PHY 303
ELECTROMAGNETIC THEORY

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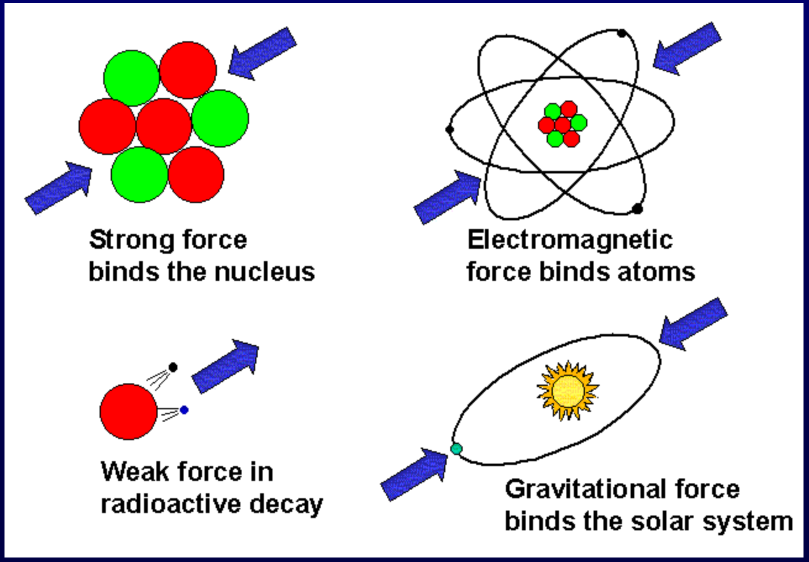
Why Study Electromagnetism ?

- **Four fundamental forces in nature:**
  - **Gravity** - matter always attracts
  - **Electromagnetic** - holds atoms together
  - **Strong nuclear** - binds atomic nucleus together
  - **Weak nuclear** - allows nuclear reactions

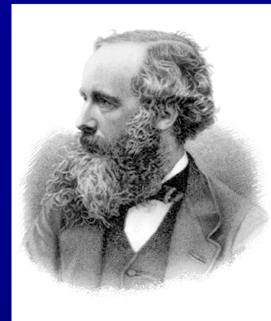
    Last two are very short-range (sub-atomic ~ 10^{-15} m)



Maxwell's Equations

The complete set of laws which govern electromagnetism

- relate electric and magnetic fields and flux
- explain the forces which act upon charges
- explain electromagnetic waves



Gauss' law:

– E field diverges and converge from charges

Gauss' law for magnetism:

- B field does not diverge or converge from a point

Faraday's law:

- E field lines encircle regions of changing **B**

Ampere's law:

- B field lines encircle regions in which current flows or **E** is changing

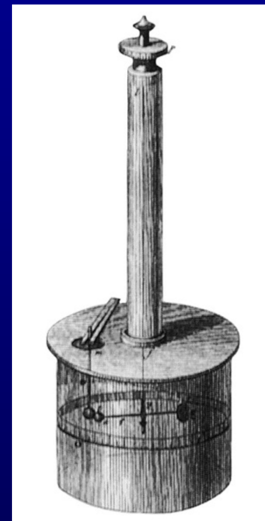
The Electric Charge: Can be Positive or Negative

- Charges exert a force proportional to separation
- Like charges repel
- Unlike charges attract
- SI Unit of charge is the Coulomb
- Ratio charge/mass for electron much larger than that of smallest ion ($m_p/m_e = 1836$)

Key concept: Coulombs Law

- Charles Augustin Coulomb (1736 - 1806)
- Describes the electric force between two charged particles Q_1 and Q_2 a distance r_{12} apart:

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r_{12}^2}\vec{r}$$



- ε_0 is the **permittivity** of free space
  ($8.85 \times 10^{-12}\ C^2\ N^{-1}\ m^{-2}$)
- Tipler uses $k = [1/(4\pi\varepsilon_0)]$, the "Coulomb Constant"
- **N.B. Force repulsive if charges of same sign**

Electric force

- The electric force is a **central force**:
  - force directed along line between charges
  - magnitude depends only on distance r

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r_{12}^2}\vec{r}$$

- The Coulomb is the charge carried past a point in a circuit by 1 Amp flowing for 1 second
- Electron only carries a charge of 1.6×10^{-19} C
- For two 1 C charges 1 m apart, force = 9×10^{9} N !!

Key concept: Electric field

- Field: a quantity that can be associated with a position
  - Either vector field (e.g. electric) or scalar (e.g. temperature)
- The electric field **E** at point P due to a charge Q is the electric force exerted by that charge on a test particle divided by the (small) charge q_0 on the test particle:

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{r}$$

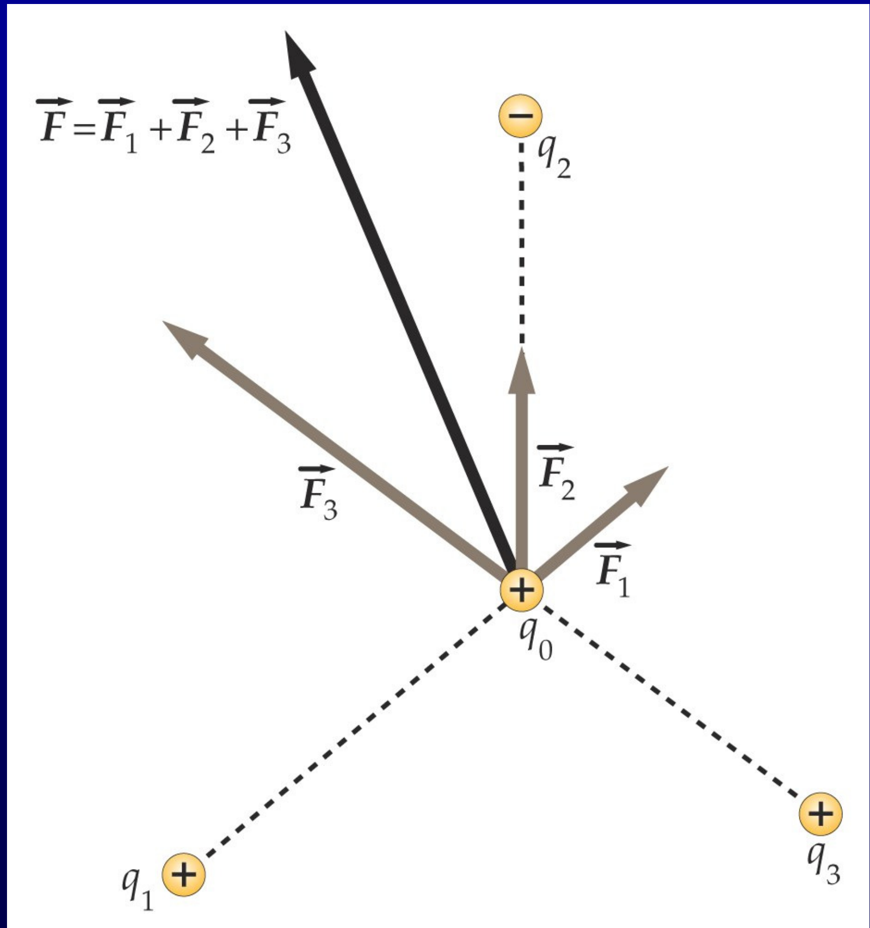
(i.e. $F = Eq_0$)

- For a distribution of charges Q_1, Q_2, ... Q_i use the **principle of superposition** to get **F** and/or **E**

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum \frac{Q_i}{r_i^2}\vec{r}_i$$

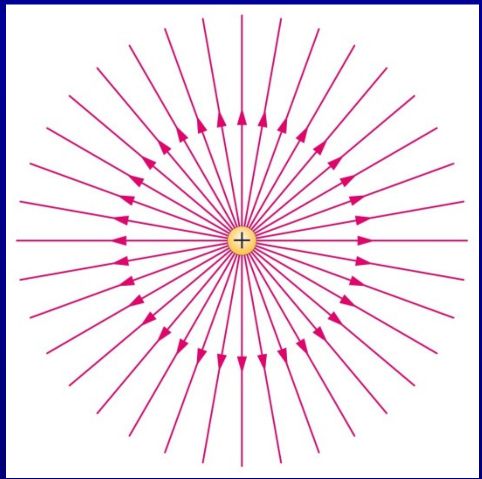
Electric field

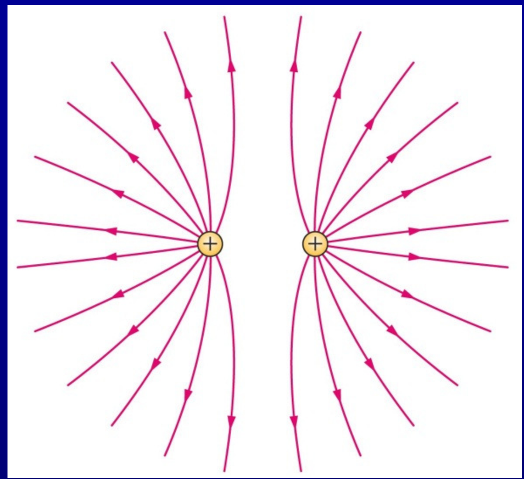
E points away from/towards a positive/negative charge

