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FINITE ELEMENT METHOD FOR SECOND ORDER NONLINEAR PARABOLIC INTERFACE PROBLEMS

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ABSTRACT. Parabolic interface problems are frequently encountered as models of real life situations and in scientific computing. In this paper, we present the error analysis of a second order nonlinear parabolic interface problem with Finite Element Method-Backward Difference Scheme (FEM-BDS). Quasi-uniform triangular elements are used for the spatial discretization and a three-step linearized scheme is proposed for the time discretization. The stability of the scheme is established and an almost optimal convergence rate is obtained. We also establish that the discrete solution reproduce the maximum principle under certain conditions. Numerical experiments are presented to support the theoretical results. It is assumed that the solution is of low regularity across the interface and the interface cannot be fitted exactly.

Keywords and phrases: Nonlinear parabolic problem, linearized implicit scheme, discrete maximum principle, almost optimal convergence
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1. INTRODUCTION

Nonlinear parabolic interface problems appear in various branches of material science, population growth, nonlinear problems of heat and mass transfer, biochemistry, multiphase flow in porous media, etc. often when two or more different materials are involved with different conductivities, diffusion constants or densities [8, 14, 22, 25]. The solutions of interface problems may have higher regularities in each individual material region than in the entire physical domain because of the discontinuities across the interface [19] and as a result of this, achieving higher order accuracy may be difficult.

Many contributions have been made towards the development of conforming finite element method (FEM) for linear parabolic interface problems eg. [2, 3, 11, 15, 16, 17] to mention recent works.

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In this work, we study the nonlinear parabolic equation

$$u_t - \nabla \cdot (a(x,u)\nabla u) + b(x,u)u = f(t,x) \quad \text{in} \quad \Omega \times (0,T] \tag{1}$$

with initial and boundary conditions

$$\left\{ \begin{array}{rcll} u(x,0) & = & u_0(x) & \text{in } \Omega \\ u(x,t) & = & 0 & \text{on } \partial\Omega \times [0,T] \end{array} \right. \tag{2}$$

and interface conditions

$$\left\{ \begin{array}{rcl} [u]_\Gamma & = & 0 \\ \left[a(x,u)\dfrac{\partial u}{\partial n}\right]_\Gamma & = & g(t,x) \end{array} \right. \tag{3}$$

where $0 < T < \infty$ and Ω is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. $\Omega_1 \subset \Omega$ is an open domain with smooth boundary $\Gamma = \partial\Omega_1$, $\Omega_2 = \Omega \setminus \bar{\Omega}_1$ is another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$, see Figure 1. The symbol $[u]$ is a jump of a quantity u across the interface Γ and is defined as the difference of the limiting values from each side of the interface. n is the unit outward normal to the boundary Γ.

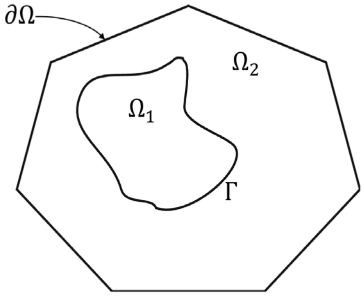

Fig. 1. A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ.

Semilinear parabolic interface problem has been discussed in [25]. With time discretization based on implicit Euler scheme, the authors obtained a convergence rate of optimal order in $H^1(\Omega)$ norm. They assumed that Ω is a convex polygon in $\mathbb{R}^2$ with C^2 boundary and the mesh can be fitted exactly to the arbitrary interface, however, it is very difficult to generate a grid which exactly follows the actual interface in practice. Convergence of the finite element solution of a class of nonlinear parabolic interface problems was studied in [28]. The author focused on the fully discrete approximation and used a linearized 2-step backward difference scheme for the time discretization while piecewise linear interpolation was used to approximate the interface. With the assumption that the coefficient $a(u)$ is positive and smooth with respect to $u \in \mathbb{R}$ but not continuous across the interface, the author proved a convergence rate of almost optimal order in the L^2-norm.

In [21], we studied the finite element solution of a class of nonlinear parabolic interface problems. We obtained regularity estimates which were used to establish convergence rates of almost optimal order in $H^1(\Omega)$-norm for both semi and full discretizations of the problem. Implicit Euler scheme was used for the time discretization and the implementation was based on predictor-corrector method due to the nonlinear terms. This made the scheme computationally time consuming. Anti-symmetric interior penalty discontinuous Galerkin method was proposed in [26] for the solution of nonlinear parabolic interface problem. Again the time discretization was based on a second-order linearized backward difference scheme. Use was made of over-penalyzed method to improve the L^2-norm error to optimal order with the assumption that the diffusion coefficient is only continuous on each sub-domain and the interface could be fitted exactly (using triangles with curved edges). In [4], we analyzed a semidiscrete scheme for a class of nonlinear parabolic interface problems and we presented the solution of a second-order nonlinear parabolic interface problem with nonlinear source term on a quasi-uniform triangular elements in [5]. A four-step linearized implicit scheme was proposed for the time discretization and convergence rate of almost optimal order in L^2-norm was obtained because the mesh could perfectly match the interface.

The discretization of (1)$-$(3) results to a system of nonlinear algebraic equations as a result of $a(x, u)$ and $b(x, u)$. To avoid this difficulty, we propose a linearized 3-step time discretization scheme for the problem. Earlier work on this subject had focused on the convergence in much weaker norm, ie, L^2-norm however, in this work, we establish the stability of the scheme and show that almost optimal order of convergence in the $H^1(\Omega)$-norm could be obtained when the mesh cannot exactly fit the interface. In terms of matrices arising in the scheme, we show that the scheme preserves the maximum principle under certain conditions. Numerical experiments are presented to support the theoretical results.

For our analysis, we impose the following

Assumption 1:

(1) Ω is a bounded convex polygonal domain in $\mathbb{R}^2$, the interface Γ and the boundary $\partial\Omega$ are piecewise smooth, Lipschitz continuous and 1-dimensional.

