, we have

$$\|u - \pi_h u\|_{H^m(\Omega)} \leq Ch^{3-m}(\|u\|_{H^3(\Omega_1)} + \|u\|_{H^3(\Omega_2)}) , \qquad m = 0, 1, 2.$$

The weak form of (3.1)−(3.3) is obtained as: given $u_0, u_1 \in H^3(\Omega)$, find $u : [0, T] \to H_0^1(\Omega) \cap H^3(\Omega)$ such that

$$(u_{tt}, v) + A(u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall\, v(t) \in H_0^1(\Omega), \text{ a.e. } t \in [0, T] \tag{3.5}$$

where

$$(\phi, \psi) = \int_\Omega \phi\psi \, dx \qquad A(\phi, \psi) = \int_\Omega [a(x,t)\nabla\phi \cdot \nabla\psi + b(x,t)\phi\psi] \, dx$$

$$\langle \phi, \psi \rangle_\Gamma = \int_\Gamma \phi\psi \, d\Gamma$$

For the time discretization, the interval $[0, T]$ is divided into M equally spaced subintervals:

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk$, $k = T/M$ being the time step. Let

$$u^n = u(t_n, x) \ , \quad f^n = f(t_n, x) \quad \text{and} \quad g^n = g(t_n, x) \ .$$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the centered difference quotient defined by

$$\partial w_n \ = \ \frac{w^{n+1} - 2w^n + w^{n-1}}{k^2} \ , \quad n = 1, 2, \ldots, (M-1)$$

The fully discrete finite element approximation to (3.3) is defined as follows: Given U_h^0 and U_h^1, find $U_h^n \in S_h$, such that $\forall \ v_h \in S_h$,

$$4(\partial U_h^n, v_h)_h + A_h(U_h^{n+1} + 2U_h^n + U_h^{n-1}, v_h) = (f^{n+1} + 2f^n + f^{n-1}, v_h)_h + \langle g_h^{n+1} + 2g_h^n + g_h^{n-1}, v^h\rangle_{\Gamma_h}$$
$$\forall\ v_h \in S_h, \quad n = 1, 2, \ldots, (M-1). \quad (3.6)$$

Where $A_h(\phi, \psi)$, $(\xi, \phi)_h$ and $\langle \phi, \psi\rangle_{\Gamma_h}$ are defined as

$$A_h(\phi, \psi) = \sum_{K \in \mathfrak{T}_h} \int_K [a\nabla\phi \cdot \nabla\psi + b\phi\psi]\ dx\ , \quad (\xi, \phi)_h = \sum_{K \in \mathfrak{T}_h} \int_K \xi\phi\ dx,$$

$$\langle \phi, \psi\rangle_{\Gamma_h} = \sum_{K \in \mathfrak{T}_h^\star} \int_{K_I} \phi\psi\ ds$$

and $s \in K_I$.

The analysis of this work is done with the assumption that $\dfrac{\partial^i u}{\partial t^i}$ exists (for $i = 1, \ldots, 4$). It can be shown using Taylor expansion that

$$\|u^n - 2u^{n-1} + u^{n-2}\|_{L^2(\Omega)} \le k^2\lambda_0 \quad (3.7)$$

Let $P_h : H^3(\Omega) \cap H_0^1(\Omega) \to S_h$ be the elliptic projection of the exact solution u in S_h defined by

$$A_h(P_h\nu, \phi) = A(\nu, \phi) \quad \forall\ \phi \in S_h,\ t \in [0, T] \quad (3.8)$$

For this projection, we have

Lemma 3.3

Let $a_i(x,t)$, $b_i(x,t)$ be continuous on $\Omega_i \times (0,T]$, $i=1,2$ and $\|u_{ttt}\| < \infty$. Assume that $u \in H^3(\Omega) \cap H_0^1(\Omega)$ and let $P_h u$ be defined as in (3.8), then

$$\|(P_h u - u)_{tt}\|_{L^2(\Omega)} + h\|(P_h u - u)_{tt}\|_{H^1(\Omega)} \leq Ch^3 \left(\|u\|_{H^3(\Omega)} + \|u_t\|_{H^3(\Omega)} + \|u_{tt}\|_{H^3(\Omega)} \right)$$

Proof.