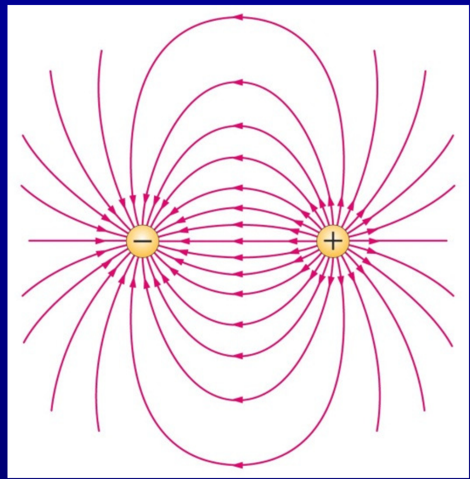


Use whatever coordinate system is convenient

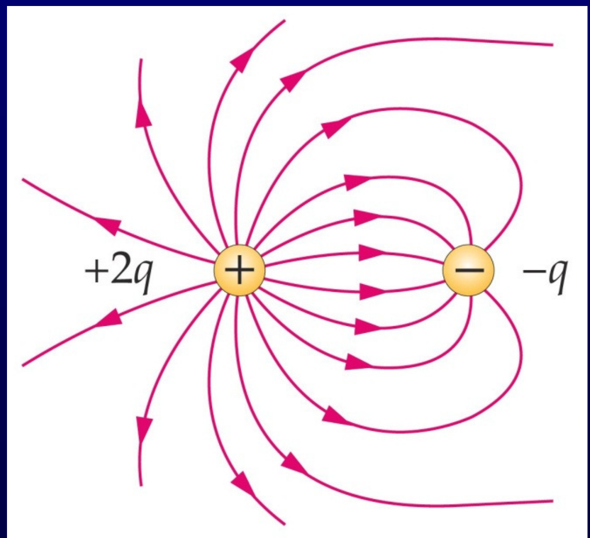
Drawing electric field lines





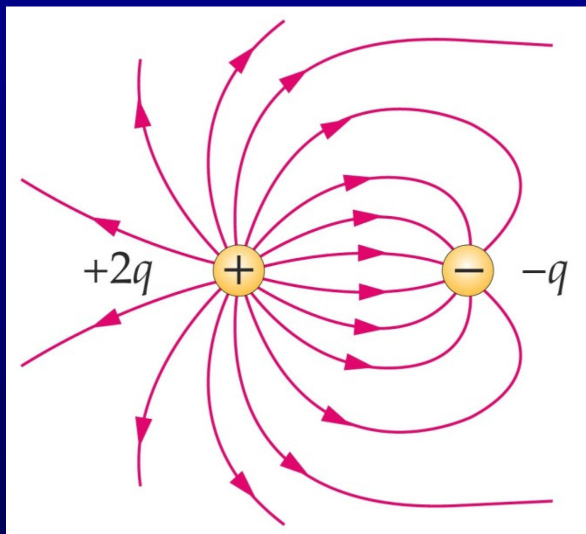


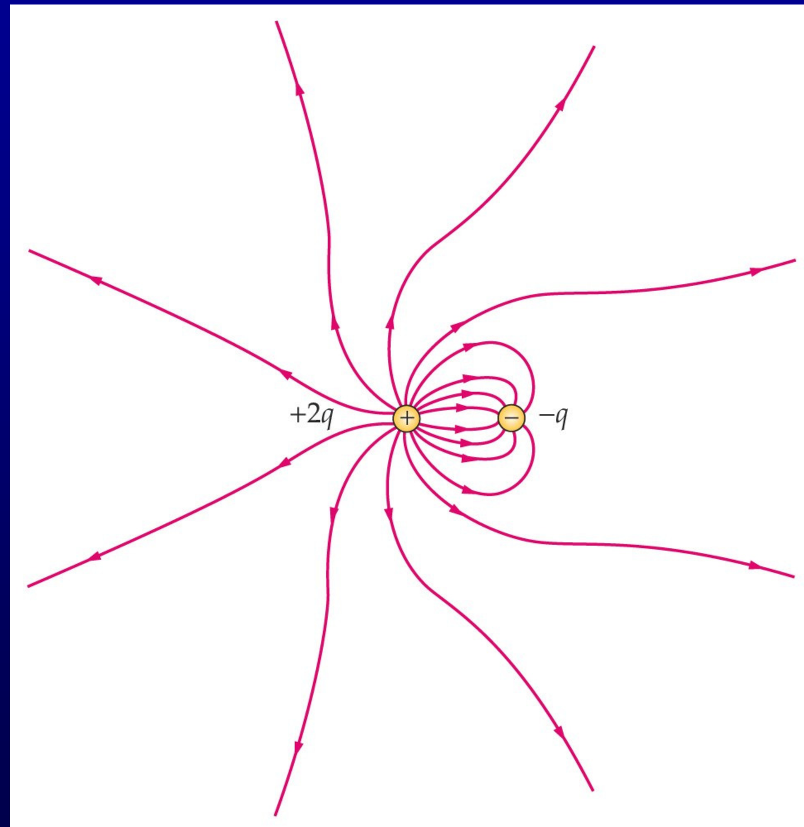
Field lines do not cross



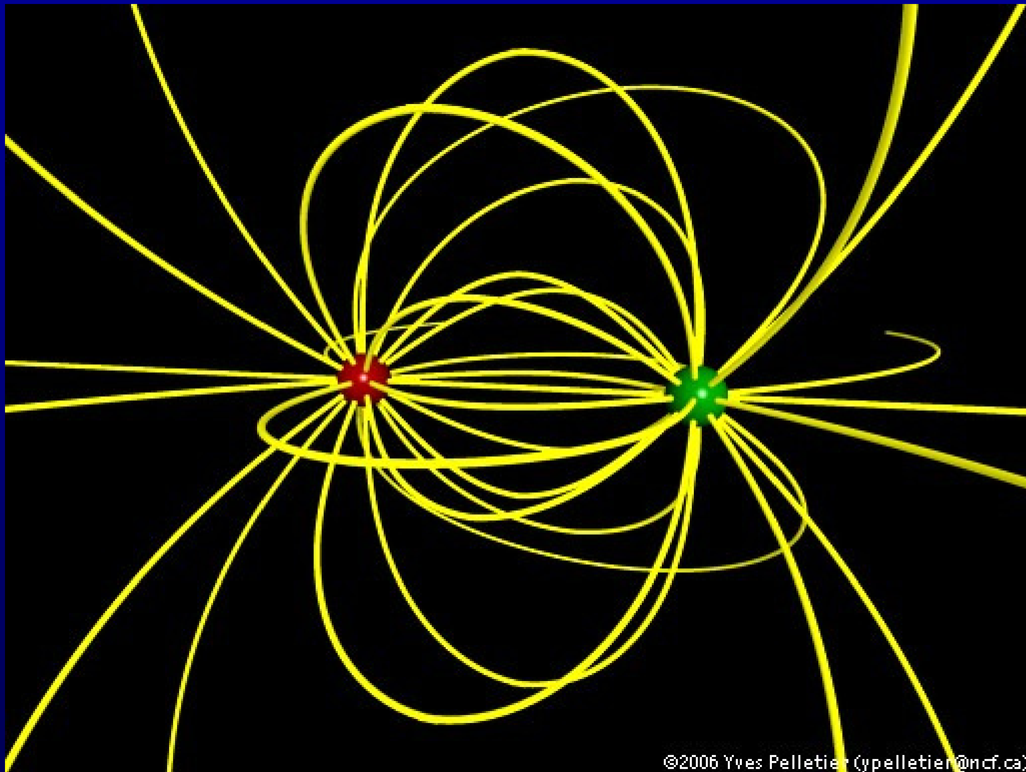
Use line spacing to indicate field strength

Move out a bit…





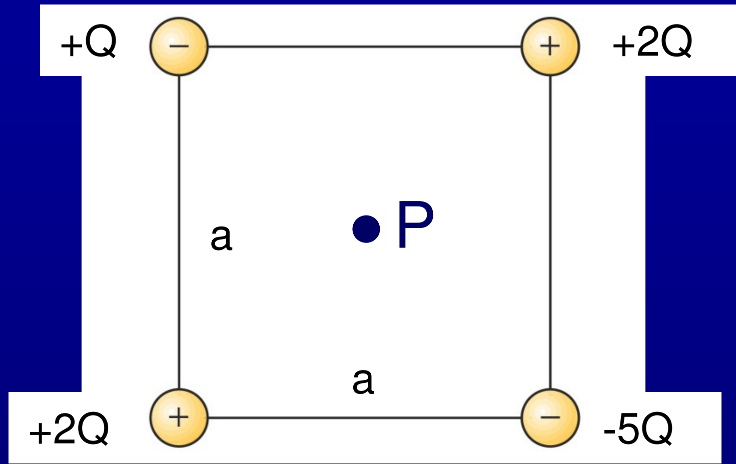
In Practice, think in 3-D



Calculating the E field

Q) Find E at the centre, P of a square of length a

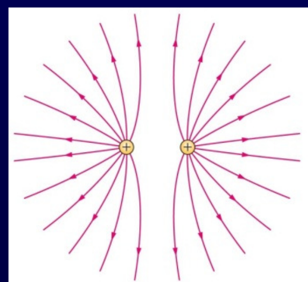
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum \frac{Q_i}{r_i^2}\vec{r}_i$$



Answer: $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{12Q}{a^2}$$ **towards -5Q**

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{\left(a/\sqrt{2}\right)^2}[Q+5Q]\right] = \frac{1}{4\pi\varepsilon_0}\left[\frac{12Q}{a^2}\right]$$

For the two +2Q charges, E cancels (like the case below)

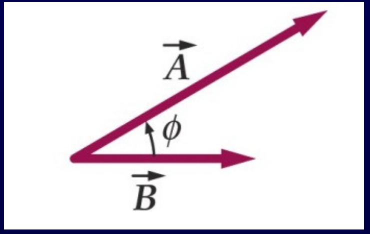


Key concept: Gauss's Law

- The **net electric flux (Φ_{net})** through any **closed surface** is equal to the **net charge enclosed** by the surface (Q_{inside}) divided by ε_0:

$$\Phi_{net} = \int_S E_n dA = \int_S E.\vec{n} dA = \frac{1}{\varepsilon_0} Q_{inside}$$

- Thus the net flux is the **dot product** of E and the unit vector normal to the surface integrated over the surface (a "surface integral")
- Dot product: **A.B** = AB cosø



Gaussian surface

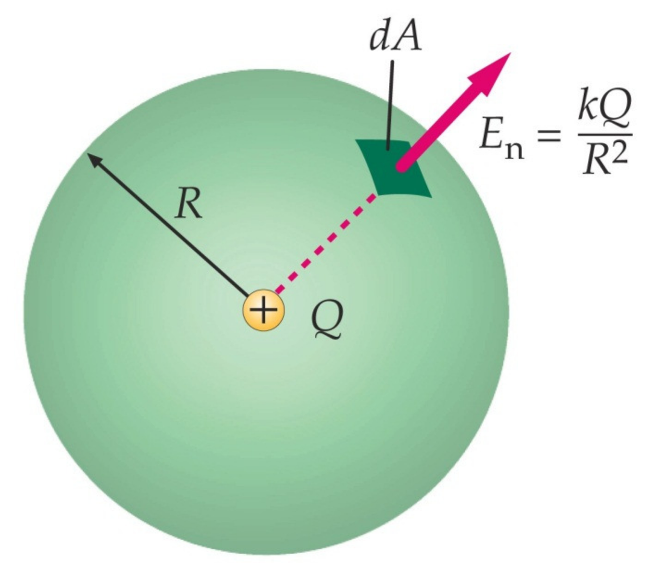
- A **Gaussian surface** is any closed surface over which the flux is evaluated (use whatever surface is easiest).
- This method to find E is only useful in practice for symmetrical surfaces (sphere, infinite plane…)

Example: Find E for a point charge

$$\Phi_{net} = \int_S E_n dA = E\int_s dA = E(4\pi r^2)$$

$$\Phi_{net} = \frac{Q_{inside}}{\varepsilon_0} = E(4\pi r^2)$$

$$\Rightarrow E = \frac{Q_{inside}}{4\pi r^2 \varepsilon_0}$$



Directed outward if positive charge

NB. Get the same answer by using Coulomb's Law

Key concept: Electric potential energy

- The electric potential energy U(r) of test-charge q_0 at distance r from a point charge is the work done against the electric force to move q_0 from infinity to distance r from the point charge

$$U(r) = -\int_{\infty}^{r} F.dl = -q_0 \int_{\infty}^{r} E.dl$$

A line integral or path integral

- We define U(r) = 0 when r = ∞
- Electric force is a **conservative force**: the work done is independent of path chosen for line integral

Key concept: Electric potential

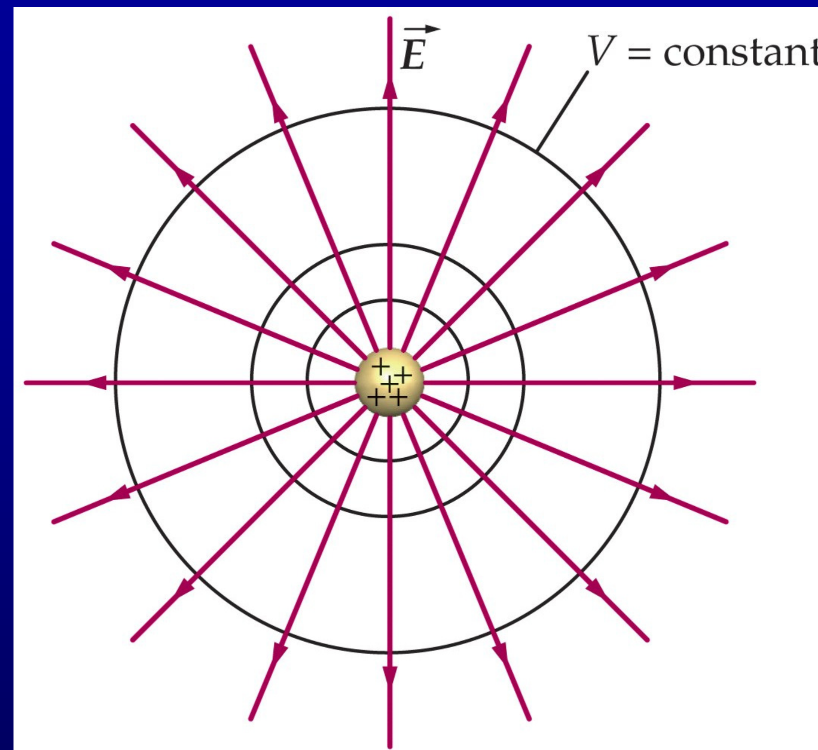
- The electric potential V is the electric potential energy U of a test-particle at that point divided by its charge

$$V = \frac{U}{q_0}$$

- Unit: Volt (Count Alessandro Volt (1745-1827))
- Hence from definition of U and E, $U(r) = -q_0 \int_{\infty}^{r} E.dl$

$$V = -\int_{\infty}^{r} E.dl$$ **and hence** $$E = -\frac{dV}{dr}$$

Equipotential surface



- V is constant on an equipotential surface
- E is normal to an equipotential surface – no work done moving charge around that surface

Conductors in electrostatic equilibrium

- E is normal to the surface of a conductor and has magnitude, $E_n = \sigma/\varepsilon_0$

- E is zero inside a conductor, i.e. the net charge density within a conductor is zero

- This is true unless an external energy source is applied to maintain a field (a conductor comes to equilibrium very quickly, e.g. nanoseconds for copper).

Example of forces involved

Q) What is the force binding a crystal of Salt?