(2) $g(t,x) \in L^2(0,T;H^2(\Gamma)) \cap H^1(0,T;H^{1/2}(\Gamma))$, $f(t,x) \in H^1(0,T;H^{-1}(\Omega))$. Functions $a(x,\xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ and $b(x,\xi) : \Omega \times \mathbb{R} \to \mathbb{R}$ are measurable with respect to $x \in \Omega_i$ $(i = 1, 2)$, and satisfy

$$a_i(x,\xi) \geq \mu_1, \quad b_i(x,\xi) \geq \mu_1, \quad \|a_i(x,0)\|_{L^\infty(\Omega)} \leq \mu_2,$$
$$\|b_i(x,0)\|_{L^\infty(\Omega)} \leq \mu_3,$$
$$|a_i(x,\xi) - a_i(x,\psi)| + |b_i(x,\xi) - b_i(x,\psi)| \leq \mu_4 \|\xi - \psi\|_{L^2(\Omega_i)},$$

for $\xi, \psi \in \mathbb{R}$, $x \in \Omega_i$, $t \in \mathbb{R}^+$ with positive constants μ_1, μ_2 and μ_3 independent of t, x, ξ, ψ.

In this study, we use the standard notations and properties of Sobolev spaces as contained in [1]. Other tools used in this paper are the linear theories of interface and non-interface problems, as well as approximation properties of linear interpolation and projection operators.

We shall need the following space

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

which is equipped with the norm

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \quad \forall\, v \in X$$

The weak form of (1)−(3) is:
Find $u(t) \in H_0^1(\Omega)$, $t \in (0,T]$ such that

$$(u_t, v) + A(u : u, v) = (f, v) + \langle g, v\rangle_\Gamma \quad \forall\, v(t) \in H_0^1(\Omega),\ t \in (0,T] \tag{4}$$

where

$$(\phi, \psi) = \int_\Omega \phi\psi\, dx, \qquad \langle \phi, \psi\rangle_\Gamma = \int_\Gamma \phi\psi\, ds,$$
$$A(\xi : \phi, \psi) = \int_\Omega \left[a(x,\xi)\nabla\phi \cdot \nabla\psi + b(x,\xi)\phi\psi\right]\, dx.$$

For (4), we have the following regularity estimates

Lemma 1: Suppose the conditions of Assumption 1 are satisfied for every $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $b : \Omega \times \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ and $g \in H^1(0,T;H^{1/2}(\Gamma))$, there exists a constant C depending on μ_1, μ_2, μ_3, μ_4, T and Ω such that

$$\|u\|_{L^2(0,T;X)} \leq C\left(\|g\|_{H^1(0,T;H^{1/2}(\Gamma))} + \|u_0\|_X + \|f\|_{H^1(0,T;L^2(\Omega))}\right),$$

for $u(t) \in X \cap H_0^1(\Omega)$.
Proof: It follows from [21].

This paper is organized as follows. In Section 2, we describe a finite element discretization of the problem and state some auxiliary results. In Section 3, we give the discrete version of (4) then establish the stability and convergence rate of almost optimal order of the scheme. Discrete maximum principle of the scheme is established in Section 4 and conclusion is made in Section 5. Throughout this paper, C is a generic positive constant (which is independent of the mesh parameter h and the time step size k) and may take on different values at different occurrences.

2. FINITE ELEMENT DISCRETIZATION

We adopt the standard finite element discretization used in [2, 9]. $\mathcal{T}_h$ denotes a partition of Ω into disjoint triangles K (called elements) such that no vertex of any triangle lies on the interior or side of another triangle. The domain Ω_1 is approximated by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries respectively.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ;

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$$

$K \in \mathcal{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$. The triangulation $\mathcal{T}_h$ of the domain Ω satisfies the following conditions

- $\bar{\Omega} = \bigcup\limits_{K\in\mathcal{T}_h} \bar{K}$
- If $\bar{K}_1, \bar{K}_2 \in \mathcal{T}_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.
- Each $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h, and has at most two vertices lying on Γ_h.
- For each element $K \in \mathcal{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles respectively. It is assumed that, for some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \leq h \leq C_1 \bar{r}_K \quad \forall\, h \in (0, h_0)$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$.

The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x) \ ,$$

where each basis function ϕ_j, $(j = 1, 2, \dots, N_h)$ is a pyramid function with unit height. For the approximation g_h of g, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h, then

$$g_h(x,t) = \sum_{j=1}^{n_h} \beta_j(t)\psi_j(x) \ .$$

We recall some existing results which will be used in our analysis.
Lemma 2: Let $\Omega_h^\star$ be the union of all interface elements, $f \in H^2(\Omega)$ and $g \in H^2(\Gamma)$, we have

$$\begin{aligned} \|v\|_{H^1(\Omega_h^\star)} &\le Ch^{1/2}\|v\|_X \quad \forall\, v \in X & (5)\\ |\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}| &\le Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_h^\star)} \quad \forall\, v_h \in S_h & (6)\\ |(f,\phi) - (f,\phi)_h| &\le Ch^2\|f\|_{H^2(\Omega)}\|\phi\|_{H^1(\Omega)} \quad \forall\, \phi \in S_h & (7) \end{aligned}$$

Proof: See [24] for (5), See [9] for (6) and [27, Chapter 6] for (7).

For $u \in H^1(\Omega)$, the boundary value of u (ie $u_{|\partial\Omega}$) is defined on $H^{1/2}(\partial\Omega)$ the trace space of $H^1(\Omega)$. Similarly, the trace space on the interface Γ is $H^{1/2}(\Gamma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous and satisfies the embedding

$$\|z\|_{L^2(\partial\Omega)} \le \|z\|_{H^{1/2}(\partial\Omega)} \le c_0\|z\|_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega) \tag{8}$$

See [1, 12, 6] for more information on trace operator.

