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Chaos control of 4D chaotic systems using recursive backstepping nonlinear controller

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Abstract

This paper examines chaos control of two four-dimensional chaotic systems, namely: the Lorenz–Stenflo (LS) system that models low-frequency short-wavelength gravity waves and a new four-dimensional chaotic system (Qi systems), containing three cross products. The control analysis is based on recursive backstepping design technique and it is shown to be effective for the 4D systems considered. Numerical simulations are also presented.
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1. Introduction

Control mechanism that enable a system to maintain a desired dynamical behaviour (the "goal" or "target") even when intrinsically chaotic have many applications ranging from biology to engineering [1–4]. Thus, it is of considerable interest and potential utility, to devise control techniques capable of achieving the desired type of behaviour in nonlinear and chaotic systems. The control of chaos and bifurcation is concerned with using some designed control input(s) to modify the characteristics of a parameterized nonlinear system. The control can be static or dynamic feedback control, or open-loop control. The objective can be the stabilization and reduction of the amplitude of bifurcation orbital solutions, optimization of a performance index near bifurcation, reshapening of the bifurcation diagram or a combination of these.

For over a decade, there has been intense research activities devoted to the design of effective control techniques [1–33]. A large number of the proposed methods are based on the Ott, Grebogi and Yorke (OGY) closed-loop feedback method [5] and the Pyragas time-delayed auto-synchronization (TDAS) method [6]. The OGY method seeks to use small perturbation to place chaotic orbits onto unstable periodic orbits [5]; and have been applied to some experimental systems [7–12] including the stabilization of pattern dynamics in a Taylor vortex flow with hourglass geometry [11] and control of chaotic Taylor–Coutte flow [12]. On the other hand, Pyragas TDAS method uses continuous time-delayed feedback [6]; and has been shown to be an efficient method that has been realized experimentally in electronic chaos oscillators [13], mechanical pendulums [14], lasers [15] and chemical systems [16].

Despite the successful implementation of these two basic schemes, some drawbacks have been identified. For instance, the restriction to stabilization of unstable periodic orbits (UPO) to stable periodic orbits (PO) in the OGY method neglects

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the fact that steady state solutions represent the most practical operation mode in many chaotic systems such as electronic oscillators [17] or lasers [18]. The TDAS, on the other hand depends on the torsion of neighbouring trajectories in the phase space [19]. In addition, stability analysis of delayed feedback systems is very difficult. To address these drawbacks and many others, numerous linear [20–23] and nonlinear [24–33] control methods have emerged over the years. In particular, backstepping recursive nonlinear control scheme has been employed recently for controlling and tracking chaotic systems [27–33]; because backstepping design can guarantee global stability, tracking and transient performance for a broad class of strick-feedback nonlinear systems [32,33]. The technique is a systematic design approach and consists in a recursive procedure that skillfully interlaces the choice of a Lyapunov function with the control.

In this paper, a simple backstepping-based control scheme is proposed for controlling four-dimensional (4D) chaotic systems. The 4D chaotic systems considered here are the Lorenz–Stenflo system (LS) [34–38] and a new 4D chaotic system recently proposed by Qi et al. [39]. Based on active control technique, we have recently studied the synchronization behaviour of these two systems [40]. The rest of the paper is organized as follows: In Section 2, we consider the control of the Lorenz–Stenflo system and in Section 3, we treat the Qi system. The paper is concluded in Section 4.

2. Controlling Lorenz–Stenflo system

2.1. The Lorenz–Stenflo system

Here, we consider the following four coupled nonlinear autonomous first order differential equations:

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) + \gamma x_4 \\
\dot{x}_2 &= x_1(r - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 \\
\dot{x}_4 &= -x_1 - \alpha x_4
\end{aligned} \tag{1}$$

which were formulated by Stenflo [34] from a low-frequency short-wavelength gravity wave equation. In (1), the dots denote time derivatives, r (>0), α (>0), γ (>0) and β (>0) are, respectively, the Rayleigh number, Prandtl number, rotation number and geometric parameter. System (1), named Lorenz–Stenflo (LS) system is similar to the famous Lorenz equations, but differ from it by the introduction of the new control parameter γ, and a new state variable x_4, describing the flow rotation. Thus, the generalized system (1) reduces to the Lorenz system in the absence of γ and x_4.