- Typical distance d between ions?
  - d is about 1Å (10^{-10} m)
    Thus force between postive/negative ion $\approx 2 \times 10^{-8}$ N
- How many bonds in one square meter?
  - About $1/d^2 \approx 10^{20}$ bonds
- Force to break them?
  $= 10^{20} \times 2 \times 10^{-8} = 2 \times 10^{12}$ N

Oversimplified, but clearly crystals are strong!

Key concept: Electric field

- Field: a quantity that can be associated with a position
  - Either vector field (e.g. electric) or scalar (e.g. temperature)
- The electric field **E** at point P due to a charge Q is the electric force exerted by that charge on a test particle divided by the (small) charge q_0 on the test particle:

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{r}$$

(i.e. $F = Eq_0$)

- For a distribution of charges Q_1, Q_2, ... Q_i use the **principle of superposition** to get **F** and/or **E**

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum\frac{Q_i}{r_i^2}\vec{r}_i$$

Calculating the E field

- Coulomb's law can be used to find E for a continuous charge distribution
- "Continuous" may, for example, mean a line (or ring), a surface or a volume. Charge is distributed:
  - Line or ring – use linear charge density, λ - unit C m^{-1}
  - Surface – use surface charge density, σ - unit C m^{-2}
  - Volume – use volume charge density, ρ - unit C m^{-3}
- <u>Always</u> draw the charge distribution, including the point at which E is required
- Use Coulomb's law for each element
- Sum using vectors or (more usually) integration

Example problem

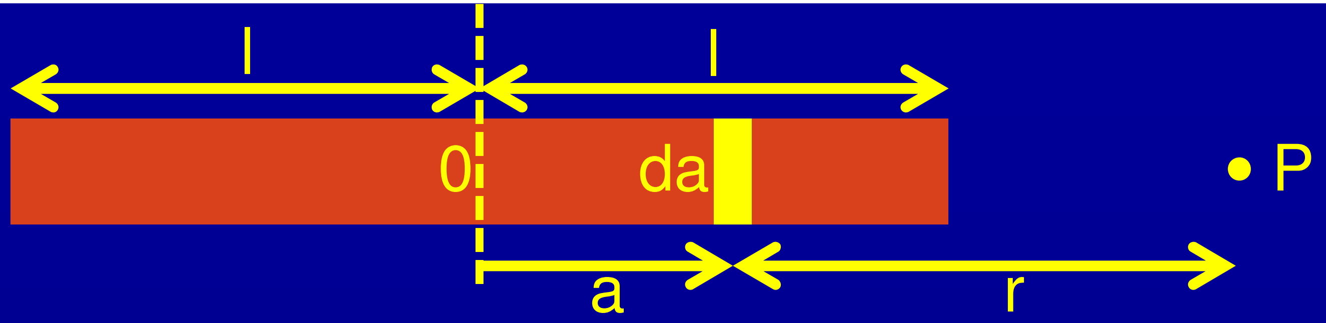
Q) Show that E_z at distance z along the z-axis from a long, straight, uniform line charge of length 2l centred at the origin and oriented along the z-axis is given by

$$E_z = \left[\frac{\lambda l}{2\pi\varepsilon_0 \left(z^2 - l^2\right)} \right]$$

where z > l

Clue: Question contains the word “line” and the answer contains λ (i.e., use linear charge density)

Draw the problem:



Find E_z at point P (z>l)

Total length 2l, charge distributed as λ C m^{-1}

Field due to element da is:

where $r = z\text{-}a$ and $dq = \lambda da$

$$dE_z = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$

$$\Rightarrow dE_z = \frac{1}{4\pi\varepsilon_0}\frac{\lambda da}{(z-a)^2}$$

$$\Rightarrow E_z = \frac{\lambda}{4\pi\varepsilon_0}\int_{-1}^{l}\frac{da}{(z-a)^2}$$

$$\Rightarrow E_z = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{z-a}\right]_{-l}^{l} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{2l}{z^2-l^2}\right] = \frac{\lambda l}{2\pi\varepsilon_0\left(z^2-l^2\right)}$$

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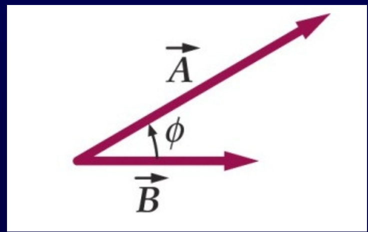


Gaussian surface

- A **Gaussian surface** is any closed surface over which the flux is evaluated (use whatever surface is easiest).
- This method to find E is only useful in practice for symmetrical surfaces (sphere, infinite plane…)

Using Gauss's Law to find E

- Select the appropriate Gaussian Surface carefully (e.g. sphere, cylinder) – try to ensure E is normal to the surface used
- Use the appropriate charge distribution (λ, σ, ρ)
- Insert into the equation and solve the integral

$$\Phi_{net} = \int_S E_n dA = \int_S E.\vec{n} dA = \frac{Q_{inside}}{\varepsilon_0}$$

- Always true, but only really useful in practice if you have a symmetrical situation (otherwise it can be hard to integrate)

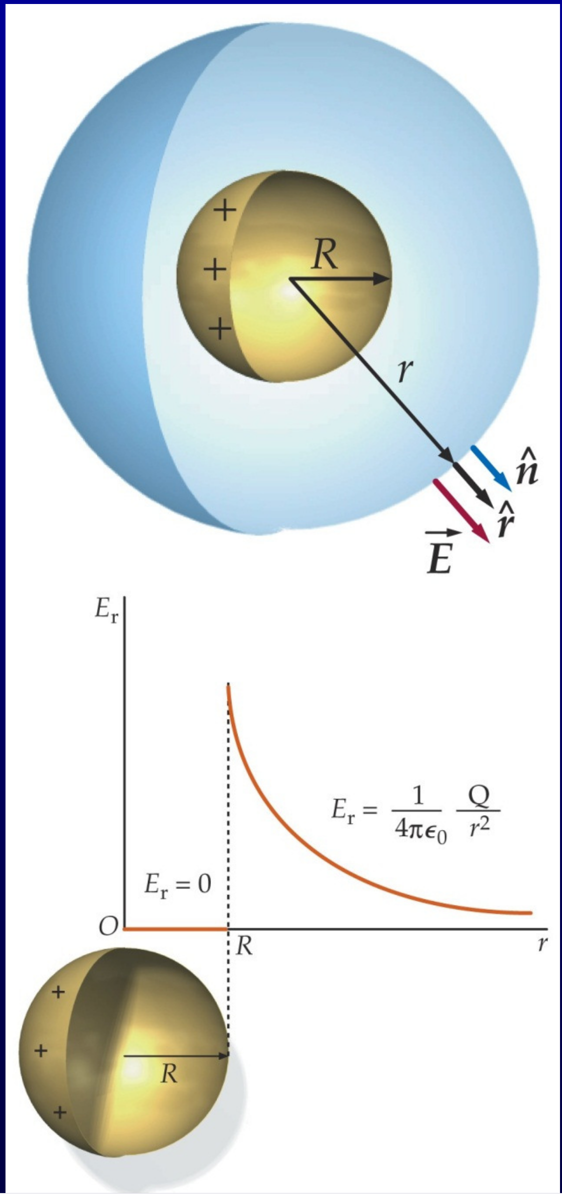
Classic example: charged spheres

Q1. Determine **E** inside and outside a thin, uniformly charged sphere of radius R and total charge Q

Q2. Determine **E** inside and outside a uniform, spherical distribution of charge of radius R and total charge Q

These are NOT the same – the sphere in Q2 is full of charge whereas the sphere in Q1 is empty, (i.e., all the charge is on the surface)

1) Determine **E** inside and outside a thin, uniformly charged sphere of radius R and total charge Q



Field E is only radial to the surface (question says “uniformly charged”)

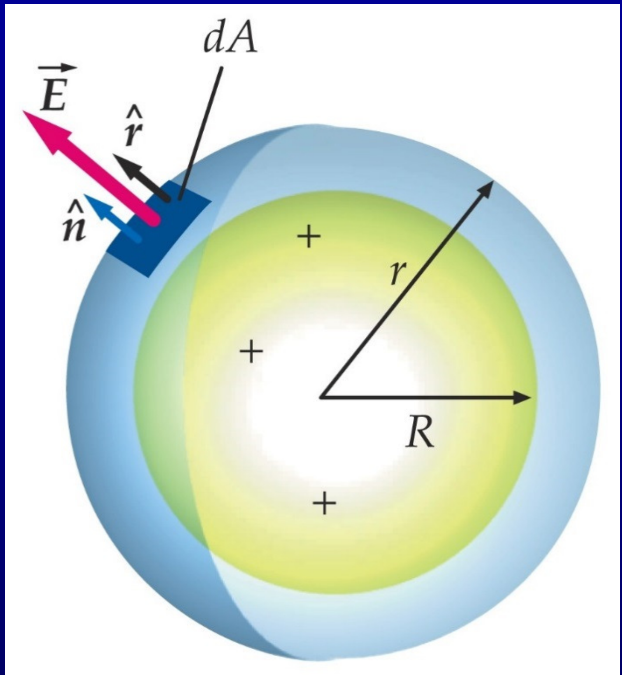
For r>R, Q_{inside}= Q
Use a spherical gaussian surface

$$\Phi_{net} = \int_S E_n dA = E\int_s dA = \frac{Q}{\varepsilon_0}$$

$$\int_S dA = 4\pi r^2 \Rightarrow E = \frac{Q}{4\pi r^2 \varepsilon_0}(r > R)$$

For r<R, Q_{inside}= 0 $\Rightarrow E = 0(r < R)$

Q2) Determine **E** inside and outside a uniform, spherical distribution of charge of radius R and total charge Q



For r>R, Q_{inside}=Q

$$\Rightarrow E = \frac{Q}{4\pi r^2 \varepsilon_0} (r > R)$$

For r<R, at radius r only a fraction of the total volume, and hence charge, is contributing to Q_{inside} . The total charge:

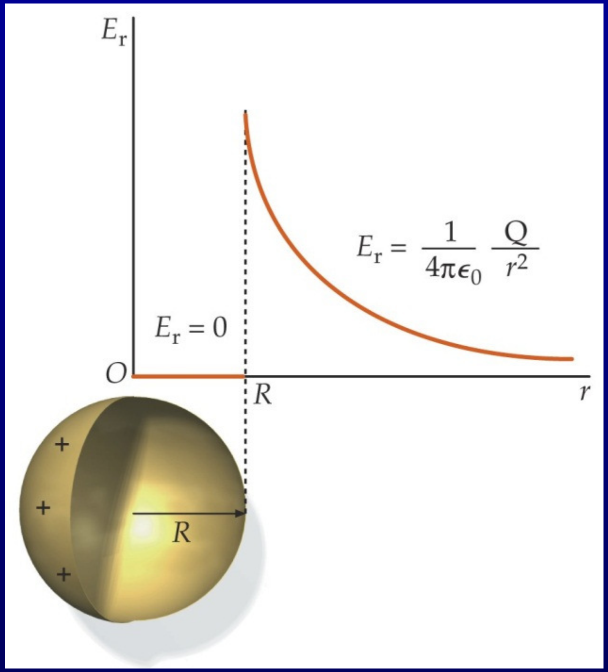
(where ρ is the volume charge density)