Let $P_h : X \cap H_0^1(\Omega) \to S_h$ be the elliptic projection of the exact solution ν in S_h defined by

$$A_h(u : P_h\nu, \phi) = A(u : \nu, \phi) \quad \forall\, \phi \in S_h,\ t \in [0,T] \tag{9}$$

It follows that there exists $C > 0$, such that

$$\|P_h\nu\|_{H^1(\Omega)} \le C\|\nu\|_{H^1(\Omega)} \qquad \forall\, \nu \in H^1(\Omega)$$

For this projection, we have
Lemma 3: Let $a(x,u)$ and $b(x,u)$ satisfy Assumption 1. Assume

that $u \in X \cap H_0^1$ and let $P_h u$ be defined as in (9), then

$$\begin{aligned}
\|P_h u - u\|_{H^1(\Omega)} &\leq Ch\left(1 + \frac{1}{|\log h|}\right)^{1/2} \|u\|_X \\
\|P_h u - u\|_{L^2(\Omega)} &\leq Ch^2\left(1 + \frac{1}{|\log h|}\right) \|u\|_X \\
\|(P_h u - u)_t\|_{H^1(\Omega)} &\leq Ch\left(1 + \frac{1}{|\log h|}\right)^{1/2} (\|u\|_X + \|u_t\|_X) \\
\|(P_h u - u)_t\|_{L^2(\Omega)} &\leq Ch^2\left(1 + \frac{1}{|\log h|}\right) (\|u\|_X + \|u_t\|_X)
\end{aligned}$$

Proof It can be proved in the similar version to [5, Lemmas 2.5 and 2.5] but with little modification due to different assumptions on $a(x, u)$ and $b(x, u)$.

Remark 1: The term $|\log h|$ in Lemma 3 is due to the fact that the mesh cannot perfectly fit the interface. However, with the assumption that the interface can be fitted exactly using interface elements with curved edges, optimal convergence rate is possible (see [23] for example).

3. ERROR ESTIMATE

In this section, we establish the stability of the proposed fully discrete scheme and obtain almost optimal order error estimate in $H^1(\Omega)$-norm.

The interval [0,T] is divided into M equally spaced (for simplicity) subintervals:

$$0 = t_0 < t_1 < \ldots < t_M = T$$

with $t_n = nk$, $k = T/M$ being the time step. Let

$$u^n = u(x, t_n) \quad \text{and} \quad g^n = g(x, t_n)\ .$$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward difference quotient defined by

$$\partial^3 w^n = \frac{11w^n - 18w^{n-1} + 9w^{n-2} - 2w^{n-3}}{6k}\ , \quad n = 3, 4, 5, \ldots, M$$

The fully discrete finite element approximation to (4) is defined as follows: Let $U_h^0 = P_h u_0$, find $U_h^n \in S_h$, such that

$$\begin{aligned}
&(\partial^3 U_h^n, v_h)_h + A_h(3U_h^{n-1} - 3U_h^{n-2} + U_h^{n-3} : U_h^n, v_h) \\
&= (f(t_n, x), v_h)_h + \langle g_h^n, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h, \quad n = 4, 5, \ldots, M \quad (10)
\end{aligned}$$

where $A_h(\eta : \phi, \psi)$ and $(\psi, v_h)_h$ are defined as

$$A_h(\eta : \phi, \psi) = \sum_{K \in \mathcal{T}_h} \int_K [a(x,\eta)\nabla\phi \cdot \nabla\psi + b(x,\eta)\phi\psi]\, dx$$

$$(\psi, \phi)_h = \sum_{K \in \mathcal{T}_h} \int_K \psi\phi\, dx$$

$\forall\ \phi, \psi \in H^1(\Omega),\ t \in [0, T]$ and are obtained by numerical quadrature. See [10] for information on numerical integration on finite elements.

(10) is zero-stable. To see this, we obtain the first characteristic polynomial

$$\rho(y) \quad = \quad \frac{11}{6}y^3 - 3y^2 + \frac{3}{2}y - \frac{1}{3}\ .$$

The roots of this polynomial have modulli less than one and the roots with modulus one are simple. See Lambert [20] for more information on zero-stability.

The analysis of this work is done with the assumption that $\dfrac{\partial^4 u}{\partial t^4}$ exists. It can be shown using Taylor expansion that

$$\left\{ \begin{array}{rcl} \|U_h^n - 2U_h^{n-1} + U_h^{n-2}\|_{L^2(\Omega)} & \leq & (\Delta t)^2\lambda_0 \\ \|U_h^n - 3U_h^{n-1} + 3U_h^{n-2} - U_h^{n-3}\|_{L^2(\Omega)} & \leq & (\Delta t)^3\lambda_1, \end{array} \right. \tag{11}$$

where $\lambda_0, \lambda_1 \geq 0$. We have the following stability estimate:

Lemma 4: Suppose the conditions of Assumption 1 are satisfied for a, b, f and g. Let $\frac{k}{h^2}$ be sufficiently small, there exists a constant C independent of $h \in (0, 1)$ and k such that, for the solution of (10),

$$\begin{aligned} \|U_h^n\|^2_{H^1(\Omega)} &\leq C\|U_h^2\|^2_{H^1(\Omega)} \\ &\quad + C\int_{t_2}^{t_n} \left[\|g\|^2_{H^{1/2}(\Gamma)} + h^2\|g\|^2_{H^2(\Gamma)} + \|f\|^2_{L^2(\Omega)} + k^2\right]\, dt \\ &\qquad n = 3, 4, 5, \ldots, M. \end{aligned} \tag{12}$$

Proof: Take $v_h = \partial^3 U_h^n$ in (10),

$$\begin{aligned} &\|\partial^3 U_h^n\|^2_{L^2(\Omega)} + \frac{\mu_1}{k}\|U_h^n\|^2_{H^1(\Omega)} \\ &\quad \leq \left(\frac{\lambda_0 k}{3} + \frac{\lambda_1}{2}\right)\mu_1 k\|U_h^n\|_{H^1(\Omega)} + \frac{\mu_1}{k}\|U_h^{n-1}\|_{H^1(\Omega)}\|U_h^n\|_{H^1(\Omega)} \\ &\qquad + \|f(t_n, x)\|_{L^2(\Omega)}\|\partial^3 U_h^n\|_{L^2(\Omega)} + C\|g^n\|_{H^{1/2}(\Gamma)}\|\partial^3 U_h^n\|_{H^1(\Omega)} \\ &\qquad + Ch^2\|g^n\|_{H^2(\Gamma)}\|\partial^3 U_h^n\|_{H^1(\Omega)} \end{aligned}$$

