3 CONIC SECTIONS

Introduction: If a plane is made to cut a cone it gives a curve. This locus is generally referred to as conic section. Conic sections have interesting geometric properties. We discuss some of this and also illustrate some applications of conics in Physics.

Objectives: At the end of this lecture/chapter, the students/ reader should be able to:

i. Describe the various conic sections.
ii. Derive a conic as locus.
iii. Find the equation of a parabola.
iv. Find and describe the focus, directix and vertex of a parabola.
v. Obtain the coordinates of the center, vertices and foci of an ellipse and a hyperbola.
vi. Find the eccentricity and the length of the latus rectum of an ellipse and a hyperbola.
vii. Find the equations of the tangent and normal to a given conic section.

Pre-test

1. Find the equation of the locus of point which moves in the plane such that the ratio of its distance from the point (-1, 2) and the line $3x + 4y = 13$ is $\sqrt{10}$.

The general conic section is of the form

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

where the coefficients or parameters a, b, c, d, e, f are real numbers with at least one of $a, b,$ and c non-zero.

A conic section is a curve obtained as the intersection of the surface of a cone with a plane. The three types of conic sections are the parabola, the ellipse and the hyperbola.

Pictures of Conic Sections

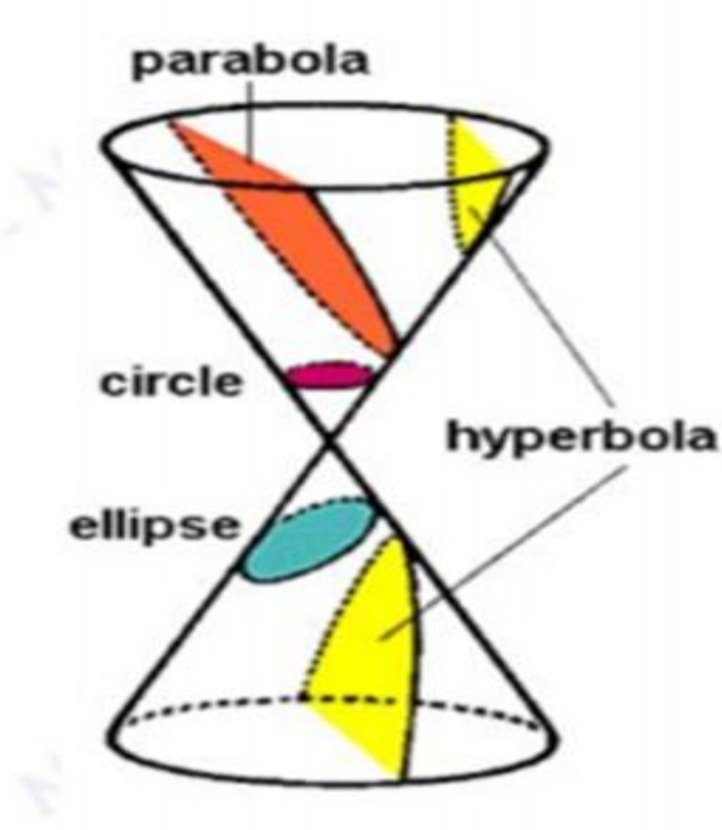


Fig. 3.1

3.1 CONIC AS LOCUS

A conic section is the curve traced out by a point P which moves in the plane in such a way that the ratio of its distance from a fixed point F to its distance from a fixed line (not containing F) is a constant.

The fixed point F is called its focus, the fixed line its directrix and the constant ratio its eccentricity(e).

Example 1: Find the equation of the conic section with focus at the point (-1, 2) and eccentricity e $= \sqrt{10}$ if its directrix is the line $3x + 4y = 13$.