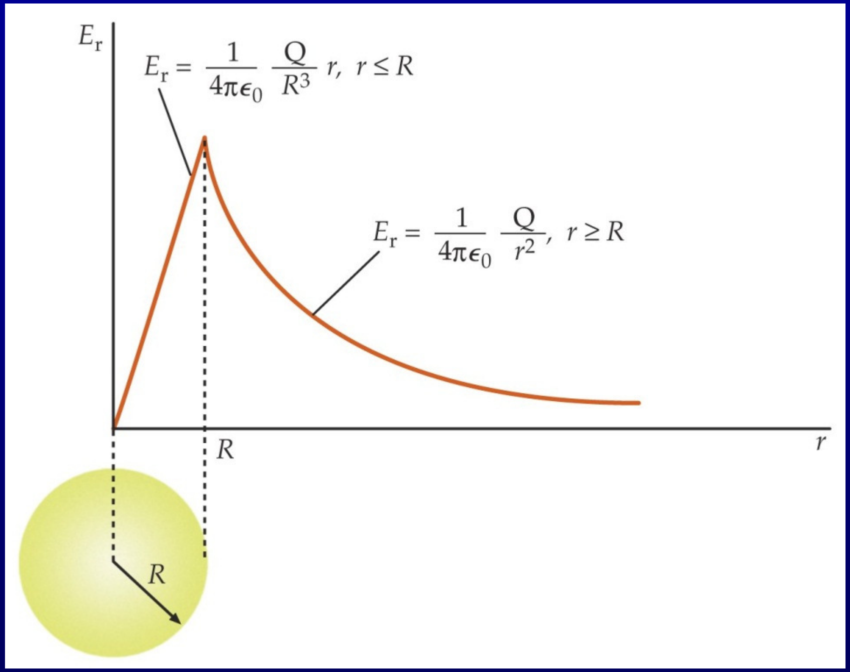
$$Q = \rho \frac{4\pi R^3}{3}$$

$$\Phi_{net} = \frac{dQ(r < R)}{\varepsilon_0} = \rho \frac{4\pi r^3}{3\varepsilon_0} = \frac{Qr^3}{\varepsilon_0 R^3} = E(4\pi r^2)$$

$$\Rightarrow E = \frac{Qr}{4\pi\varepsilon_0 R^3} (r < R)$$

Comparison of results





1) Thin shell

E is discontinuous at the surface by σ/ε_0

2) Solid charged sphere

Key concept: Electric potential energy

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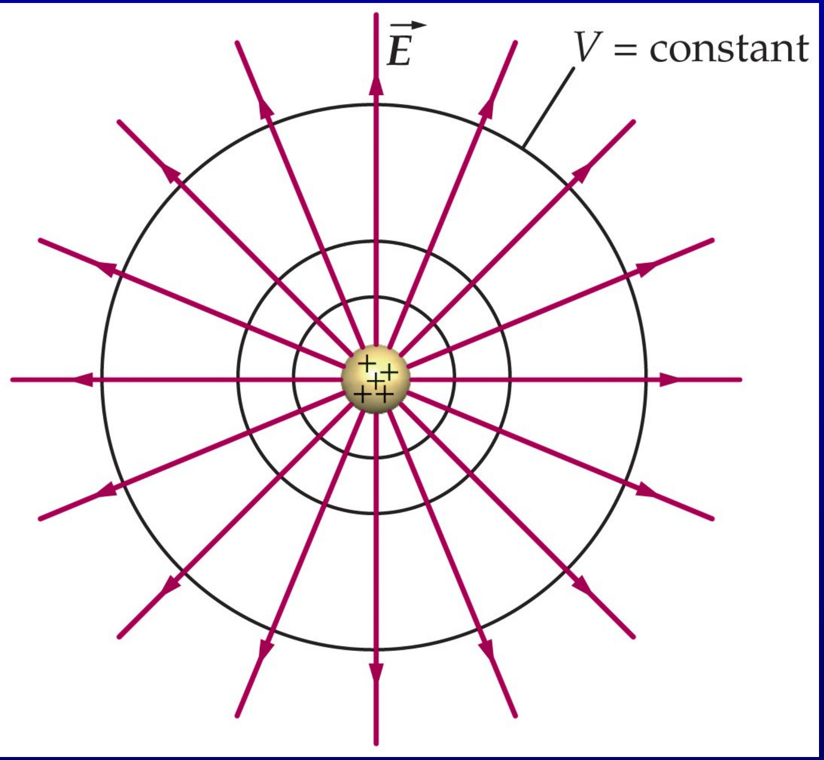
Equipotential surface



- **V is constant on an equipotential surface**
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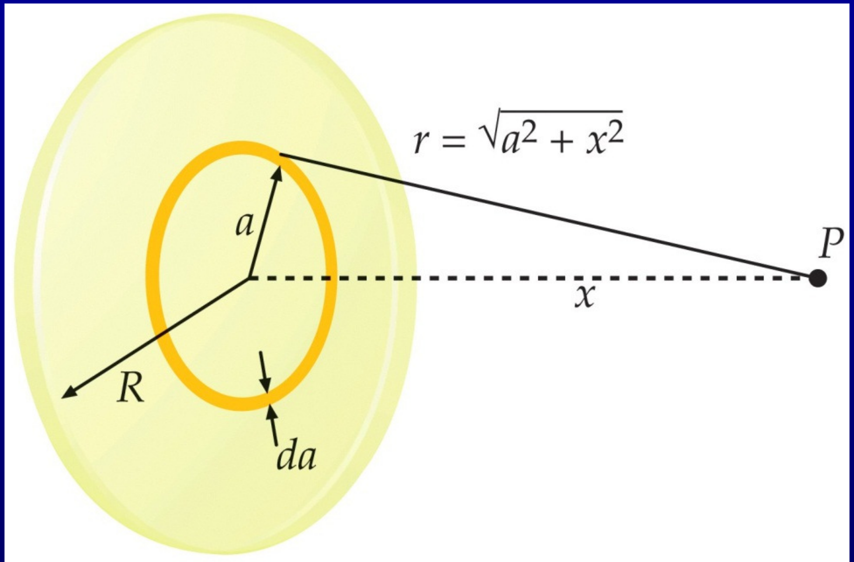
Conductors in electrostatic equilibrium

- E is normal to the surface of a conductor and has magnitude, $E_n = \sigma/\varepsilon_0$
- E is zero inside a conductor, i.e. the net charge density within a conductor is zero
- This is true unless an external energy source is applied to maintain a field (a conductor comes to equilibrium very quickly, e.g. nanoseconds for copper).

Q) Determine the electric potential on the axis of a disk of radius R that carries a total charge Q distributed uniformly on its surface.

Align disk perpendicular to a coordinate axis (say, x).



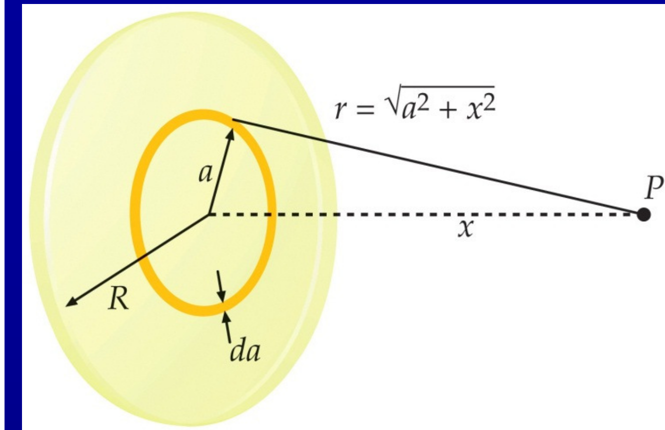
Split disk into a set of rings, radius a, width da, area $dA = 2\pi a\,da$, charge $dq = \sigma dA$.

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

Total charge $Q = \sigma \pi R^2$

From point P, ring is at distance

$$r = \sqrt{\left(a^2 + x^2\right)}$$



$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{\left(x^2+a^2\right)^{1/2}} = \frac{\sigma 2\pi a da}{4\pi\varepsilon_0\left(x^2+a^2\right)^{1/2}}$$

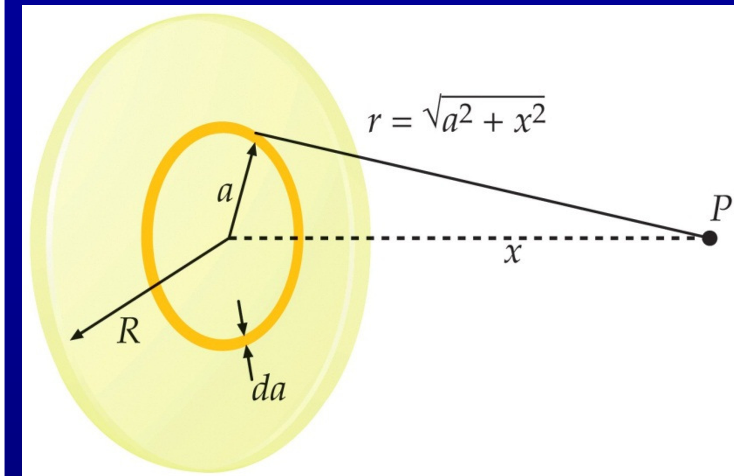
$$\Rightarrow V = \frac{\sigma\pi}{4\pi\varepsilon_0}\int_0^R \frac{2ada}{\left(x^2+a^2\right)^{1/2}}$$

Integral of the form $\int u^n du$ with u = x^2+a^2, du = 2ada and n=-1/2. When a=0, u=x^2 and when a=R, u=x^2+R^2

$$\Rightarrow V = \frac{\sigma\pi}{4\pi\varepsilon_0}\int_{x^2}^{x^2+R^2} u^{-1/2}du = \frac{\sigma\pi}{4\pi\varepsilon_0}\left[\frac{u^{1/2}}{1/2}\right]_{x^2}^{x^2+R^2}$$

$$\Rightarrow V = \frac{2\sigma\pi}{4\pi\varepsilon_0}\left(\left(x^2+R^2\right)^{1/2} - x^{1/2}\right) = \frac{2\sigma\pi}{4\pi\varepsilon_0}|x|\left(\sqrt{1+\frac{R^2}{x^2}}-1\right)$$

Sanity check…



$$V = \frac{2\sigma\pi}{4\pi\varepsilon_0}|x|\left(\sqrt{1+\frac{R^2}{x^2}}-1\right)$$

At large distance, the disk will look like a point charge

$$V = \frac{2\sigma\pi}{4\pi\varepsilon_0}|x|\left(\sqrt{1+\frac{R^2}{x^2}}-1\right) \approx \frac{2\sigma\pi}{4\pi\varepsilon_0}|x|\left(1+\frac{1}{2}\frac{R^2}{x^2}+\ldots-1\right)$$

At large distance, x>>R

$$V \approx \frac{2\sigma\pi}{4\pi\varepsilon_0}|x|\left(\frac{1}{2}\frac{R^2}{x^2}\right) = \frac{\sigma\pi R^2}{4\pi\varepsilon_0}\frac{1}{|x|} = \frac{Q}{4\pi\varepsilon_0|x|}$$

=V for a point charge

Coulomb's Law and E Summary

Coulomb's Law

$$\overrightarrow{F_{12}} = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r_{12}^2}\vec{r}$$

where ε_0 is the **permittivity** of free space

Electric field for charge Q

$$\vec{E} = \frac{\vec{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\vec{r}$$

For a charge distribution

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\sum\frac{Q_i}{r_i^2}\vec{r}_i$$

Calculating the E field

- Coulomb's law can be used to find E for a continuous charge distribution
- "Continuous" may, for example, mean a line (or ring), a surface or a volume. Charge is distributed:
  - Line or ring – use linear charge density, λ - unit C m^{-1}
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- Always draw the charge distribution, including the point at which E is required
- Use Coulomb's law for each element
- Sum using vectors or (more usually) integration

Workshop 1, Question 2

- A uniform line charge of linear charge density λ = 3.5 nC/m extends from x = 0 to x = 5 m.
- (*a*) What is the total charge? Find the electric field on the x axis at
  - (*b*) x = 6 m,
  - (*c*) x = 9 m, and
  - (*d*) x= 250 m.
- (*e*) Find the field at x = 250 m, using the approximation that the charge is a point charge at the origin, and compare your result with that for the exact calculation in Part (*d*).

Workshop 1, Question 3

- A single point charge q = +2 μC is at the origin. A spherical surface of radius 3.0 m has its center on the x axis at x = 5 m.
- (a) Sketch electric field lines for the point charge. Do any lines enter the spherical surface?
- (b) What is the net number of lines that cross the spherical surface, counting those that enter as negative?
- (c) What is the net flux of the electric field due to the point charge through the spherical surface?

Workshop 1, Question 4

- Two large parallel conducting plates separated by 10 cm carry equal and opposite surface charge densities so that the electric field between them is uniform. The difference in potential between the plates is 500 V. An electron is released from rest at the negative plate.
- (a) What is the magnitude of the electric field between the plates? Is the positive or negative plate at the higher potential?
- (b) Find the work done by the electric field on the electron as the electron moves from the negative plate to the positive plate. Express your answer in both electron volts and joules.
- (c) What is the change in potential energy of the electron when it moves from the negative plate to the positive plate?
- (d) What is its kinetic energy when it reaches the positive plate?