We use (5), (6), (8) and (11) to obtain the last inequality. By inverse estimate [7, Theorem 4.5.11] and Young's inequality, we obtain, for $\frac{k}{h^2}$ sufficiently small,

$$\begin{aligned}\|U_h^n\|^2_{H^1(\Omega)} &\leq (1+Ck)\|U_h^{n-1}\|^2_{H^1(\Omega)} \\ &\quad + Ck\left[\|g^n\|^2_{H^{1/2}(\Gamma)} + h^2\|g^n\|^2_{H^2(\Gamma)} + \|f(t_n,x)\|^2_{L^2(\Omega)} + k^2\right]\end{aligned}$$

(12) follows by iteration on n.

Remark 2: The scheme (10) is not self-starting. The initial two values can be obtained using lower-order time discretization schemes:

$$\begin{aligned}&(\partial^1 U_h^1, v_h)_h + A_h(U_h^0 : U_h^1, v_h) \\ &\qquad = (f(t_1, x, U_h^0), v_h)_h + \langle g_h^1, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \qquad (13)\end{aligned}$$

$$\begin{aligned}&(\partial^2 U_h^2, v_h)_h + A_h(2U_h^1 - U_h^0 : U_h^2, v_h) \\ &\qquad = (f(t_2, x, 2U_h^1 - U_h^0), v_h)_h + \langle g_h^2, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h, \qquad (14)\end{aligned}$$

where

$$\begin{aligned}\partial^1 w^1 &= \frac{w^1 - w^0}{k} \\ \partial^2 w^2 &= \frac{3w^2 - 4w^1 + w^0}{2k}\end{aligned}$$

However, this doesn't affect the stability of the scheme. In fact, using (13)−(14) together with (10), (12) becomes

$$\begin{aligned}\|U_h^n\|^2_{H^1(\Omega)} &\leq C\|U_h^0\|^2_{H^1(\Omega)} \\ &\quad + C\int_{t_0}^{t_n}\left[\|g\|^2_{H^{1/2}(\Gamma)} + h^2\|g\|^2_{H^2(\Gamma)} + \|f\|^2_{L^2(\Omega)} + k^2\right] dt \\ &\qquad\qquad n = 1, 2, 3, \ldots, M.\end{aligned}$$

The main result below establishes the convergence of the scheme (10) to the exact solution in $H^1(\Omega)$-norm.

Theorem 1: Let u^n and U_h^n be the solutions of (4) and (10) respectively at t_n. Suppose that the conditions of Assumption 1 are satisfied for every a, b, f, g and $\dfrac{\partial^4 u}{\partial t^4}$ is defined for $\Omega \times [0,T]$. There exists a positive constant C independent of $h \in (0, h_0)$ and

$k \in [0, k_0)$ such that

$$\begin{aligned}\|u^n - U_h^n\|_{H^1(\Omega)} \;\le\; & C\sum_{i=0}^{2}\|u^i - U_h^i\|_{H^1(\Omega)} \\ & + \left[k^3 + h\left(1 + \frac{1}{|\log h|}\right)^{1/2}\right] C(u,g,f), \\ & \qquad\qquad\qquad n = 3,4,5,\ldots\end{aligned}$$

Proof: Let $z^n = P_h u^n - U_h^n$. From (4) and (10) using (9), we have

$$\begin{aligned}&(\partial^3 z^n, v_h)_h + A_h(3U_h^{n-1} - 3U_h^{n-2} + U_h^{n-3}; z^n, v_h) \\ &\quad = (\partial^3(P_h u^n - u^n), v_h)_h + (\partial^3 u^n - u_t^n, v_h) + (\partial^3 u^n, v_h)_h - (\partial^3 u^n, v_h) \\ &\qquad + (f(t_n,x), v_h) - (f(t_n,x), v_h)_h + \langle g^n, v_h\rangle_\Gamma - \langle g_h^n, v_h\rangle_{\Gamma_h} \\ &\qquad + A_h(3U_h^{n-1} - 3U_h^{n-2} + U_h^{n-3} : P_h u^n, v_h) - A_h(u^n : P_h u^n, v_h)\end{aligned}$$

After a simple calculation using Young's inequality with $v_h = \partial^3 z^n$, we have

$$\begin{aligned}&\|\partial^3 z^n\|_{L^2(\Omega)}^2 + \frac{\mu_1}{2k}\|z^n\|_{H^1(\Omega)}^2 \\ &\qquad \le \frac{3\mu_1}{k}\sum_{j=0}^{2}\left\{\|z^{n-j-1}\|_{H^1(\Omega)}^2 + \|z^{n-j} - z^{n-j-1}\|_{H^1(\Omega)}^2\right\} \\ &\qquad\quad + B_1 + B_2 + B_3 \qquad (15)\end{aligned}$$

where

$$\begin{aligned}B_1 &= (\partial^3(P_h u^n - u^n), \partial^3 z^n)_h + (\partial^3 u^n - u_t^n, \partial^3 z^n) + (\partial^3 u^n, \partial^3 z^n)_h \\ &\quad -(\partial^3 u^n, \partial^3 z^n), \\ B_2 &= (f(t_n,x), \partial^3 z^n) - (f(t_n,x), \partial^3 z^n)_h + \langle g^n, \partial^3 z^n\rangle_\Gamma \\ &\quad -\langle g_h^n, \partial^3 z^n\rangle_{\Gamma_h} + \mu_1\lambda_1 k^3\|z^n\|_{L^2(\Omega)}\|\partial^3 z^n\|_{L^2(\Omega)}, \\ B_3 &= A_h(3U_h^{n-1} - 3U_h^{n-2} + U_h^{n-3} : P_h u^n, \partial^3 z^n) \\ &\quad -A_h(u^n : P_h u^n, \partial^3 z^n).\end{aligned}$$