Some dynamical behaviours of the Lorenz–Stenflo equation are reported in [34–38,40], including the familiar period-doubling route to chaos [35,37]; and adaptive control and synchronization [38] and synchronization based on active control [40]. With the following parameters: $\alpha = 1.0$, $\beta = 0.7$, $\gamma = 1.5$ and $r = 26.0$, the LS system exhibits the chaotic motion.

2.2. Design of backstepping control

Let us consider an LS system given by:

$$\begin{aligned}
\dot{x}_1 &= \alpha(x_2 - x_1) + \gamma x_4 \\
\dot{x}_2 &= x_1(r - x_3) - x_2 \\
\dot{x}_3 &= x_1 x_2 - \beta x_3 + u(t) \\
\dot{x}_4 &= -x_1 - \alpha x_4
\end{aligned} \tag{2}$$

where $u(t)$ is a control function. Here, we aim at determining the controller $u(t)$ which is required to drive system (2) to a desired behaviour. We first define error states e_i $(i = 1,2,3,4)$

$$e_1 = x_1 - x_{1\mathrm{d}}, \quad e_2 = x_2 - x_{2\mathrm{d}}, \quad e_3 = x_3 - x_{3\mathrm{d}}, \quad x_4 = x_4 - x_{4\mathrm{d}}, \tag{3}$$

where $x_{1\mathrm{d}}$, $x_{2\mathrm{d}}$, $x_{3\mathrm{d}}$, and $x_{4\mathrm{d}}$ are desired states. For simplicity, let $x_{1\mathrm{d}} = 0$, $x_{2\mathrm{d}} = c_1 e_1$, $x_{3\mathrm{d}} = c_2 e_1 + c_3 e_2$, and $x_{4\mathrm{d}} = c_4 e_1 + c_5 e_2 + c_6 e_3$, where the c_i's are arbitrary control parameters to be chosen later. Using Eq. (3) in (2), it follows that the error dynamic equation can be written as

$$\begin{aligned}
\dot{e}_1 &= (\alpha(c_1 - 1) + \gamma c_4)e_1 + (\alpha + \gamma c_5)e_2 + \gamma(c_6 e_3 + e_4) \\
\dot{e}_2 &= e_1(r - e_3 - c_2 e_1 - c_3 e_2) - e_2 - c_1 e_1 - c_1 \dot{e}_1 \\
\dot{e}_3 &= e_1(e_2 + c_1 e_1) - \beta(e_3 + c_2 e_1 + c_3 e_2) - c_2 \dot{e}_1 - c_3 \dot{e}_2 + u(t) \\
\dot{e}_4 &= -e_1 - \alpha(e_4 + c_4 e_1 + c_5 e_2 + c_6 e_3) - (c_4 \dot{e}_1 + c_5 \dot{e}_2 + c_6 \dot{e}_3)
\end{aligned} \tag{4}$$

Considering the following Lyapunov function:

$$V = \frac{1}{2}\sum_{i=1}^{4} k_i e_i^2, \tag{5}$$

the time derivative of Eq. (5) is

$$\dot{V} = \sum_{i=1}^{4} k_i e_i \dot{e}_i = k_1 e_1 \dot{e}_1 + k_2 e_2 \dot{e}_2 + k_3 e_3 \dot{e}_3 + k_4 e_4 \dot{e}_4 \tag{6}$$

Substituting (4) in (6) and choosing the control parameters as: $c_1 = c_2 = c_4 = c_5 = c_6 = 0$ and $c_3 = 1$, we obtain the Lyapunov first derivative

$$\dot{V} = k_1 e_1(\alpha e_2 - \alpha e_2 + \gamma e_4) + k_2 e_2[e_1(r - e_3 - e_2) - e_2] + k_3 e_3[e_1 e_2 - \beta(e_3 + e_2) - [e_1(r - e_3 - e_2) - e_2] + u(t)] - k_4 e_4(e_1 + \alpha e_4). \tag{7}$$

To make $\dot{V}$ negative definite, we must choose the k_i's such that $\dot{V}$ is zero. Let $k_1 = k_2 = k_4 = 0$ and $k_3 = 1$. It follows that

$$u(t) = \beta(e_3 + e_2) + [e_1(r - e_3 - e_2) - e_2] - e_1 e_2 \tag{8}$$

satisfies the condition.