<u>Solution</u>

Let $P(\xi, \eta)$ be a variable point on the conic.

$$|PF| = \sqrt{(\xi + 1)^2 + (\eta - 2)^2}$$

Distance of P from the directrix K

$$= \frac{|3\xi + 4\eta - 13|}{\sqrt{3^2 + 4^2}}$$

By definition,

$$\frac{|\mathrm{PF}|}{|\mathrm{PK}|} = \mathrm{e}$$

$$\Rightarrow \frac{|\mathrm{PF}|^2}{|\mathrm{PK}|^2} = \mathrm{e}^2$$

$$(\xi + 1)^2 + (\eta - 2)^2 = 10\left(\frac{(3\xi + 4\eta - 13)^2}{25}\right)$$

$$5(\xi^2 + 2\xi + 1 + \eta^2 - 4\eta + 4) = 2(9\xi^2 + 16\eta^2 + 24\xi\eta - 78\xi - 104\eta + 169)$$

$$\Rightarrow 13\xi^2 + 48\xi\eta + 27\eta^2 - 166\xi - 188\eta + 313 = 0$$

Using x, y instead of ξ, η we have

$$13x^2 + 48xy + 27y^2 - 166x - 188y + 313 = 0$$

3.2 PARABOLA

A parabola is a locus of points, equidistant from a given point, called the focus, and from a given line called the directrix.

In other words, its distance from a fixed point called the focus, and its distance from a fixed line called its directrix are in constant ratio (eccentricity). This implies that the eccentricity (e) =1.

<u>Equation of a Parabola</u>

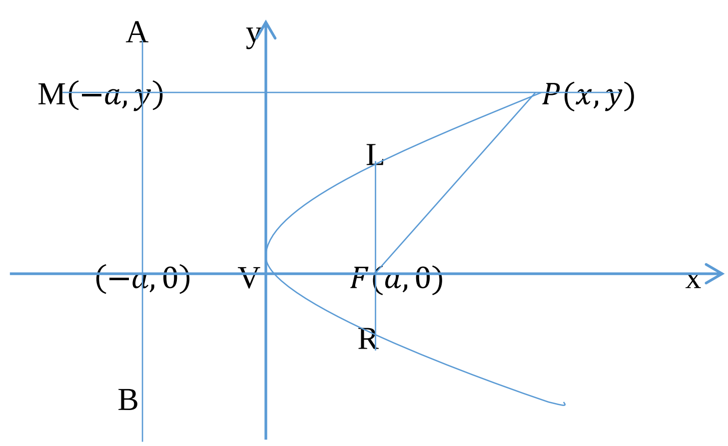


Fig. 3.2

In the figure above, the focus is F(a, 0) and the directrix is line AB. The distance of AB from the y-axis is a.

Let P(x, y) be an arbitrary point on the parabola, then by definition

$$|PF| = |PM|$$

i. e $\sqrt{(x-a)^2 + y^2} = \sqrt{(x+a)^2}$

$\Rightarrow (x-a)^2 + y^2 = (x+a)^2$

$x^2 - 2ax + a^2 + y^2 = x^2 + 2ax + a^2$

$y^2 = 4ax$

The equation of the parabola is

$$y^2 = 4ax \qquad (1)$$

The focus is $(a, 0)$.

The directrix is the line $x = -a$.

Vertex is $(0, 0)$

The line segment through the focus, and perpendicular to the axis of symmetry, and with endpoints L and R is called the lactus rectum. The length is $|4a|$.

The graph is symmetric with respect to the x −axis

The parabola opens leftward if < 0 .

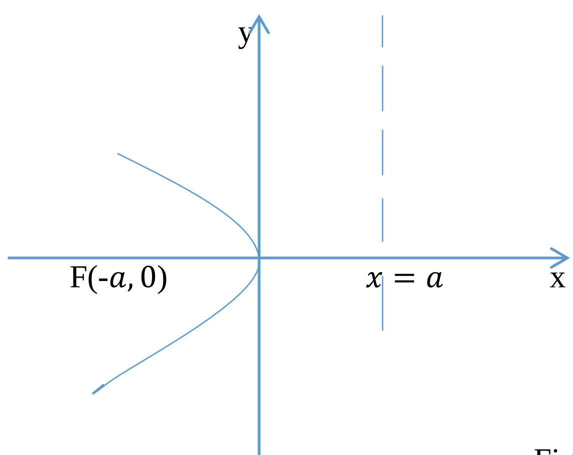


Fig. 3.3

If we interchange x and y in (1), we obtain

$x^2 = 4ay$ (2)