Workshop: Problem 4

Electric field is uniform in this case

$$\text{a)}\ E_x = -\frac{dV}{dx} = -\frac{V}{x} = -5000\ \text{V/m} = \text{-}\ 5\ \text{kV/m}$$

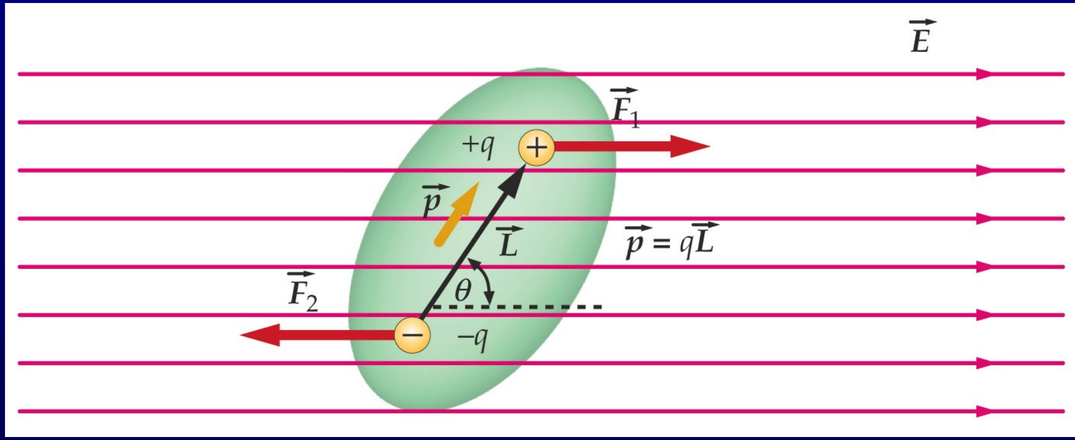
Higher potential at positive plate

$$\text{b)}\ W = qV = 8\times10^{-17}\ \text{J} = 500\ \text{eV}$$

$$\text{c)}\ U = -qV = -500\ \text{eV};\ \text{E}_{\text{kin}} = 500\text{eV}$$

Electric dipole

- Consider two charges, +Q, -Q, distance L apart
- Placed in an E-field, the field will cause the dipole to rotate into the direction of the field
  - E causes a torque $\tau = p \times E$, where p is the "dipole moment" (p = LQ in this case)



- The E-field created by an electric dipole falls off as r^{-3} (rather than as r^{-2} for a point charge)

Q) Determine the electric potential V, a) inside and b) outside a uniform, spherical distribution of charge of radius R and total charge Q.

Use $$V = -\int_{\infty}^{r} E.dl$$

a) For r>R use appropriate E and an integrating variable, x

$$E = \frac{Q}{4\pi r^2 \varepsilon_0}(r > R)$$

$$\Rightarrow V = -\int_{\infty}^{r} E.dl = -\int_{\infty}^{r} \frac{Q}{4\pi\varepsilon_0 x^2} dx$$

$$\Rightarrow V = \frac{Q}{4\pi\varepsilon_0 r}(r > R)$$

Looks like a point charge at the origin

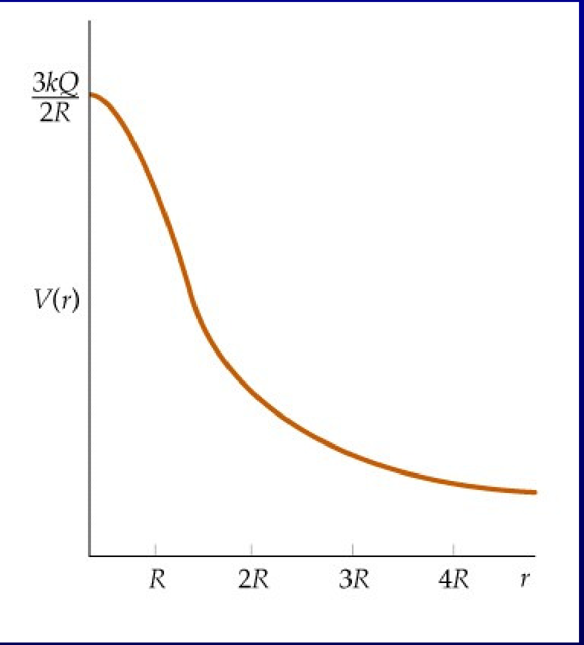
b) Inside, need to split the integral in two: 1) for coming up to R from infinity, and 2) then going inside the sphere

$$E = \frac{Qr}{4\pi\varepsilon_0 R^3}(r < R)$$

$$\Rightarrow V = -\int_{\infty}^{R} \frac{Q}{4\pi\varepsilon_0 x^2}dx + -\int_{R}^{r} \frac{Qx}{4\pi\varepsilon_0 R^3}dx$$

$$\Rightarrow V = \frac{Q}{4\pi\varepsilon_0 R} + \frac{Q}{4\pi\varepsilon_0 R^3}\left[\frac{-x^2}{2}\right]_R^r$$

$$\Rightarrow V = \frac{Q}{4\pi\varepsilon_0 R}\left[\frac{3}{2} - \frac{r^2}{2R^2}\right](r < R)$$



(Must give the same answer at r=R !)

Unit outline

Electrostatic potential energy
Capacitance
Capacitors

Lecture 1

Potential energy stored in capacitors
Dielectrics
Capacitors in circuits
Electric current
Ohm’s Law

Lecture 2

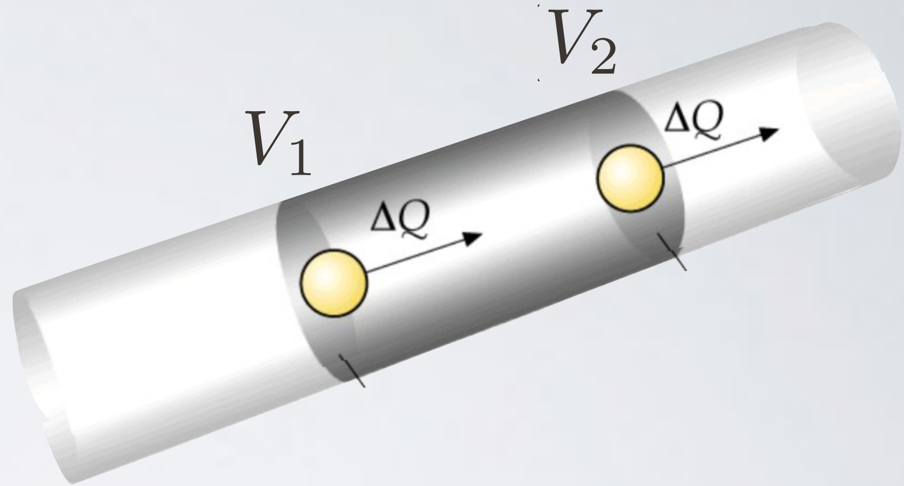
Energy in electric circuits
Resistors
Kirchoff’s Laws
RC circuits

Lecture 3

Follow-up lecture

Energy in electric circuits

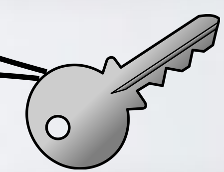
$$\Delta U = \Delta Q(V_2 - V_1) = \Delta Q V$$



Over time Δt,

$$P = \frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t} V = IV$$

Eq 25-13



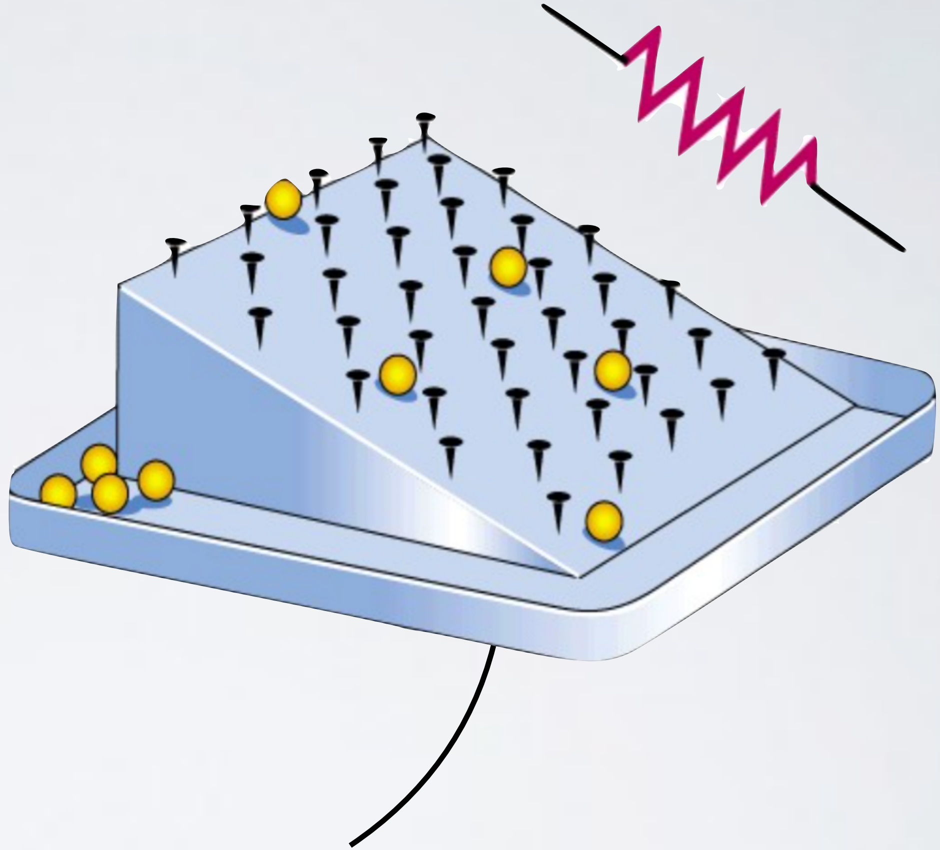
Power dissipated via *Joule heating* in resistive conductor

$$P = I^2 R = \frac{V^2}{R}$$

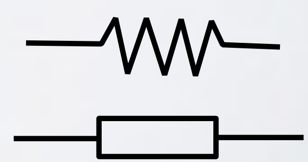
Eq 25-14

Energy in electric circuits

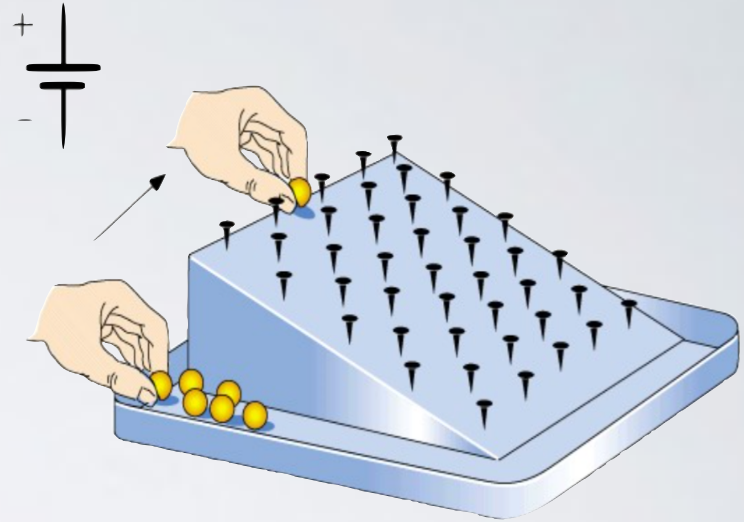
Joule heating can be thought of as resulting from the friction imposed by the atoms in the conductor on the moving charges



A resistor is a *passive* element in which energy is dissipated, e.g. a lightbulb or an electrical heating element



Energy in electric circuits



A battery does work on charges, raising them through the potential between its terminals

The work done per unit charge is called the emf (electromotive force) $\mathcal{E}$

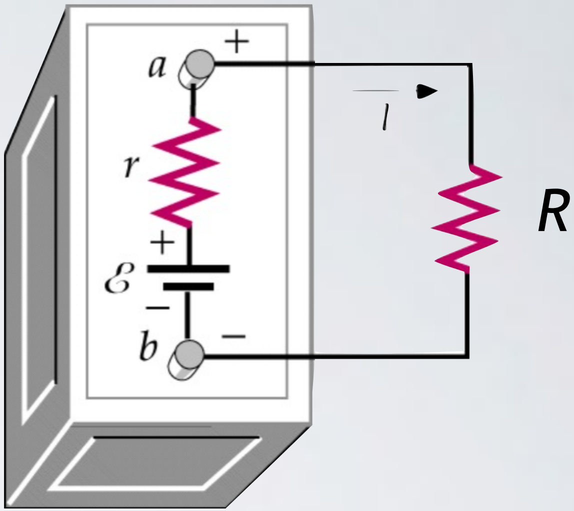
It has units of V

Eq 25-15

$$P = \frac{\Delta Q \mathcal{E}}{\Delta t} = I\mathcal{E}$$

Power supplied by ideal source of emf

Energy in electric circuits



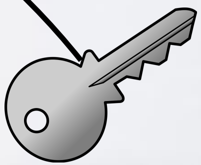
In a real battery, the internal resistance is not 0 Ω. Rather, it has a finite value *r*