It follows from Lemma 3.2 and an argument similar to (Adewole, 2017, Lemma 3.5). □

Main Results

In this section, we establish the stability of the proposed scheme and prove error estimates of optimal order in $L^2(\Omega)$ norm. For this presentation, we consider the case $\Gamma = \Gamma_h$.

Lemma 4.1

Let $a_i(x,t)$, $b_i(x,t)$ and $f_i(x,t)$ be continuous on $\Omega_i \times (0,T]$, $i = 1,2$. Suppose $g(x,t) \in L^2(0,T; H^{1/2}(\Gamma))$ and $k \in (0,1)$, there exists a constant C independent of k and h such that

$$\|U_h^n\|_{L^2(\Omega)}^2 \leq C \left[\|U_h^0\|_{L^2(\Omega)}^2 + \|U_h^1\|_{L^2(\Omega)}^2 + \int_{t_2}^{t_n} \left(\|f\|_{L^2(\Omega)}^2 + \|g\|_{H^{1/2}(\Gamma)}^2 \right) dt \right] \tag{4.1}$$

for $n = 2, \ldots, M$.

Proof.

Let $v_h = U_h^{n+1} + 2U_h^n + U_h^{n-1}$ in (3.6) and use (3.4) and (3.7)

$$\frac{3}{k^2}\|U_h^{n+1}\|^2_{L^2(\Omega)} + \mu\|U_h^{n+1} + 2U_h^n + U_h^{n-1}\|^2_{H^1(\Omega)}$$
$$\leq \frac{16}{k^2}\left(\|U_h^n\|^2_{L^2(\Omega)} + \|U_h^{n-1}\|^2_{L^2(\Omega)}\right)$$
$$+ \|f^{n+1} + 2f^n + f^{n-1}\|_{L^2(\Omega)}\|U_h^{n+1} + 2U_h^n + U_h^{n-1}\|_{L^2(\Omega)}$$
$$+c_0\|g_h^{n+1} + 2g_h^n + g_h^{n-1}\|_{H^{1/2}(\Gamma)}\|U_h^{n+1} + 2U_h^n + U_h^{n-1}\|_{H^1(\Omega)}.$$

Using Young's inequality and the fact that $k \in (0,1)$, we have

$$\frac{3}{k^2}\|U_h^{n+1}\|^2_{L^2(\Omega)} \leq \frac{16}{k^2}\left(\|U_h^n\|^2_{L^2(\Omega)} + \|U_h^{n-1}\|^2_{L^2(\Omega)}\right) + \frac{1}{2k\mu}\|f^{n+1} + 2f^n + f^{n-1}\|^2_{L^2(\Omega)}$$
$$+ \frac{c_0^2}{2k\mu}\|g_h^{n+1} + 2g_h^n + g_h^{n-1}\|^2_{H^{1/2}(\Gamma)}, \qquad n = 1, 2, \ldots, (M-1).$$

(4.1) follows by iteration on n. □

The main result below establishes the convergence of the fully discrete solution to the exact solution in the $L^2(\Omega)$ norm.

Theorem 4.2

Let u^n *and* U_h^n *be the solutions of* (3.5) *and* (3.6) *at* t_n *respectively. Suppose* $g(x,t) \in L^2(0,T;H^3(\Gamma))$ *and* $a_i(x,t)$, $b_i(x,t)$, $f_i(x,t)$, $\frac{\partial^4 u}{\partial t^4}$ *are continuous on* $\Omega_i \times (0,T]$, $i = 1,2$. *There exists a positive constant* C *independent of* h *and* k *such that*

$$\|u^n - U_h^n\|_{L^2(\Omega)} \le C\left[k^2 + h^3\right]\mathfrak{A}_n$$

where

$$\mathfrak{A}_n = \max\left\{\sqrt{\|u^n\|_X^2 + \int_0^{t_n}\left(\|u\|_X^2 + \|u_t\|_X^2 + \|u_{tt}\|_X^2\right)\,dt},\ \sqrt{\|u_{tt}^n\|_{L^2(\Omega)}^2 + \int_0^{t_n}\|\frac{\partial^4 u}{\partial t^4}\|_{L^2(\Omega)}^2\,dt}\right\}$$

Proof.