2.3. Numerical results

For the purpose of numerical simulation, we fix $\alpha = 1.0$, $\beta = 0.7$, $\gamma = 1.5$ and $r = 26.0$ to place the system in chaotic motion. Fig. 1 illustrates a typical chaotic orbits in the uncontrolled state. When the control is switched on it is clear from Fig. 2 that the chaotic behaviour has been controlled as soon as $u(t)$ is activated at $t = 150$. Thus, the control law given by Eq. (8) is effective.

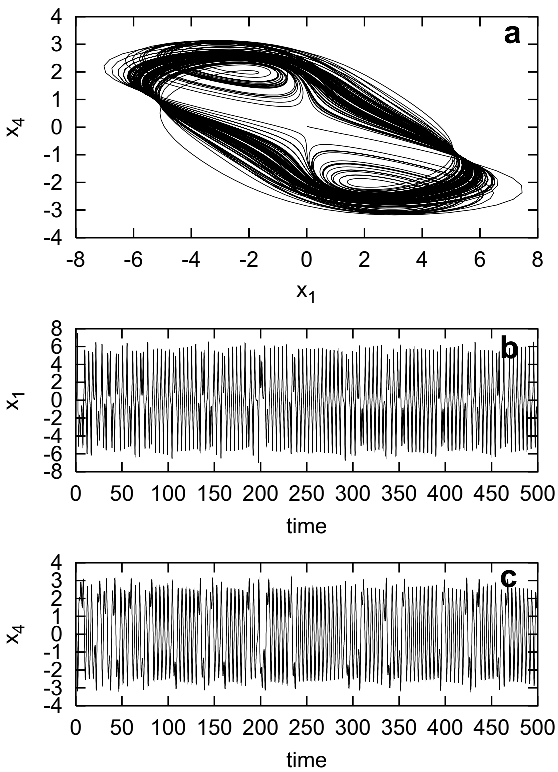

Fig. 1. Chaotic dynamics of the Lorenz–Stenflo system in the uncontrolled state: (a) the chaotic phase portrait, (b) time series of the x_1 variable and (c) time series of the x_4 variable. The parameters of the system are: $\alpha = 1.0$, $\beta = 0.7$, $\gamma = 1.5$ and $r = 26.0$.

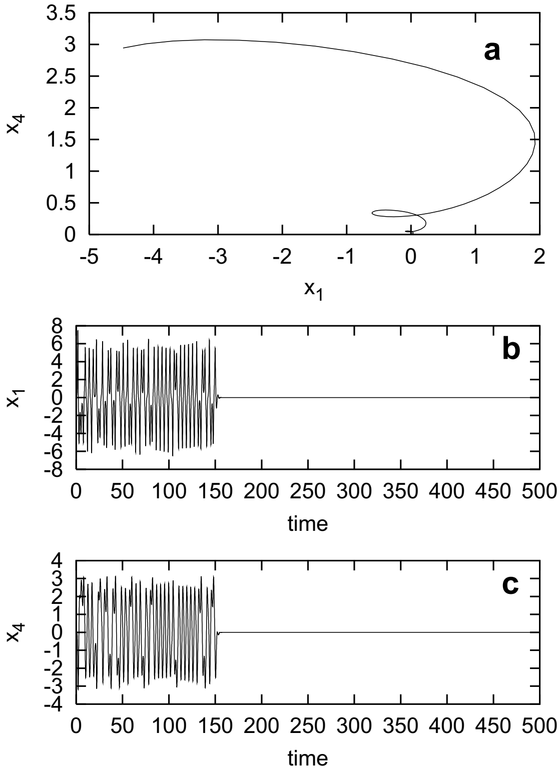

Fig. 2. Dynamics of the Lorenz–Stenflo system in controlled state when $u(t)$ has been activated: (a) the controlled phase portrait, (b) time series of the x_1 variable and (c) time series of the x_4 variable. The parameters of the system are as in Fig. 1.