Therefore, if the current in the circuit is *I*, the real terminal voltage is

$$V_a - V_b = \mathcal{E} - Ir$$

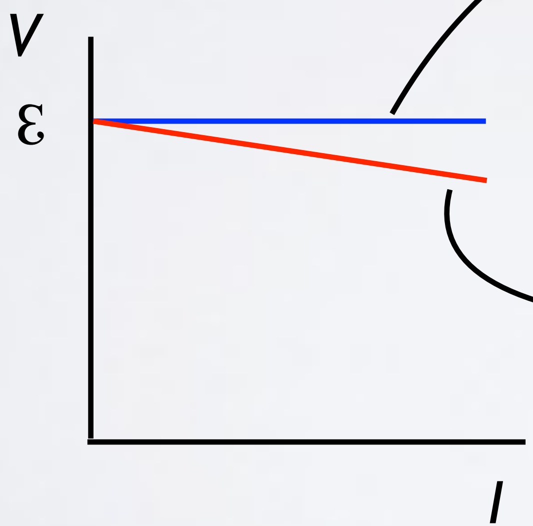
Eq 25-16

This means that the higher the current that flows in the circuit, the less voltage is delivered by the battery



Energy in electric circuits

$$V_a - V_b = \mathcal{E} - Ir$$



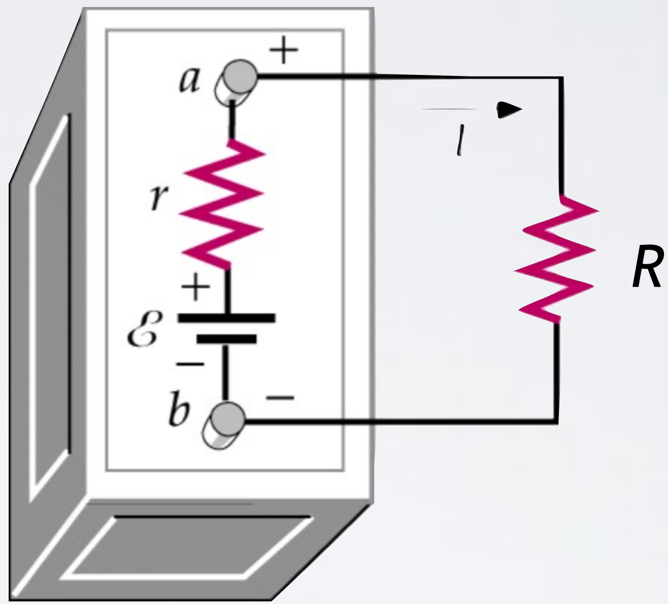
Ideal battery produces a terminal voltage of ε whatever the current

Real battery produces a terminal voltage that is reduced from ε by a value dependent on the current

Energy in electric circuits

The current that flows in a circuit which also contains a resistor *R*:

The voltage between points *a* and *b*…



…is equal to the voltage that falls through this resistor *R*…

…and is equal to the emf raised, minus the voltage that falls owing to the battery's internal resistance *r*

$$IR = V_a - V_b = \mathcal{E} - Ir$$

Energy in electric circuits

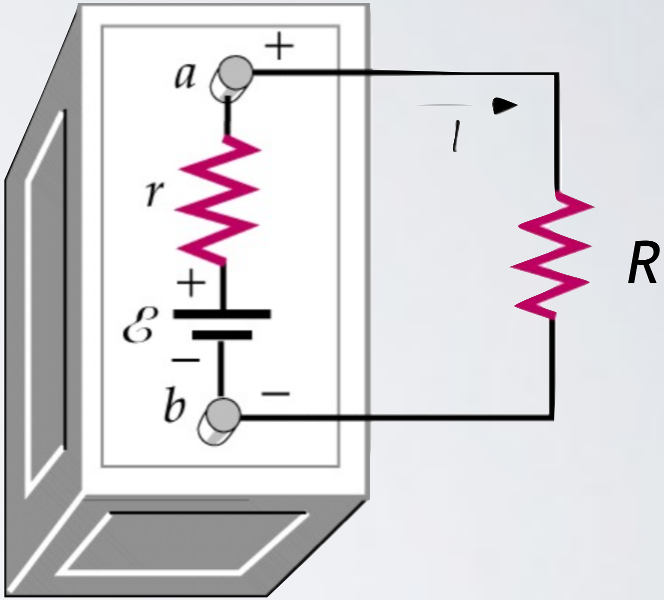
The current that flows in a circuit which also contains a resistor *R*:

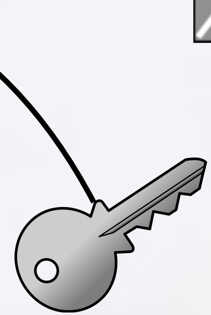
$$IR = V_a - V_b = \mathcal{E} - Ir$$

So $$\mathcal{E} = I(R + r)$$

Or $$I = \frac{\mathcal{E}}{R + r}$$

Eq 25-17

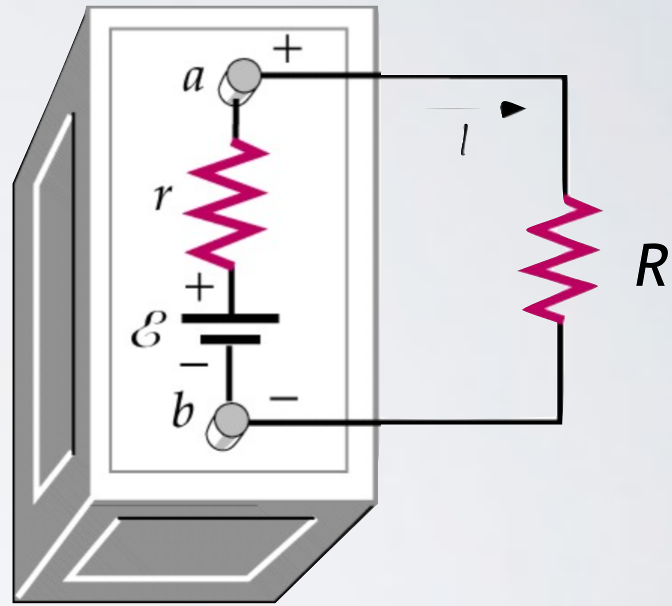




Energy in electric circuits

Worked example: for a battery with emf $\mathcal{E}$ and the internal resistance *r*, what value of resistance *R* maximises the power delivered? What is the power?

$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$



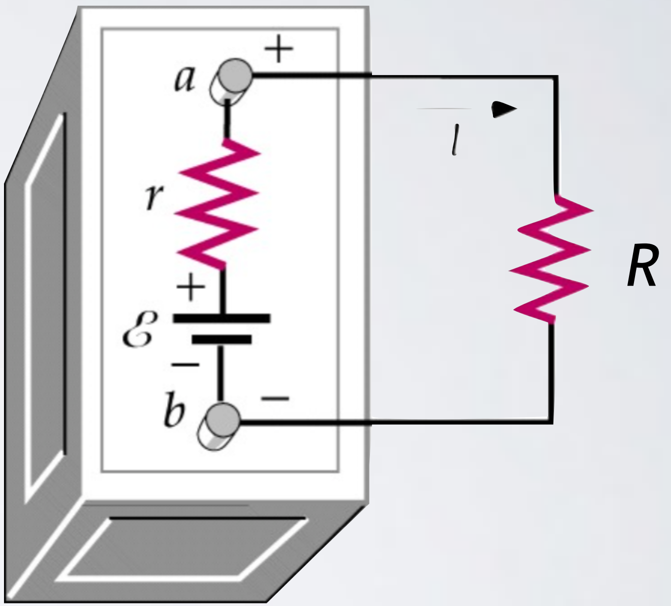
Finding max P w.r.t. R: $$\frac{dP}{dR} = 0$$

Use quotient rule: $$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Energy in electric circuits

Worked example: for a battery with emf $\mathcal{E}$ and the internal resistance *r*, what value of resistance *R* maximises the power delivered? What is the power?

$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$



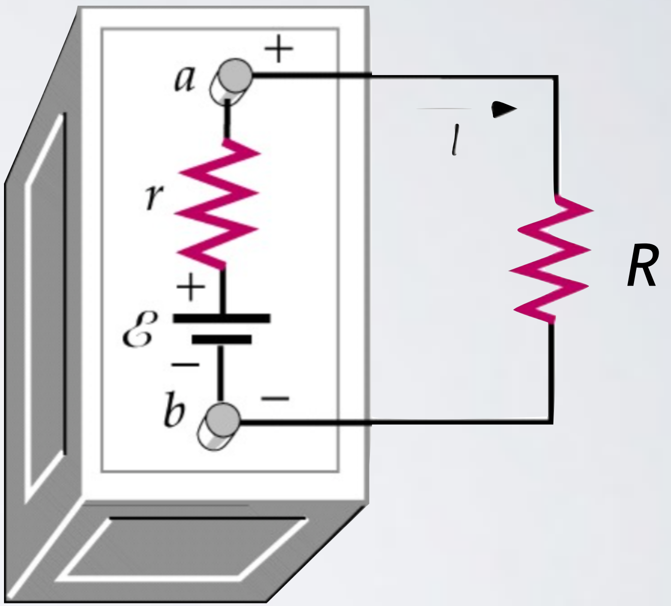
Finding max P w.r.t. R: $$\frac{dP}{dR} = 0$$

$$\frac{dP}{dR} = \frac{\mathcal{E}^2 (R+r)^2 - 2\mathcal{E}^2 R (R+r)}{(R+r)^4}$$

Energy in electric circuits

Worked example: for a battery with emf $\mathcal{E}$ and the internal resistance *r*, what value of resistance *R* maximises the power delivered? What is the power?

$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$



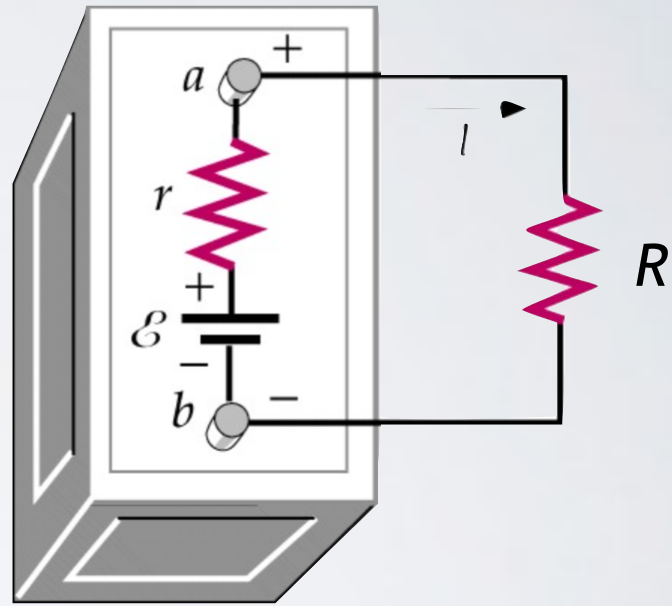
Finding max P w.r.t. R:

$$\frac{dP}{dR} = 0$$

$$\frac{dP}{dR} = \frac{\mathcal{E}^2(R+r) - 2\mathcal{E}^2 R}{(R+r)^3}$$

Energy in electric circuits

Worked example: for a battery with emf ℰ and the internal resistance *r*, what value of resistance *R* maximises the power delivered? What is the power?



$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

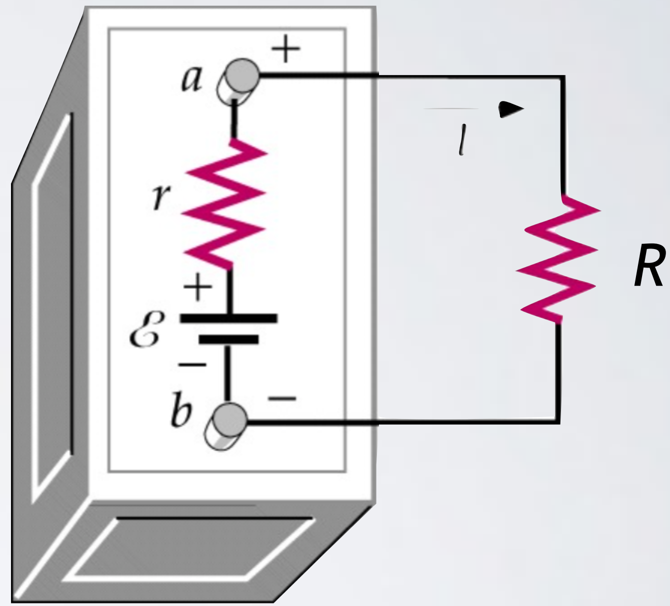
Finding max P w.r.t. R:

$$\frac{dP}{dR} = 0$$

$$\frac{dP}{dR} = \frac{\mathcal{E}^2 (r - R)}{(R+r)^3} = 0 \Rightarrow r - R = 0$$

Energy in electric circuits

Worked example: for a battery with emf ℰ and the internal resistance *r*, what value of resistance *R* maximises the power delivered? What is the power?