$$\begin{aligned}B_1 &\le \varepsilon\|\partial^3(P_h u^n - u^n)\|_{L^2(\Omega)}^2 + \frac{3}{4\varepsilon}\|\partial^3 z^n\|_{L^2(\Omega)}^2 + \varepsilon\|\partial^3 u^n - u_t^n\|_{L^2(\Omega)}^2 \\ &\quad + C\varepsilon h^2\|\partial^3 u^n\|_X^2 \qquad (16)\end{aligned}$$

use is made of inverse estimate and (7) to obtain the last inequality. Using Lemma 2 with the fact that $D^\alpha z^n = 0$ for $|\alpha| = 2$, we have

$$\begin{aligned}B_2 &\le Ch^2\|f(t_n,x)\|_{H^2(\Omega)}\|\partial^3 z^n\|_{H^1(\Omega)} + Ch^2\|g^n\|_{H^2(\Gamma)}\|\partial^3 z^n\|_{H^1(\Omega)} \\ &\quad + \mu_1\lambda_1 k^3\|z^n\|_{L^2(\Omega)}\|\partial^3 z^n\|_{L^2(\Omega)}.\end{aligned}$$

By inverse estimate,

$$\begin{aligned} B_2 \leq\ & Ch\|f(t_n,x)\|_{H^2(\Omega)}\|\partial^3 z^n\|_{L^2(\Omega)} + Ch\|g^n\|_{H^2(\Gamma)}\|\partial^3 z^n\|_{L^2(\Omega)} \\ & + \mu_1\lambda_1 k^3\|z^n\|_{L^2(\Omega)}\|\partial^3 z^n\|_{L^2(\Omega)} \\ \leq\ & C\varepsilon h^2\|f(t_n,x)\|^2_{H^2(\Omega)} + \frac{3}{4\varepsilon}\|\partial^3 z^n\|^2_{L^2(\Omega)} + C\varepsilon h^2\|g^n\|_{H^2(\Gamma)} \\ & + \mu_1^2\lambda_1^2 k^6\varepsilon\|z^n\|^2_{L^2(\Omega)}. \end{aligned} \tag{17}$$

By Assumption 1

$$\begin{aligned} B_3 \leq\ & |a(x, 3U_h^{n-1} - 3U_h^{n-2} + U_h^{n-3}) - a(x,u^n)| \\ & \times\|P_h u^n\|_{H^1(\Omega)}\|\partial^3 z^n\|_{H^1(\Omega)}\ dx \\ \leq\ & \mu_4\|(3U_h^{n-1} - 3U_h^{n-2} + U_h^{n-3}) - u^n\|_{L^2(\Omega)} \\ & \times\|P_h u^n\|_{H^1(\Omega)}\|\partial^3 z^n\|_{H^1(\Omega)}\ dx \\ \leq\ & \mu_4\lambda_1 k^3\|P_h u^n\|_{H^1(\Omega)}\|\partial^3 z^n\|_{H^1(\Omega)} \\ & + \mu_4\|P_h u^n - u^n\|_{L^2(\Omega)}\|P_h u^n\|_{H^1(\Omega)}\|\partial^3 z^n\|_{H^1(\Omega)} \\ & + \mu_4\|z^n\|_{L^2(\Omega)}\|P_h u^n\|_{H^1(\Omega)}\|\partial^3 z^n\|_{H^1(\Omega)} \\ \leq\ & C\|z^n\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} + \frac{3}{4}\|\partial^3 z^n\|^2_{H^1(\Omega)} \\ & + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\|u^n\|^2_X\|u^n\|^2_{H^1(\Omega)} + Ck^6\|u^n\|^2_{H^1(\Omega)} \end{aligned} \tag{18}$$

Substitute (16)−(18) into (15), with $\varepsilon = 6$ and $\frac{k}{h^2}$ sufficiently small,

$$\begin{aligned} \frac{\mu_1}{2k}\|z^n\|^2_{H^1(\Omega)} \leq\ & \frac{3\mu_1}{k}\sum_{j=0}^{2}\left\{\|z^{n-j-1}\|^2_{H^1(\Omega)} + \|z^{n-j} - z^{n-j-1}\|^2_{H^1(\Omega)}\right\} \\ & + 6\|\partial^3(P_h u^n - u^n)\|^2_{L^2(\Omega)} + 6\|\partial^4 u^n - u_t^n\|^2_{L^2(\Omega)} \\ & + Ch^4\left(1 + \frac{1}{|\log h|}\right)^2\|u^n\|^2_X\|u^n\|^2_{H^1(\Omega)} \\ & + Ch^2\left(\|g^n\|^2_{H^2(\Gamma)} + \|f(t_n x)\|^2_{H^2(\Omega)} + \|\partial^3 u^n\|^2_X\right) \\ & + C\left(k^6 + \|u^n\|^2_{H^1(\Omega)}\right)\|z^n\|^2_{L^2(\Omega)} + Ck^6\|u^n\|^2_{H^1(\Omega)}\ , \end{aligned}$$

therefore,

$$(1 - c_2 k)\|z^n\|^2_{H^1(\Omega)} \leq \quad C\sum_{j=0}^{2}\left\{\|z^{n-j-1}\|^2_{H^1(\Omega)} + \|z^{n-j} - z^{n-j-1}\|^2_{H^1(\Omega)}\right\}$$

$$
\begin{aligned}
&+ C\left[k\|\partial^3(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^3 u^n - u^n_t\|^2_{L^2(\Omega)}\right] \\
&+ Ch^4 k\left(1 + \frac{1}{|\log h|}\right)^2 \|u^n\|^2_X \|u^n\|^2_{H^1(\Omega)} \\
&+ Ch^2 k\left(\|g^n\|^2_{H^2(\Gamma)} + \|f(t_n x)\|^2_{H^2(\Omega)} + \|\partial^3 u^n\|^2_X\right) + Ck^7\|u^n\|^2_{H^1(\Omega)}
\end{aligned}
$$