Letting $z^n = P_h u^n - U_h^n$ in (3.6) and using (3.8), we have

$$(\partial z^n, v_h) + \frac{1}{4} A_h(z^{n+1} + 2z^n + z^{n-1}, v_h) = (\partial(P_h u^n - u^n), v_h) + (\partial u^n - u_{tt}^n, v_h) - (u_{tt}^{n+1} - 2u_{tt}^n + u_{tt}^{n-1}, v_h) \quad (4.2)$$

With $v_h = k\left(z^{n+1} + 2z^n + z^{n-1}\right)$, we obtain

$$\begin{aligned} \frac{2}{3k}\|z^{n+1}\|_{L^2(\Omega)}^2 + \frac{\mu_1 k}{4}\|z^{n+1} + 2z^n + z^{n-1}\|_{H^1(\Omega)}^2 \\ \leq \frac{4}{k}\left(\|z^n\|_{L^2(\Omega)}^2 + \|z^{n-1}\|_{L^2(\Omega)}^2\right) + \frac{3k^2}{4\varepsilon}\|z^{n+1} + 2z^n + z^{n-1}\|_{L^2(\Omega)}^2 \\ + \varepsilon\|\partial(P_h u^n - u^n)\|_{L^2(\Omega)}^2 + \varepsilon\|\partial u^n - u_{tt}^n\|_{L^2(\Omega)}^2 + \varepsilon k^4 \|\frac{\partial^4 u^n}{\partial t^4}\|^2 \end{aligned}$$

where $\mu_1 = \min\{a, b\}$ and $\varepsilon > 0$. We take $\varepsilon = \frac{3}{\mu_1}$ and obtain for $k \in (0, 1)$,

$$\begin{aligned} \|z^{n+1}\|_{L^2(\Omega)}^2 \leq C\Big(\|z^n\|_{L^2(\Omega)}^2 + \|z^{n-1}\|_{L^2(\Omega)}^2 + k\|\partial(P_h u^n - u^n)\|_{L^2(\Omega)}^2 \\ + k\|\partial u^n - u_{tt}^n\|_{L^2(\Omega)}^2 + k^5\|\frac{\partial^4 u^n}{\partial t^4}\|^2\Big), \quad (4.3) \end{aligned}$$

Proof cont.

$n = 2, \ldots, M$. After a simple calculation, we have

$$\begin{aligned}
\|z^n\|^2_{L^2(\Omega)} &\leq C\left(\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)}\right) \\
&\quad + C\int_0^{t_n} \|(u - P_h u)_{tt}\|^2_{L^2(\Omega)}\, dt + Ck^4 \int_0^{t_n} \|\frac{\partial^4 u}{\partial t^4}\|^2_{L^2(\Omega)}\, dt \\
&\leq C\left[\|z^0\|^2_{L^2(\Omega)} + \|z^1\|^2_{L^2(\Omega)} + h^6 \int_0^{t_n} \left(\|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X\right)\, dt\right] \\
&\quad + Ck^4 \int_0^{t_n} \|\frac{\partial^4 u}{\partial t^4}\|^2_{L^2(\Omega)}\, dt
\end{aligned}$$

where use is made of Lemma 3.3 to obtain the last inequality. Taking $U_h^0 = P_h u_0$, $U_h^1 = U_h^0 + kP_h u_1 + \frac{k^2}{2} P_h[\nabla \cdot (a(x,0)\nabla u_0) - b(x,0)u_0 + f(x,0)]$ and using triangle inequality,

$$\begin{aligned}
\|u^n - U_h^n\|^2_{L^2(\Omega)} &\leq Ch^6\left[\|u^n\|^2_X + \int_0^{t_n} \left(\|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X\right)\, dt\right] \\
&\quad + Ck^4\left[\|u^n_{tt}\|^2_{L^2(\Omega)} + \int_0^{t_n} \|\frac{\partial^4 u}{\partial t^4}\|^2_{L^2(\Omega)}\, dt\right].
\end{aligned}$$