Focus is $(0, a)$

Directrix is $y = -a$

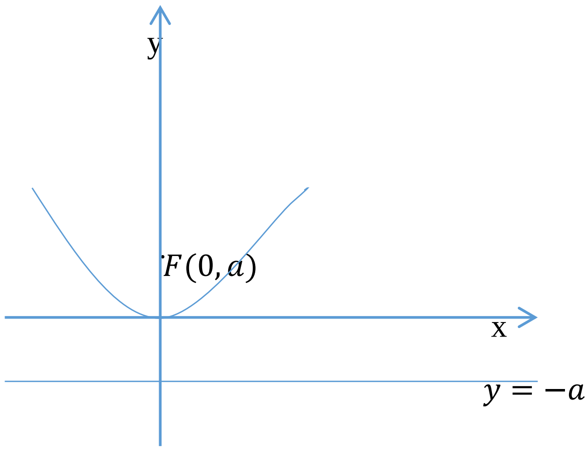


Fig. 3.4

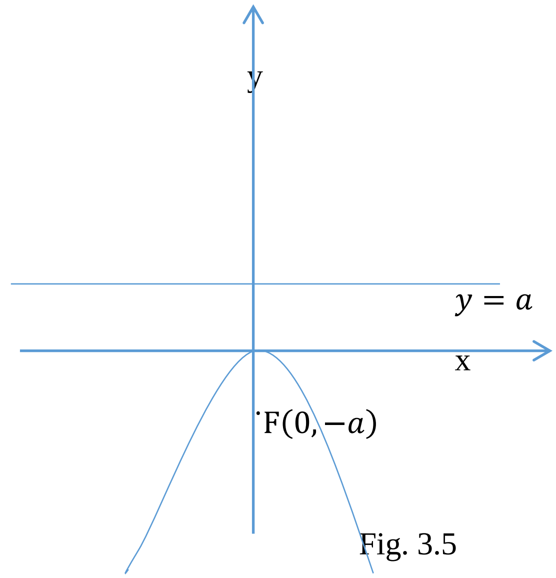


Fig. 3.5

Example 2: Find the focus, directrix and the length of the lactus rectum of the parabola $y^2 + 10x = 0$ and sketch the graph.

<u>Solution</u>

$$y^2 + 10x = 0$$

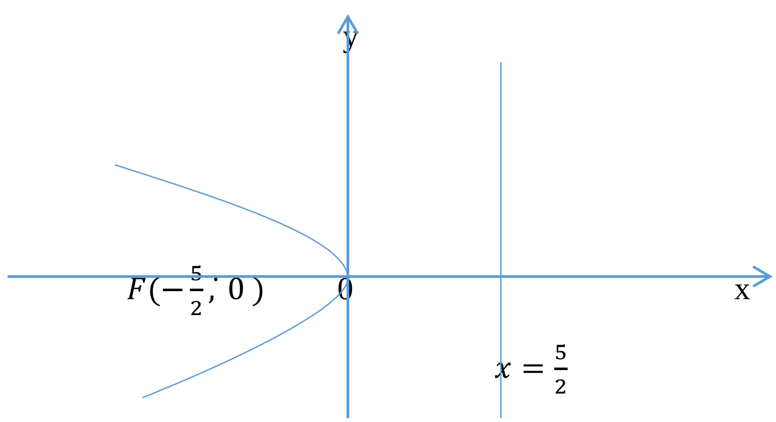
$$y^2 = -10x$$

If we compare this with equation (1),

$$4a = -10 \text{ i.e } a = -\frac{5}{2}$$

Focus is $\left(-\frac{5}{2}, 0\right)$ and the directrix is $x = \frac{5}{2}$.

Length of the latus rectum = 10



Example 3: Find the equation of the parabola whose vertex is the origin and whose focus is (3, 0).

Solution

$$y^2 = 4(3)x$$

$$= 12x$$

The case of translation

Here we consider the case where the vertex (0,0) of the parabola $y^2 = 4ax$ is translated to the point (x_1, y_1). In this case, the equation becomes

$(y - y_1)^2 = 4a(x - x_1)$ (3)

So that the focus now is $(a + x_1, y_1)$ and the directrix is $x = -a + x_1$.

Equation (3) is called the standard or canonical form.