where $c_2 = C\left(k^6 + \|u^n\|^2_{H^1(\Omega)}\right)$.
For $0 < k < \min\left\{1, \dfrac{1}{c_2}\right\}$, there is a $C > 0$ such that
$(1 - c_2 k)^{-1} \leq C$, and therefore

$$
\begin{aligned}
\|z^n\|^2_{H^1(\Omega)} \;\leq\;& C\sum_{j=0}^{2}\left\{\|z^{n-j-1}\|^2_{H^1(\Omega)} + \|z^{n-j} - z^{n-j-1}\|^2_{H^1(\Omega)}\right\} \\
&+ C\left[k\|\partial^3(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^3 u^n - u^n_t\|^2_{L^2(\Omega)}\right] \\
&+ Ch^4 k\left(1 + \frac{1}{|\log h|}\right)^2 \|u^n\|^2_X \|u^n\|^2_{H^1(\Omega)} \\
&+ Ch^2 k\left(\|g^n\|^2_{H^2(\Gamma)} + \|f(t_n x)\|^2_{H^2(\Omega)} + \|\partial^3 u^n\|^2_X\right) \\
&+ Ck^7\|u^n\|^2_{H^1(\Omega)},
\end{aligned}
$$

for $n = 3, \ldots, M$. By iteration on n, we have

$$
\begin{aligned}
\|z^n\|^2_{H^1(\Omega)} \;\leq\;& C\sum_{i=0}^{2}\|z^i\|^2_{H^1(\Omega)} + C\sum_{j=3}^{n}\sum_{i=0}^{2}\|z^{j-i} - z^{j-i-1}\|^2_{H^1(\Omega)} \\
&+ Ck\sum_{j=3}^{n}\|\partial^3(u^j - P_h u^j)\|^2_{L^2(\Omega)} + Ck^7\sum_{j=3}^{n}\|u^j\|^2_{H^1(\Omega)} \\
&+ Ch^4 k\left(1 + \frac{1}{|\log h|}\right)^2 \sum_{j=3}^{n}\|u^j\|^2_X\|u^j\|^2_{H^1(\Omega)} \\
&+ Ch^2\sum_{j=3}^{n}\left(\|g^j\|^2_{H^2(\Gamma)} + \|f(t_j, x)\|^2_{H^2(\Omega)} + \|\partial^3 u^j\|^2_X\right) \\
&+ Ck\sum_{j=3}^{n}\|\partial^3 u^j - u^j_t\|^2_{L^2(\Omega)}
\end{aligned}
$$

Using the discrete version of Gronwall's inequality, we obtain

$$\begin{aligned}
\|z^n\|^2_{H^1(\Omega)} &\leq C\sum_{i=0}^{2}\|z^i\|^2_{H^1(\Omega)} + Ck\sum_{j=3}^{n}\|\partial^3(u^j - P_h u^j)\|^2_{L^2(\Omega)} \\
&\quad + Ch^4k\left(1+\frac{1}{|\log h|}\right)^2\sum_{j=3}^{n}\|u^j\|^2_X\|u^j\|^2_{H^1(\Omega)} \\
&\quad + Ch^2\sum_{j=3}^{n}\left(\|g^j\|^2_{H^2(\Gamma)} + \|f(t_j,x)\|^2_{H^2(\Omega)} + \|\partial^3 u^j\|^2_X\right) \\
&\quad + Ck\sum_{j=3}^{n}\|\partial^3 u^j - u^j_t\|^2_{L^2(\Omega)} + Ck^7\sum_{j=3}^{n}\|u^j\|^2_{H^1(\Omega)}
\end{aligned}$$

After a simple calculation, we have

$$\begin{aligned}
\|z^n\|^2_{H^1(\Omega)} &\leq C\sum_{i=0}^{2}\|z^i\|^2_{H^1(\Omega)} + Ck^6\int_{t_2}^{t_n}\|u\|^2_{H^1(\Omega)}\,dt \\
&\quad + C\int_{t_2}^{t_n}\|(u-P_h u)_t\|^2_{L^2(\Omega)}\,dt + Ck^6\int_{t_2}^{t_n}\|\frac{\partial^4 u}{\partial t^4}\|^2_{L^2(\Omega)}\,dt \\
&\quad + Ch^4\left(1+\frac{1}{|\log h|}\right)^2\int_{t_2}^{t_n}\|u\|^2_X\|u\|^2_{H^1(\Omega)}\,dt \\
&\quad + Ch^2\int_{t_2}^{t_n}\left[\|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + \|f\|^2_{H^2(\Omega)}\right]\,dt
\end{aligned}$$

By triangle inequality and Lemma 3,

$$\begin{aligned}
&\|u^n - U^n_h\|^2_{H^1(\Omega)} \\
&\quad \leq 2\|u^n - P_h u^n\|^2_{H^1(\Omega)} + 2\|z^n\|^2_{H^1(\Omega)} \\
&\quad \leq C\sum_{i=0}^{2}\|z^i\|^2_{H^1(\Omega)} + Ch^2\left(1+\frac{1}{|\log h|}\right)\|u^n\|_X \\
&\quad\quad + Ck^6\left(\|u\|^2_{H^1(\Omega)} + \int_{t_2}^{t_n}\|\frac{\partial^4 u}{\partial t^4}\|^2_{L^2(\Omega)}dt\right) \\
&\quad\quad + Ch^4\left(1+\frac{1}{|\log h|}\right)^2\int_{t_2}^{t_n}\left[\|u\|^2_X\|u\|^2_{H^1(\Omega)} + \|u\|^2_X + \|u_t\|^2_X\right]\,dt \\
&\quad\quad + Ch^2\int_{t_2}^{t_n}\left[\|u_t\|^2_X + \|g\|^2_{H^2(\Gamma)} + \|f\|^2_{H^2(\Omega)}\right]\,dt.
\end{aligned}$$

It is obvious that

$$h^4\left(1+\frac{1}{|\ln h|}\right)^2 \le h^2\left(1+\frac{1}{|\ln h|}\right) \quad \Leftrightarrow \quad 0 < h < 0.58857838891.$$

The result follows taking $U_h^0 = P_h u_0$.