3. Controlling Qi system

3.1. The Qi system

The second model system which we study is the following 4D autonomous system described by [39]

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4 \\ \dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4 \\ \dot{x}_3 &= -cx_3 + x_1x_2x_4 \\ \dot{x}_4 &= -dx_4 + x_1x_2x_3, \end{aligned} \tag{9}$$

where $x_1,\ x_2,\ x_3$ and x_4 are the state variables of the system and $a,\ b,\ c$ and d are all positive real constant parameters. System (9) was recently introduced by Qi et al. [39] and it has been shown to exhibit complex dynamical behaviour including the familiar period-doubling route to chaos as well as hopf bifurcations [39]. In Ref. [40], we presented an active control based synchronization scheme for the Qi system operated in the chaotic mode. For the system parameters: $a = 30,\ b = 10,\ c = 1$ and $d = 10$, the Qi model exhibits chaotic motion.

3.2. Design of backstepping control

Following Section 2, we choose a Qi system given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2x_3x_4 \\ \dot{x}_2 &= b(x_1 + x_2) - x_1x_3x_4 \\ \dot{x}_3 &= -cx_3 + x_1x_2x_4 + u(t) \\ \dot{x}_4 &= -dx_4 + x_1x_2x_3, \end{aligned} \tag{10}$$

where $u(t)$ is the control function to be determined. Using the definition of the error states as in Eq. (3), it follows that the error dynamics of system (10) can be written as

$$\begin{aligned}
\dot{e}_1 &= a(e_2 + c_1e_1 - e_1) + (e_2 + c_1e_1)(e_3 + c_2e_1 + c_3e_2)(e_4 + c_4e_1 + c_5e_2 + c_6e_3)\\
\dot{e}_2 &= b(e_1 + e_2 + c_1e_1) - e_1(e_3 + c_2e_1 + c_3e_2)(e_4 + c_4e_1 + c_5e_2 + c_6e_3) - c_1\dot{e}_1\\
\dot{e}_3 &= -c(e_3 + c_2e_1 + c_3e_2) + e_1(e_2 + c_1e_1)(e_4 + c_4e_1 + c_5e_2 + c_6e_3) - c_2\dot{e}_1 - c_3\dot{e}_2 + u(t)\\
\dot{e}_4 &= -d(e_4 + c_4e_1 + c_5e_2 + c_6e_3) + e_1(e_2 + c_1e_1 + c_3e_2)(e_3 + c_2e_1 + c_3e_2) - c_4\dot{e}_1 - c_5\dot{e}_2 - c_6\dot{e}_3.
\end{aligned} \tag{11}$$

Substituting Eq. (11) in Eq. (6) and choosing the control parameters as in the previous case, i.e. $c_1 = c_2 = c_4 = c_5 = c_6 = 0$ and $c_3 = 1$, we have

$$\begin{aligned}
\dot{V} &= k_1e_1[a(e_2 - e_1) + e_2e_4(e_2 + e_3)] + k_2e_2[b(e_1 + e_2) - e_1e_4(e_2 + e_3)] + k_3e_3[-c(e_2 + e_3) + e_1e_2e_4 - b(e_1 + e_2)\\
&\quad + e_1e_4(e_2 + e_3) + u(t)] + k_4e_4[2e_1e_2(e_2 + e_3) - ce_4]
\end{aligned}$$

To make $\dot{V}$ negative definite, we choose the k_i's such that $\dot{V}$ is zero. Let $k_1 = k_{\,2} = k_4 = 0$ and $k_3 = 1$. Thus, it follows that

$$u(t) = c(e_2 + e_3) - e_1e_2e_4 + b(e_1 + e_2) - e_1e_4(e_2 + e_3) \tag{13}$$

satisfies the required condition.