Example 4: Write down the equation of the parabola $x^2 - 4x - 12y + 40 = 0$ in its canonical form and hence find

(i) the vertex;

(ii) the focus;

(iii) the directrix.

Solution

$$x^2 - 4x - 12y + 40 = 0$$

$$x^2 - 4x = 12y - 40$$

$$x^2 - 4x + 4 = 12y - 36$$

$$(x - 2)^2 = 12(y - 3)$$

(i) Vertex is (2, 3)

(ii) $4a = 12 \Rightarrow a = 3$

Focus is $(x_1, a + y_1) = (2, 6)$

(iii) Directrix is $y = -3 + 3 = 0$

Example 5: Find the vertex, focus and the directrix of the parabola $4(y + 5)^2 - 3x + 2 = 0$.

Solution

$$(y+5)^2 = \frac{3x-2}{4}$$

$$= \frac{3}{4}\left(x - \frac{2}{3}\right)$$

$\Rightarrow 4a = \frac{3}{4}, a = \frac{3}{16}$

Vertex is $\left(\frac{2}{3}, -5\right)$

Focus is $\left(\frac{3}{16} + \frac{2}{3}, -5\right) = \left(\frac{41}{48}, -5\right)$

Directrix is $x = -\frac{3}{16} + \frac{2}{3} = \frac{23}{48}$

Equations of the tangent and normal to $y^2 = 4ax$ at the point (x_1, y_1)

$$y^2 = 4ax$$

Gradient of the tangent to $y^2 = 4ax$ at (x_1, y_1) is $\frac{dy}{dx}|_{(x_1,y_1)}$

Differentiating (1) implicitly with respect to x.

$$2y\frac{dy}{dx} = 4a$$

$$\frac{dy}{dx} = \frac{4a}{2y}$$

$$\frac{dy}{dx}|_{(x_1,y_1)} = \frac{4a}{2y_1} = \frac{2a}{y_1}$$

Using the slope-one point of the equation of a line, we obtain the equation of the tangent as

$$(y - y_1) = \frac{2a}{y_1}(x - x_1)$$

$$yy_1 - {y_1}^2 = 2ax - 2ax_1$$

Since (x_1, y_1) is on $y^2 = 4ax$ it implies that

$${y_1}^2 = 4ax_1$$

Thus

$$yy_1 - 4ax_1 = 2ax - 2ax_1$$

$$yy_1 = 2ax + 2ax_1$$

$$yy_1 = 2a(x + x_1)$$

Equation of the tangent to $y^2 = 4ax$ at (x_1, y_1) is

$yy_1 = 2a(x + x_1)$ (4)

Since the gradient of the tangent is $\frac{2a}{y_1}$, the gradient of the normal $= -\frac{y_1}{2a}$.

Equation of the normal is

$$(y - y_1) = -\frac{y_1}{2a}(x - x_1)$$

$$2ay - 2ay_1 = -xy_1 + xx_1$$

$$2ay + xy_1 = 2ay_1 + xx_1 \qquad (5)$$

Example 6: Find the equation of the tangent to the parabola $y^2 = 12x$ at the point (3,6).

Solution

$$y(6) = 2(3)(x + 3)$$

$$6y = 6x + 18$$

$$y = x + 3$$

Example 7: Find the equation of the normal to the parabola $y^2 + 3y - 3x + 5 = 0$ at the point $(1, -1)$.

Solution

Differentiate implicitly with respect to x.

$$2y\frac{dy}{dx} + 3\frac{dy}{dx} - 3 = 0$$

$$(2y + 3)\frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{3}{2y + 3}$$

$$\frac{dy}{dx}|_{(1,-1)} = 3$$

Equation of the normal at (1, -1) is

$$(y + 1) = -\frac{1}{3}(x - 1)$$

$$3y + 3 = -x + 1$$

$$3y + x + 2 = 0$$

3.3 ELLIPSE

An ellipse is the set of points in a plane the sum of whose distance from two fixed points F_1 and F_2 is a constant. This is illustrated in the figure below.