3.1. Example

Here, we present examples to verify Theorem 1. Globally continuous piecewise linear finite element functions based on triangulation described in Section 2 are used. The mesh generation and computation are done with FreeFEM++ [18].

Example 1: We consider (1)−(3) on the domain $\Omega = (-1,1) \times (-1,1)$ where Ω_1 is the region $4x^2+16y^2 < 1$, $\Omega_2 = \Omega \setminus \Omega_1$ and the interface Γ is the ellipse $4x^2+16y^2 = 1$ and therefore $\Gamma \neq \Gamma_h$.
For the exact solution, we choose

$$u = \begin{cases} \frac{1}{8}(1-4x^2-16y^2)t\exp(\sin t) & \text{in} \quad \Omega_1 \times (0,T] \\ \frac{1}{2}(1-x^2)(1-y^2)(1-4x^2-16y^2)\sin t & \text{in} \quad \Omega_2 \times (0,T] \end{cases}$$

The source function f, interface function g and the initial data u_0 are determined from the choice of u with $b = 0$ and

$$a = \begin{cases} 5 & \text{in} \quad \Omega_1 \times (0,T] \\ \dfrac{1}{1+u^2} & \text{in} \quad \Omega_2 \times (0,T] \end{cases}$$

Errors in H^1-norm at $t = 3$ for various step size h time step k are presented in Table 1. The data indicate that

$$\|\text{Error}\|_{H^1(\Omega)} \approxeq 1.07457\times 10^{-9} + 1.50469\mathfrak{h}^{0.9823} \quad \text{when } k \text{ is constant}$$

and

$$\|\text{Error}\|_{H^1(\Omega)} \approxeq 4.32100 \times 10^{-2} + 1.14347 \times 10^{-3} k^{3.2045} \quad \text{when } h \text{ is constant}$$

where $\mathfrak{h} = h\left(1+\frac{1}{|\log h|}\right)^{1/2}$.

Table 1. Error estimates in H^1-norm for Example 1.

h	Error ($k = 0.001$)	k	Error ($h = 2.429 \times 10^{-2}$)
0.1124	$2.118326186 \times 10^{-1}$	0.025	$4.321003490 \times 10^{-2}$
0.05807	$1.073763139 \times 10^{-1}$	0.020	$4.321003060 \times 10^{-2}$
0.02994	$5.369577398 \times 10^{-2}$	0.010	$4.321002696 \times 10^{-2}$
0.02063	$3.593520624 \times 10^{-2}$	0.005	$4.321002653 \times 10^{-2}$

4. DISCRETE MAXIMUM PRINCIPLE (DMP)

Here, we investigate the DMP of the proposed scheme and show that the DMP is preserved under certain assumptions.

With $v_h = \phi_i$ in (10), we have

$$\mathbf{M}\frac{\frac{11}{6}\mathbf{u}^n - 3\mathbf{u}^{n-1} + \frac{3}{2}\mathbf{u}^{n-2} - \frac{1}{3}\mathbf{u}^{n-3}}{k} + \mathbf{K}\mathbf{u}^n \;=\; \mathbf{l}^n \qquad (19)$$

where

$$M_{ij} = \int_\Omega \phi_j\phi_i \, dx \qquad K_{ij} = \int_\Omega [a^n \nabla\phi_j \cdot \nabla\phi_i + b^n \phi_j\phi_i] \, dx$$

$$l_i^n = \int_\Omega f(t_n, x)\phi_i \, dx \;+\; \int_{\Gamma_h} g_h(t_n, x)\phi_i \, ds$$

$a^n = a(x, 3\mathbf{u}^{n-1} - 3\mathbf{u}^{n-2} + \mathbf{u}^{n-3})$, $\quad b^n = b(x, 3\mathbf{u}^{n-1} - 3\mathbf{u}^{n-2} + \mathbf{u}^{n-3})$.

Let $\mathbf{A} = \mathbf{M} + \dfrac{6}{11}k\mathbf{K}$, (19) becomes

$$\mathbf{A}\mathbf{u}^n \;=\; \mathbf{M}\left[\frac{18}{11}\mathbf{u}^{n-1} - \frac{9}{11}\mathbf{u}^{n-2} + \frac{2}{11}\mathbf{u}^{n-3}\right] + \frac{6}{11}k\mathbf{l}^n \qquad (20)$$

Let $\Omega_{ij} := \text{supp } \phi_i \cap \text{supp } \phi_j$. If $\text{meas}(\Omega_{ij}) > 0$ then for regular meshes [13, pp 157],

$$\int_\Omega \phi_i\phi_j \, dx \le \text{meas}(\Omega_{ij}) \le ch^2 \, , \qquad \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, dx \le -K_0$$

with some constants $K_0 > 0$ independent of i, j, h and $i \neq j$.

Lemma 5: Suppose $a(x, u)$ and $b(x, u)$ satisfy Assumption 1 for $(x, u) \in \Omega \times \mathbb{R}$. Let $\alpha = \mu_4\|u^n\|_{L^2(\Omega)} + \mu_2$ and $\beta = \mu_4\|u^n\|_{L^2(\Omega)} + \mu_3$. Let

$$h < \min\left\{1, \sqrt{\frac{\alpha K_0}{c\beta}}\right\} \qquad \text{and} \qquad k \ge \frac{11ch^2}{6(\alpha K_0 - \beta ch^2)} \qquad (21)$$

then

$$A_{ij} \le 0 \, , \quad (i \neq j, \; i, j = 1, 2, \ldots, N_h). \qquad (22)$$

Proof: From Assumption 1, it is not difficult to see that

$$|a(x,u)| \leq \mu_4 \|u\|_{L^2(\Omega)} + \|a(x,0)\|_{L^\infty(\Omega)}$$

$$\text{and} \quad |b(x,u)| \leq \mu_4 \|u\|_{L^2(\Omega)} + \|b(x,0)\|_{L^\infty(\Omega)}.$$

The remaining part of the proof follows the same argument as [2, Lemma 4.1] We define the following

$$u^n_{\min} := \min\{u^n_1, u^n_2, \dots, u^n_{N_h}\}, \qquad u^n_{\max} := \max\{u^n_1, u^n_2, \dots, u^n_{N_h}\}$$

$$f^{(n-1,n)}_{\min} := \inf_{\substack{x \in \Omega \\ \rho \in ((n-1)k, nk)}} f(\rho, x)\ , \qquad f^{(n-1,n)}_{\max} := \sup_{\substack{x \in \Omega \\ \rho \in ((n-1)k, nk)}} f(\rho, x)$$

for $n = 1, 2, \dots, M$.