The result follows immediately.

Numerical Experiment

Here, we present an example to verify our result. Globally continuous piecewise linear finite element functions based on triangulation described in Subsection 2 are used. The mesh generation and computation are done with FreeFEM++ (Hecht, 2012).

Example 5.1

Consider the domain $\Omega = (-1, 1) \times (0, 1)$ where the interface Γ is the line $x = 0$. $\Omega_1 = \{(x, y) \in \Omega : x < 0\}$, $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. On $\Omega \times (0, T]$, $0 < T < \infty$, we consider the problem (3.1)−(3.3) whose exact solution, is

$$u = \begin{cases} \sin(\pi x)\sin(2\pi y)\ln(1+t)\sin t & \text{in} \quad \Omega_1 \times (0, T] \\ \sin(2\pi x)\sin(\pi y)t^2\exp(-t) & \text{in} \quad \Omega_2 \times (0, T] \end{cases},$$

The source function f, interface function g and the initial data u_0, u_1 are determined from the choice of u with

Example 5.1 cont.

$$a = \begin{cases} 4 & \text{in} \quad \Omega_1 \times (0, T] \\ 2 & \text{in} \quad \Omega_2 \times (0, T] \end{cases} \qquad b = \begin{cases} 1 & \text{in} \quad \Omega_1 \times (0, T] \\ 0 & \text{in} \quad \Omega_2 \times (0, T] \end{cases}$$

Table 1 discusses the errors in L^2-norm with $k = 0.0001$. In this case, much of the errors are contributed by spatial discretization.

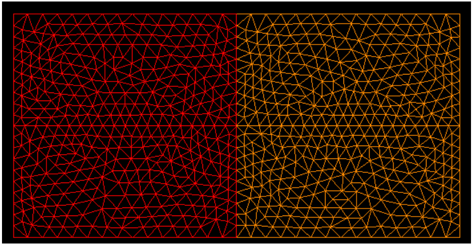

Figure: Domain of Example 5.1 with $h = 0.0945763$

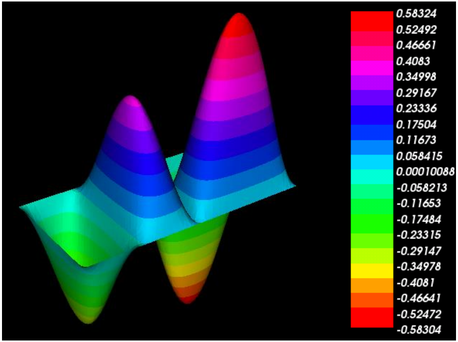

Figure: Finite element solution of Example 5.1 with $h = 0.0945763$ and $k = 0.0001$

Table: Error estimates for Example 5.1 with $k = 0.0001$.

h	$\|\text{Error}\|_{L^2(\Omega)}$	Convergence rate
0.1788870	6.64184×10^{-4}	
0.0945763	8.11113×10^{-5}	3.299
0.0504485	1.13117×10^{-5}	3.135
0.0251338	1.49383×10^{-6}	2.906

Conclusion

In this work, we investigate the convergence of approximate solution of a linear hyperbolic interface problem on quadratic element with time discretization based on modified centered difference scheme. Under certain regularity conditions on the input data, the scheme was shown to be stable and that optimal order of convergence is guaranteed when the spatial discretization meet certain requirements.

THANK YOU FOR LISTENING

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