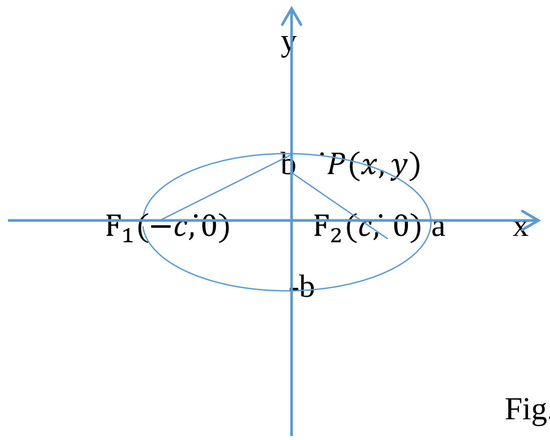


Fig. 3.6

The two fixed points are called the foci (plural of focus). To obtain the simplest equation for an ellipse, let the foci be on the x −axis at the points $(-c, 0)$ and $(c, 0)$ so that the origin is half the way between the foci.

Let the sum of the distances from a fixed point on the ellipse to the foci be $2a > 0$ and P(x, y) be point on the ellipse. Then,

$$|PF_1| + |PF_2| = 2a$$

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}$$

Squaring both sides, we get

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+a)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

squaring both sides again,

$$a^2(x^2 + 2cx + c^2) + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$\Rightarrow \ (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

From $\Delta\, PF_1F_2$ in the above figure, we see that $2c < 2a$, so $c < a$ and therefore $a^2 - c^2 > 0$.

Let $b^2 = a^2 - c^2$, then the equation becomes

$$b^2x^2 + a^2y^2 = a^2b^2$$

Divide both sides by a^2b^2

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (6)

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$.

The x −intercepts are found by setting $y = 0$. This gives

$$\frac{x^2}{a^2}=1$$

$$x^2 = a^2$$

$$\Rightarrow x = \pm a$$

Similarly, the y −intercepts are found by setting $x = 0$. This yields $y = \pm b$.

So the vertices are $(\pm a, 0)$. $(0, \pm b)$ are the co − vertices.

The line segment joining the vertices $(a, 0)$ and $(-a, 0)$ is called the major axis.

Equation (6) is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the eclipse is symmetric about both axes.

In summary, the ellipse

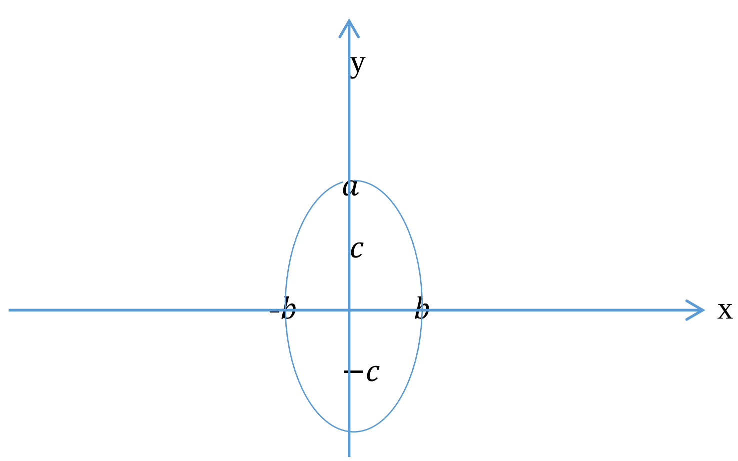
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad a \geq b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$ and vertices $(\pm a, 0)$. Co − vertices are $(0, \pm b)$.

If the foci of an ellipse are located on the y −axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in (6)

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad a \geq b > 0$$

with foci $(0, \pm c)$, vertices $(0, \pm a)$ and co-vertices $(\pm b, 0)$



a Fig. 3.7

Example 8: Sketch the graph $9x^2 + 16y^2 = 144$ and locate the foci.