$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$

Finding max P w.r.t. R:

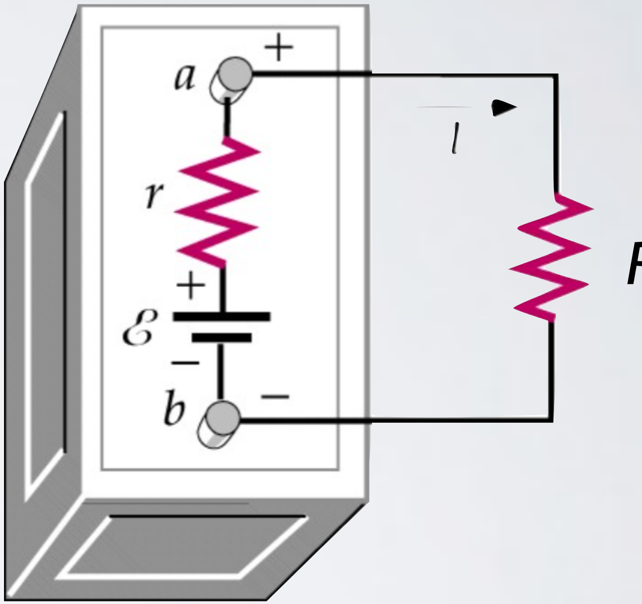
$$\frac{dP}{dR} = 0$$

$$\frac{dP}{dR} = \frac{\mathcal{E}^2 (r-R)}{(R+r)^3} = 0 \Rightarrow \underline{r = R}$$

Energy in electric circuits

Worked example: for a battery with emf $\mathcal{E}$ and the internal resistance *r*, what value of resistance *R* maximises the power delivered? What is the power?

$$P = I^2 R = \frac{\mathcal{E}^2 R}{(R+r)^2}$$



$$P_{max} = \frac{\mathcal{E}^2 r}{(2r)^2}$$

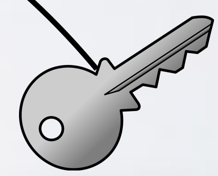
$$P_{max} = \frac{\mathcal{E}^2}{4r}$$

Energy in electric circuits

The stored energy in a battery E_{stored} is equal to the total charge it can deliver over its lifetime Q multiplied by the emf $\mathcal{E}$:

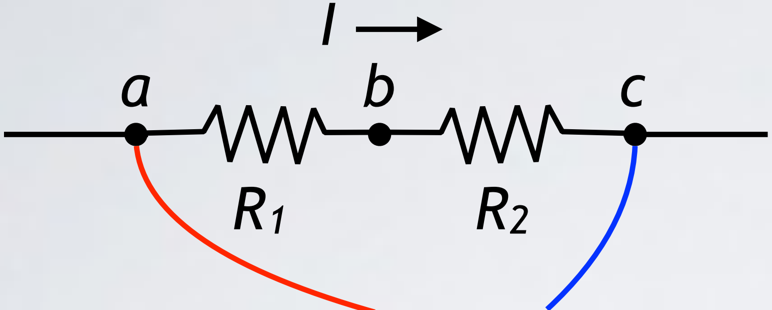
$$E_{\text{stored}} = Q\mathcal{E}$$

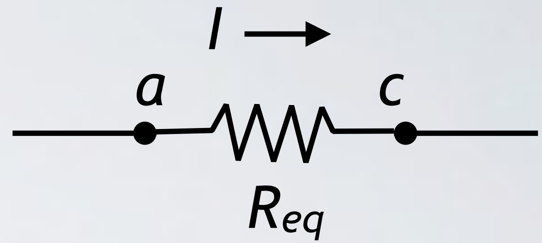
Eq 25-18



This is the total work the battery can do

Resistors in circuits… series





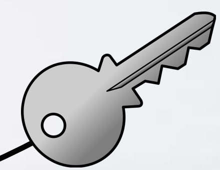
Voltage between *a* and *c* is the sum of the voltage drops over resistors 1 and 2

$$V = IR_1 + IR_2 \quad \Rightarrow \quad V = I(R_1 + R_2)$$

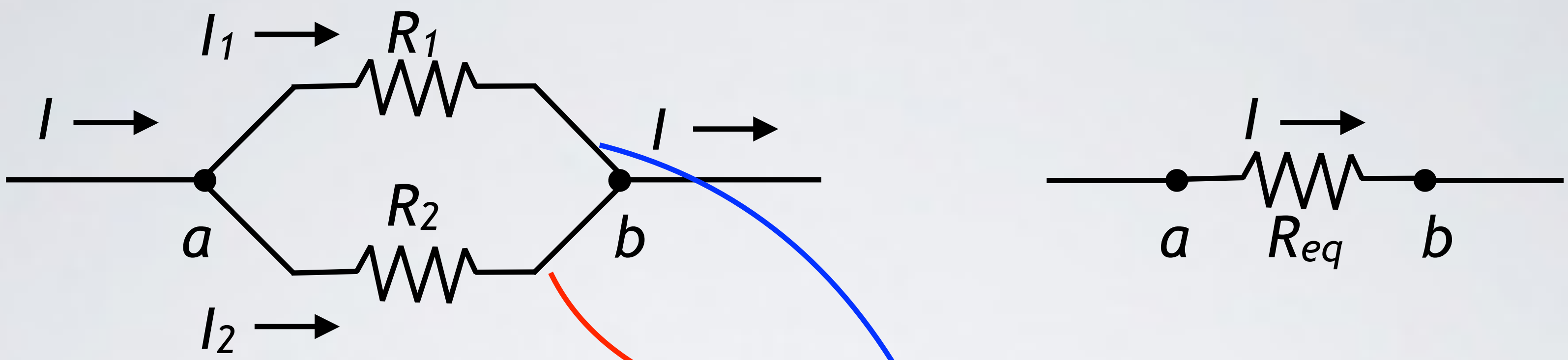
So the equivalent resistance is…

$$R_{eq} = R_1 + R_2$$

Eq 25-20



Resistors in circuits... parallel



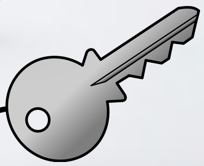
The currents in both branches I_2 *and* I_1 add up to the current I flowing in and out of the junctions (we will look at this later...). The voltage drop V across both resistors is the same.

$$I = I_1 + I_2 \quad \Rightarrow \quad \frac{V}{R_{eq}} = \frac{V}{R_1} + \frac{V}{R_2}$$

So...

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

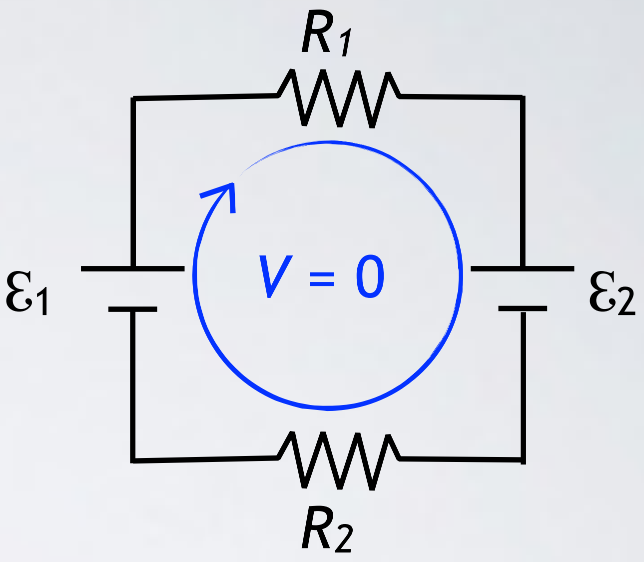
Eq 25-25



Kirchoff's Laws

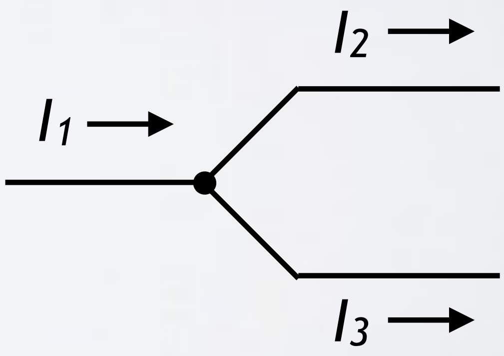
1. The sum of the voltages around a closed loop is zero!

$$\oint_c \mathbf{E} \cdot d\mathbf{r} = 0$$



2. At a junction, total current in equals total current out!

$$I_1 = I_2 + I_3$$



Current loop solution strategy

1. Replace any series or parallel resistor combos by 1 equivalent resistor

2. Assign positive current direction and draw arrows. Label currents in branches, and draw + and − signs for each emf.

3. Apply Kirchhoff's junction rule to all but 1 junction.

4. Apply Kirchhoff's voltage loop rule each time until the number of equations equals the number of unknowns. Voltage falls across a resistor (i.e. = $-IR$) and rises across a source of emf (i.e. = $+Ir$).

5. Solve away!

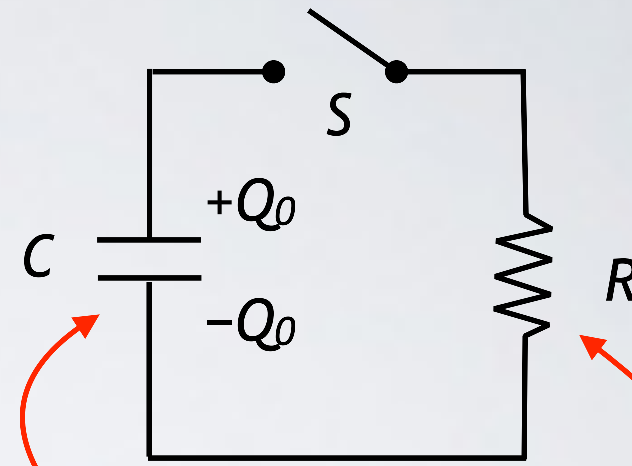
RC circuits

Discharging a capacitor:

There is initially big voltage between the plates, hence a large current flows around the circuit due to a large force acting on the charges.

As the charge gradually equalises, the voltage drops, and the force decreases. Hence the current gradually decreases.