3.3. Numerical results

In the numerical simulations, we set the parameters of the Qi system as follows: $a = 30,\ b = 10,\ c = 1$ and $d = 10$. This ensures the chaotic behaviour shown in Fig. 3 when the control is deactivated. In Fig. 4, we activate the control at $t = 5$ and it is obvious that the chaotic behaviour has been controlled as soon as control is activated.

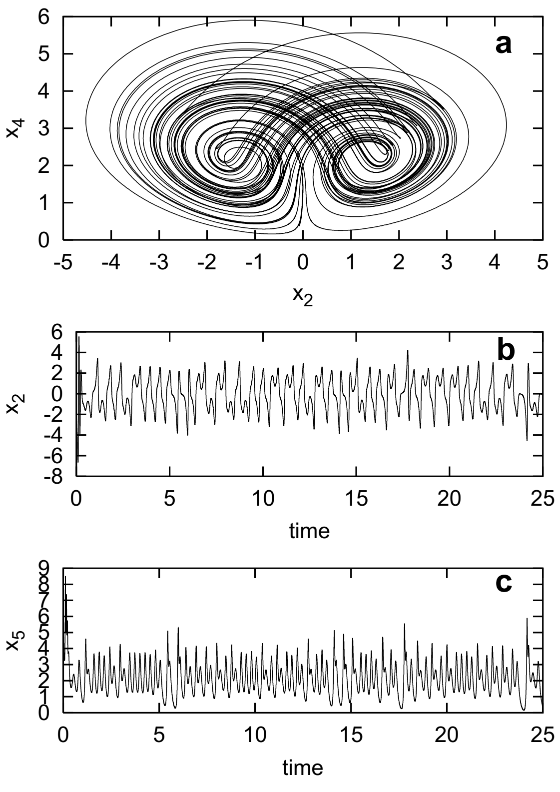

Fig. 3. Chaotic dynamics of the Qi system in the uncontrolled state: (a) chaotic phase portrait, (b) time series of the x_2 variable and (c) time series of the x_4 variable. The parameters of the system are $a = 30,\ b = 10,\ c = 1$ and $d = 10$.

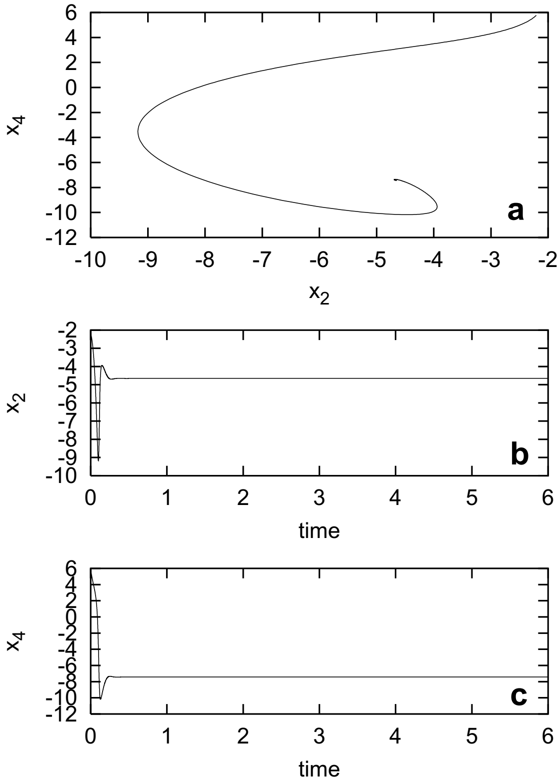

Fig. 4. Dynamics of the Qi system in controlled state when $u(t)$ has been activated: (a) the controlled phase portrait, (b) time series of the x_2 variable and (c) time series of the x_4 variable. The parameters of the system are as in Fig. 3.

4. Conclusion

This paper has examined chaos control in two different 4D chaotic systems, namely: Lorenz–Stenflo system and a new system which we call the Qi system by employing recursive backstepping approach. The presented numerical results shows that the backstepping control is very effective and can guarantee stability of the system about any desired point.

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