<u>Solution</u>

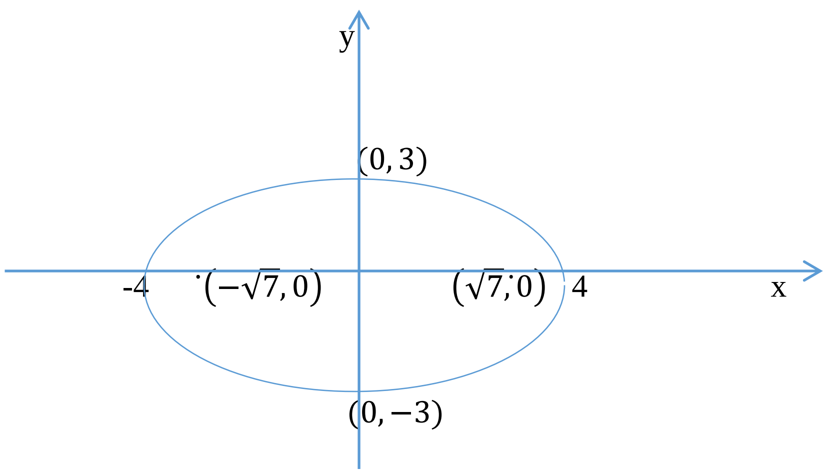
$$9x^2 + 16y^2 = 144$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$a^2 = 16, b^2 = 9, c^2 = 16 - 9 = 7$$

Foci are $\left(\pm\sqrt{7}, 0\right)$

Vertices are $(\pm 4, 0)$



<u>Change of Origin</u>

If the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is transferred to the point (x_1, y_1) in such a way that the axes are parallel to the $x$and y axes, then the new canonical form of the equation of the ellipse becomes

$$\frac{(x - x_1)^2}{a^2} + \frac{(y - y_1)^2}{b^2} = 1$$

Foci are $(\pm c + x_1, y_1)$

Vertices are $(\pm a + x_1, y_1)$ and the co-vertices are $(x_1, \pm b + y_1)$

Example 9: Write the equation of the ellipse $5x^2 + 9y^2 - 20x + 36y + 11 = 0$ in the canonical form and hence, determine:

i) the coordinates of the centre of the ellipse;

ii) the vertices of the ellipse;

iii) the two foci of the ellipse.

Solution

$$5x^2 + 9y^2 - 20x + 36y + 11 = 0$$

$$5x^2 - 20x + 9y^2 + 36y = -11$$

$$5(x^2 - 4x) + 9(y^2 + 4y) = -11$$

$$5(x^2 - 4x + 4) - 20 + 9(y^2 + 4y + 4) - 36 = -11$$

$$5(x - 2)^2 + 9(y + 2)^2 = 45$$

$$\frac{(x-2)^2}{9} + \frac{(y+2)^2}{5} = 1$$

Compare with the canonical form to obtain $a = 3, b = \sqrt{5},$ and $c = 2$.

i) Centre is $(2, -2)$

ii) Vertices are $(\pm 3 + 2, -2)$

$$= (5, -2), (-1, -2)$$

iii) Foci are $(\pm 2 + 2, -2) = (4, -2)$ and $(0, -2)$.

Eccentricity and Directrices of an Ellipse

An ellipse can also be defined as the locus of a moving point P, such that the ratio (eccentricity) of its distance from a fixed point (focus F) to its distance from a fixed line (directrix) is a constant. Consider $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,\ a \geq b > 0$. The eccentricity '$e$' is obtained using the formula

$$b^2 = a^2(1 - e^2) \qquad ; a \geq b > 0 \qquad (7)$$

and satisfies $0 < e < 1$.

In this case, the foci are $(\pm ae, 0)$ and the directrices are $x = \pm \frac{a}{e}$.

If the centre is transferred to (x_1, y_1) as in the case of equation (2), then the

foci are $(\pm ae + x_1, y_1)$

directrices are $x = \pm \frac{a}{e} + x_1$

Example 10: For the ellipse $\frac{(2x+1)^2}{4} + \frac{(y-1)^2}{2} = 1$, determine the coordinates of its centre, foci and vertices. Obtain the equations of its directrices.