Theorem 2: Let the discretization be as in Section 2 and let

(1) $A_{ij} \leq 0 \quad (i \neq j,\ i,j = 1,2,\dots,N_h)$
(2) $M_{ii} \geq 0 \quad (i = 1,2,\dots,N_h)$

then the scheme (10) satisfies

$$\begin{aligned} &\min\left\{0, \frac{18}{11}u^{n-1}_{\min} - \frac{9}{11}u^{n-2}_{\max} + \frac{2}{11}u^{n-3}_{\min}\right\} \\ &\qquad + \frac{6}{11}k \min\left\{0, f^{(n-1,n)}_{\min} + \min_{\Gamma_{((n-1)k,nk)}} g_h\right\} \\ &\qquad \leq\ u^n_i\ \leq \qquad (23) \\ &\max\left\{0, \frac{18}{11}u^{n-1}_{\max} - \frac{9}{11}u^{n-2}_{\min} + \frac{2}{11}u^{n-3}_{\max}\right\} \\ &\qquad \frac{6}{11}k \max\left\{0, f^{(n-1,n)}_{\max} + \max_{\Gamma_{((n-1)k,nk)}} g_h\right\} \end{aligned}$$

where $\Gamma_{((n-1)k,nk)} := \Gamma_h \times [(n-1)k, nk]$, $n = 3, \dots, M$.

Proof: It follows the same argument as [2, Theorem 4.2].

Remark 3: Following the same argument as above, it is not difficult to obtain from (13) and (14),

$$\begin{aligned} &\min\left\{0, u^0_{\min}\right\} + k \min\left\{0, f^{(0,1)}_{\min} + \min_{\Gamma_{(0,k)}} g_h\right\} \leq \\ &\qquad u^1_i \leq \max\left\{0, u^0_{\max}\right\} + k \max\left\{0, f^{(0,1)}_{\max} + \max_{\Gamma_{(0,k)}} g_h\right\} \end{aligned}$$

$$\min\left\{0, \frac{4}{3}u^1_{\min} - \frac{1}{3}u^0_{\max}\right\} + \frac{2}{3}k \min\left\{0, f^{(1,2)}_{\min} + \min_{\Gamma_{(k,2k)}} g_h\right\} \le u^2_i$$
$$\le \max\left\{0, \frac{4}{3}u^1_{\max} - \frac{1}{3}u^0_{\min}\right\} + \frac{2}{3}k \max\left\{0, f^{(1,2)}_{\max} + \max_{\Gamma_{(k,2k)}} g_h\right\}$$

We have the following result as a consequence of Theorem 2 and Remark 3.

Theorem 3: Let the condition of Theorem 2 hold and let $f(t,x) \ge 0$, $g(t,x) \ge 0$ and $u_0 \ge 0$. Then the discrete solution satisfies

$$u^n_i \ge 0 \qquad \forall\, n = 0, 1, \ldots, M\ ,\ i = 1, 2, \ldots, N_h$$

4. Numerical Experiment

Here, we give an example to verify Theorem 3. Globally continuous piecewise linear finite element functions based on triangulation described in Section 2 are used.

Example 2: We consider the nonlinear problem (1)−(3) on $\Omega \times (0,T]$, where $\Omega = (-1,1) \times (-1,1)$, $0 < T < \infty$ and the interface Γ is a circle centered at $(0,0)$ with radius 0.5. $\Omega_1 = \{(x,y) : x^2+y^2 < 0.25\}$, $\Omega_2 = \Omega \setminus \overline{\Omega}_1$. We choose

$$f = \begin{cases} x^2+y^2 & \text{in} \quad \Omega_1 \times (0,T] \\ 1 & \text{in} \quad \Omega_2 \times (0,T] \end{cases} , \qquad a = \begin{cases} \dfrac{u^2}{1+u^2} & \text{in} \quad \Omega_1 \times (0,T] \\ \dfrac{1}{1+u^2} & \text{in} \quad \Omega_2 \times (0,T] \end{cases}$$

$$u_0 = 0 \quad \text{in } \Omega\ , \qquad g = \exp(-t) \quad \text{on } \Gamma \times (0,T]$$

By Theorem 3, $u \ge 0$ (see Figure 2).

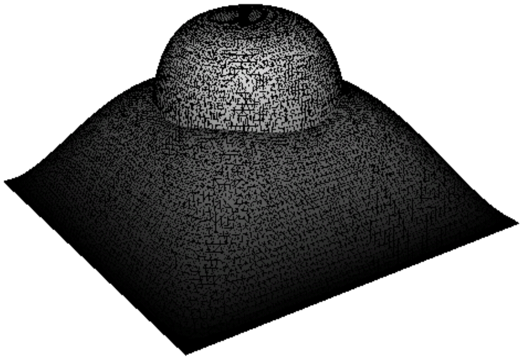

Fig. 2. The solution of Example 2 at $t = 5$ with $k = 0.01$ and $h = 0.0475216$.

4. CONCLUDING REMARKS

In this paper, we investigate the convergence of finite element solution for a nonlinear parabolic interface problem with time discretization based on three-step linearized scheme. Under certain conditions, the scheme was shown to be numerically stable and that higher order convergence in time could be obtained. The discrete solution is usually required to reproduce certain properties of the exact solution, we therefore show that the scheme preserves the maximum principle under certain conditions.

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