Solution

$$\frac{(2x+1)^2}{4}+\frac{(y-1)^2}{2}=1$$

$$\frac{\left(x+\frac{1}{2}\right)^2}{1}+\frac{(y-1)^2}{2}=1$$

$$a^2=2, a=\sqrt{2} \text{ and } b=1$$

Using (7), we see that $e=\frac{1}{\sqrt{2}}$ so that $ae=\sqrt{2}\times\frac{1}{\sqrt{2}}=1$ and $\frac{a}{e}=\sqrt{2}\times\sqrt{2}=2$

Centre is $\left(-\frac{1}{2},1\right)$

Foci= $(x_1,\pm ae+y_1)=\left(-\frac{1}{2},\pm1+1\right)=\left(-\frac{1}{2},2\right)$ and $\left(-\frac{1}{2},0\right)$

Directrices are $y=\pm\frac{a}{e}+y_1$

$$y=\pm2+1 \text{ i.e } y=3 \text{ and } y=-1$$

Tangent and Normal at a point

Equation of the tangent at $(x_1,y_1)$to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Gradient of the tangent = $\frac{dy}{dx}|_{(x_1,y_1)}$

Differentiating $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$implicitly wrt x,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\frac{dy}{dx}|_{(x_1,y_1)}=-\frac{b^2x_1}{a^2y_1}$$

The equation of the tangent is

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2yy_1-a^2{y_1}^2=-b^2xx_1+b^2{x_1}^2$$

$$a^2yy_1+b^2x{x_1}^2-a^2{y_1}^2-b^2{x_1}^2=0$$

$$\Rightarrow a^2yy_1+b^2xx_1-a^2b^2=0 \qquad (\text{Since } b^2{x_1}^2+a^2{y_1}^2=a^2b^2)$$

$$\therefore\ a^2yy_1 + b^2xx_1 = a^2b^2$$

Divide through by a^2b^2 to obtain

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

Equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad (8)$$

Gradient of the normal at $(x_1, y_1) = \frac{a^2y_1}{b^2x_1}$

Equation of the normal is

$$y - y_1 = \frac{a^2y_1}{bx_1}(x - x_1)$$

$$b^2x_1y - b^2x_1y_1 = a^2y_1x - a^2x_1y_1$$

$$\Rightarrow a^2xy_1 - b^2x_1y = (a^2 - b^2)x_1y_1$$

Equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$at the point (x_1, y_1) is

$$a^2xy_1 - b^2x_1y = (a^2 - b^2)x_1y_1 \quad (9)$$

Example 11: Find the equation of the tangent and normal to the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$ at (2,3).

<u>Solution</u>

Equation of the tangent is

$$\frac{2x}{16} + \frac{3y}{12} = 1$$

$$\frac{x}{8} + \frac{y}{4} = 1$$

$$x + 2y = 8$$

The gradient of the tangent is $-\frac{1}{2}$ so that of the normal is 2. The equation of the normal is

$$y - 3 = 2(x - 2)$$

$$y - 2x + 1 = 0$$

3.3 HYPERBOLA

A hyperbola is the set of all points in a plane the difference of whose distances from two fixed points $F_1$and F_2(the foci) is a constant.

Let the sum of the distances be $2a > 0$ and $P(x, y)$ be point on the hyperbola. Let the foci be $(\pm c, 0)$, then

$$|PF_1| - |PF_2| = 2a$$

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a$$

$$\sqrt{(x-c)^2 + y^2} = \sqrt{(x+c)^2 + y^2} - 2a$$

Squaring both sides, we get

$$x^2 - 2cx + c^2 + y^2 = x^2 + 2cx + c^2 + y^2 - 4a\sqrt{(x+c)^2 + y^2} + 4a^2$$

which simplifies to

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx$$

squaring both sides again,

$$a^2(x^2 + 2cx + c^2) + a^2y^2 = a^4 + 2a^2cx + c^2x^2$$

$$\Rightarrow\ (a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

Divide through by $c^2 - a^2$to get

$$\frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} = 1$$

Putting $b^2 = c^2 - a^2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Hence the equation of a hyperbola in the canonical form is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (9)$$

The x −intercepts are $\pm a$ and the points $(\pm a, 0)$ are the vertices of the hyperbola. If we substitute $x = 0$ in equation (9) we get $y^2 = -b^2$, which is impossible, so there is no y −intercept. The hyperbola is symmetric with respect to both axes.

From (9),

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geq 1$$

This implies that $x^2 \geq a^2$ i.e. $x \geq a$ or $x \leq -a$. This means that the hyperbola consists of two parts, called its branches.

<u>ASYMPTOTES OF THE CURVE</u>

From (9),

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 1$$

$$\therefore \ \frac{x}{a} - \frac{y}{b} = \frac{ab}{bx + ay}$$

$$= \frac{\frac{ab}{x}}{b + \frac{ay}{x}} \text{ or } \frac{\frac{ab}{y}}{\frac{by}{x} + a} \qquad (10)$$

So that as both x and y tends to infinity, the R.H.S. of (10) tends to 0. This implies that the L.H.S. also tends to zero.

It then follows that the hyperbola has two asymptotes which are given as

$$y = \frac{b}{a}x \text{ and } y = -\frac{b}{a}x$$

If the foci are on the y −axis, then the equation is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Foci are $(0, \pm c)$, vertices are $(0, \pm a)$ and the asymptotes are $y = \pm \frac{a}{b}x$.

Example 12: Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

<u>Solution</u>

$$9x^2 - 16y^2 = 144$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

So $a = 4, b = 3$ and $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$

Foci are $(\pm 5, 0)$

The asymptotes are the lines $y = \pm \frac{3}{4}x$

Example 13: Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptotes $y = 2x$.

Solution

From the question,

$$a^2 = 1 \text{ and } \frac{a}{b} = 2$$

$$\therefore b = \frac{1}{2}$$

Also, $c = \sqrt{1^2 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

Foci = $\left(0, \pm\frac{\sqrt{5}}{2}\right)$

Therefore the equation is $y^2 - 4x^2 = 1$

If the centre (origin) of the hyperbola is changed to the point (x_1, y_1), the new equation of the hyperbola is

$$\frac{(x - x_1)^2}{a^2} - \frac{(y - y_1)^2}{b^2} = 1$$

Vertices are $(\pm a + x_1, y_1)$

Foci are $(\pm c + x_1, y_1)$

Directrices and Eccentricity of a Hyperbola

In the case of a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the eccentricity $e > 1$. The eccentricity is obtained using the formula

$$b^2 = a^2(e^2 - 1) \qquad (11)$$

The foci are $(\pm ae, 0)$ and the directrices are $x = \pm\frac{a}{e}$.

Example 14: Find the foci and the equations of the directrices of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

Solution

$a = 4, b = 3$ and so $e = \frac{5}{4}$

$\therefore$ Foci $= (\pm 5, 0)$

Directrices are $x = \pm\frac{16}{5}$

Tangent and Normal at a point (x_1, y_1)

The equation of the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

At the point (x_1, y_1) is

$$\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1 \qquad (12)$$

Equation of the normal at the point (x_1, y_1) is

$$(y - y_1) = -\frac{a^2}{b^2}\frac{y_1}{x_1}(x - x_1) \quad (13)$$

Example 15: Find the equation of the tangent and normal to the hyperbola $9x^2 - 4y^2 = 36$ at the point $(-2,0)$.

<u>Solution</u>

$$9x^2 - 4y^2 = 36$$

$$\therefore \frac{x^2}{4} - \frac{y^2}{9} = 1$$

$\therefore a = 2$ and $b = 3$.

The equation of the tangent at $(-2, 0)$ is

$$\frac{-2x}{4} - \frac{0}{9} = 1$$

$$x = -2$$

Gradient of the normal =0

$\therefore$Equation of the normal is $y = 0$.

<u>Post-test</u>

1) The focus of the parabola $y^2 = 4(x - 2)$ is
2) Find the vertex of the parabola $x^2 4x - 12y + 40 = 0$
3) The equation of an ellipse is $\frac{x^2}{25} + \frac{y^2}{16} = 1$.
   (i) Find the foci of the ellipse.

   (ii) Determine the directrices of the ellipse.
   (iii) Find the vertices of the ellipse.
   (iv) Calculate the eccentricity of the ellipse.

4) For each of the following hyperbolas, find the coordinates of the centre, vertices, and foci; the length of the transverse and conjugate axes; the length of the latus rectum; the

eccentricity; and the equations of the directrices and the equations of the directrices and asymptotes. Sketch each curve.

a) $4x^2 - 9y^2 = 36$
b) $x^2 - 4y^2 + 6x + 16y - 11 = 0$

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