The resulting current profile is an exponential decrease with time



$$V = \frac{Q}{C}$$

$$V = \frac{dQ}{dt}R$$

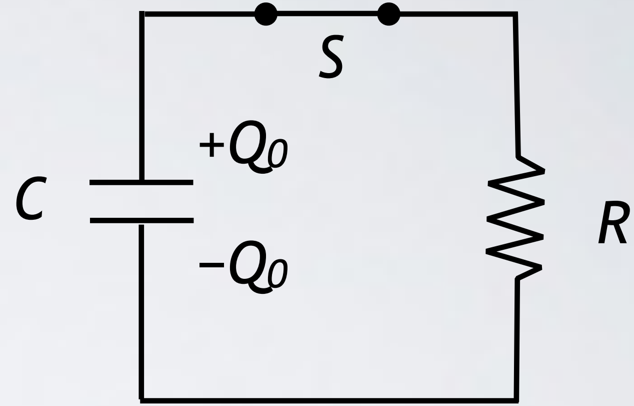
$-I$

RC circuits

Discharging a capacitor:

Applying Kirchhoff's Loop rule:

$$\frac{Q}{C} + \frac{dQ}{dt}R = 0$$



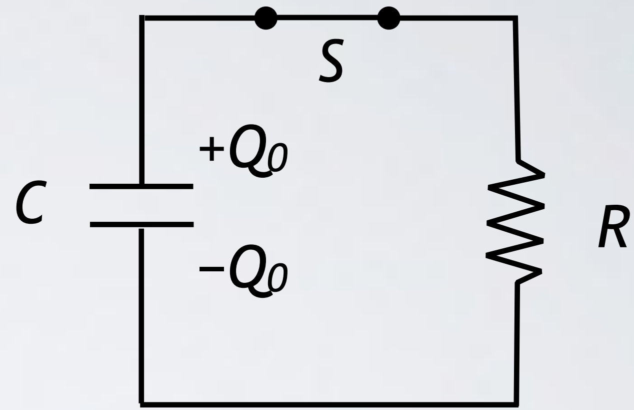
Solve by separating the variables

$$\frac{dQ}{dt} = -\frac{Q}{RC}$$

$$\frac{dQ}{Q} = -\frac{1}{RC}dt$$

RC circuits

Discharging a capacitor:



Integrating:

$$\frac{dQ}{Q} = -\frac{1}{RC}dt$$

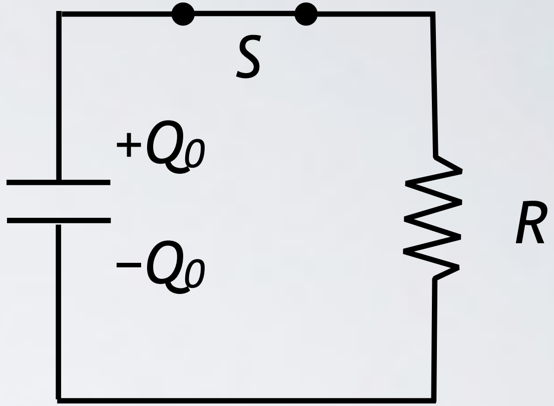
$$\int_{Q_0}^{Q'} \frac{dQ}{Q} = -\frac{1}{RC}\int_0^{t'} dt$$

$$\ln\frac{Q'}{Q_0} = -\frac{t'}{RC}$$

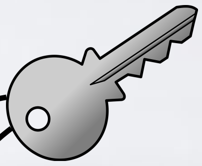
RC circuits

Discharging a capacitor:

$$\ln \frac{Q'}{Q_0} = -\frac{t'}{RC}$$



$$Q(t) = -Q_0 \exp^{-\frac{t}{RC}}$$



$$Q(t) = Q_0 \exp^{-\frac{t}{\tau}}$$

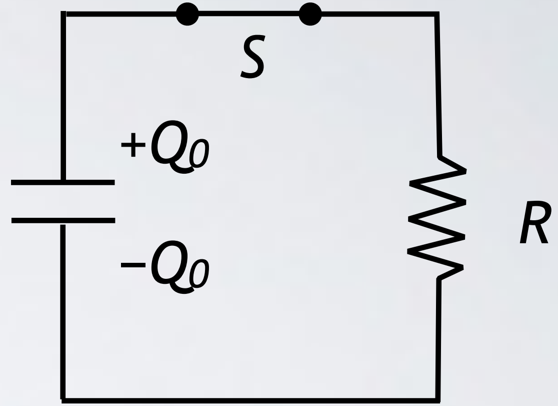
$$\tau = RC$$

Eq 25-36

Eq 25-35

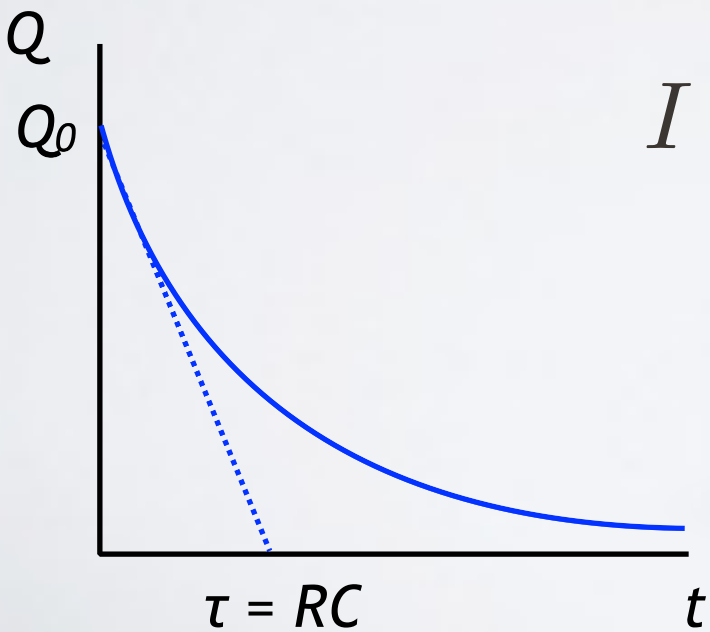
RC circuits

Discharging a capacitor:



$$Q(t) = Q_0 \exp^{-\frac{t}{\tau}}$$

$$\tau = RC$$



$$I = \frac{dQ}{dt} = \frac{Q_0}{RC} \exp^{-\frac{t}{RC}}$$

$$I = I_0 \exp^{-\frac{t}{\tau}}$$

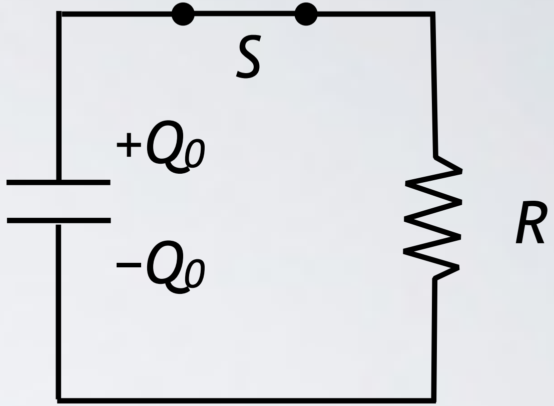
Eq 25-37

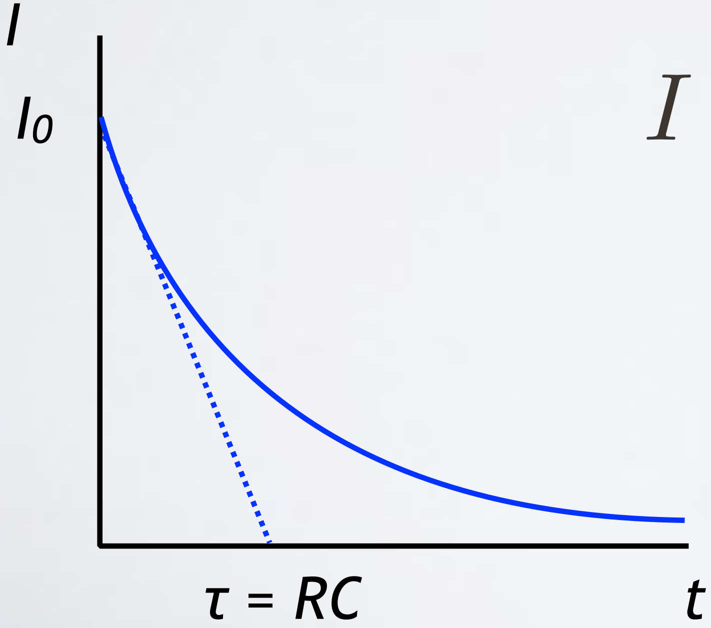
RC circuits

Discharging a capacitor:

$$Q(t) = Q_0 \exp^{-\frac{t}{\tau}}$$

$$\tau = RC$$





$$I = \frac{dQ}{dt} = \frac{Q_0}{RC} \exp^{-\frac{t}{RC}}$$

$$I = I_0 \exp^{-\frac{t}{\tau}}$$

Eq 25-37

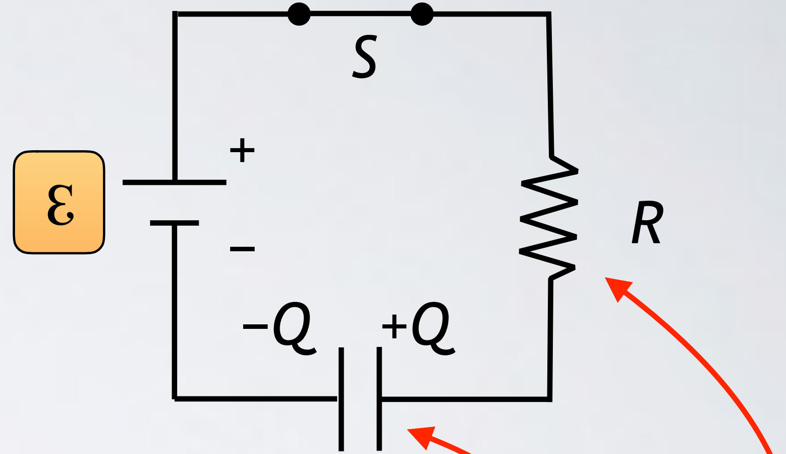
RC circuits

Charging a capacitor:

There is initially zero voltage between the plates, hence a large current flows around the circuit since there is no force opposing the emf of the battery.

As the charge gradually builds up, the voltage increases, opposing the emf. Hence the current gradually decreases.

The resulting current profile is an exponential decrease with time. Again! (But in opp. dir.)

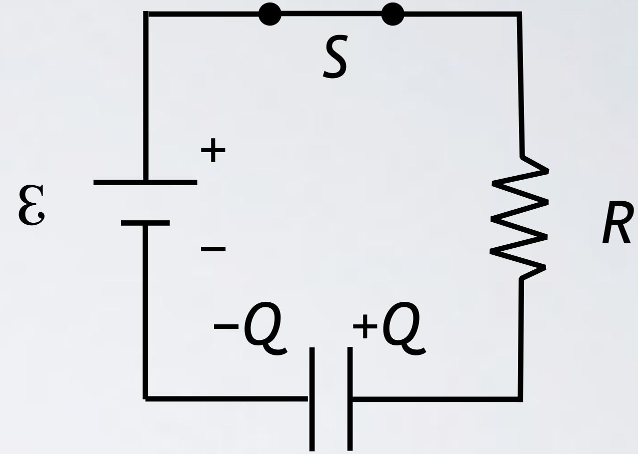


$$V = -\frac{Q}{C}$$

$$V = -\frac{dQ}{dt}R$$

RC circuits

Charging a capacitor:



Applying Kirchhoff's Loop rule:

$$\mathcal{E} - \frac{dQ}{dt}R - \frac{Q}{C} = 0$$

Similar considerations as previously yield:

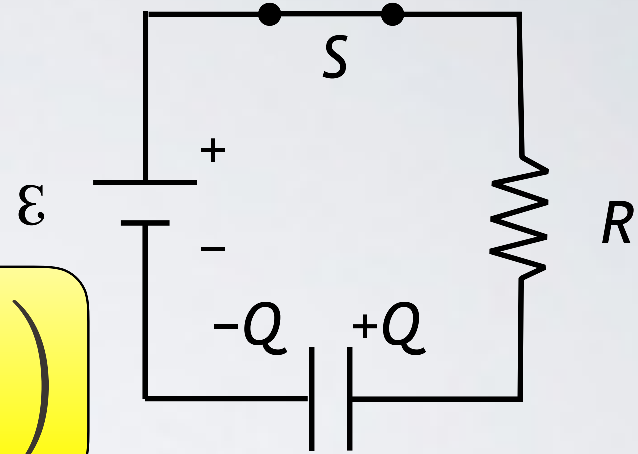
Eq 25-40

$$Q(t) = C\mathcal{E}\left(1 - \exp^{-\frac{t}{\tau}}\right)$$

RC circuits

Charging a capacitor:

Finding the current:



$$Q(t) = C\mathcal{E}\left(1 - \exp^{-\frac{t}{\tau}}\right)$$

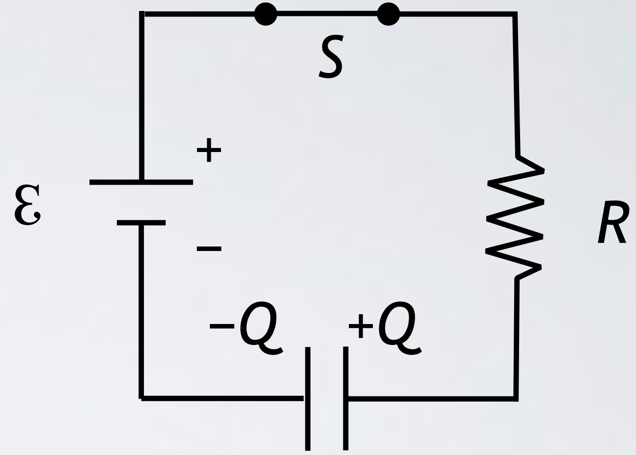
$$I = \frac{dQ}{dt} = C\mathcal{E}\left(\frac{1}{RC}\exp^{-\frac{t}{RC}}\right)$$

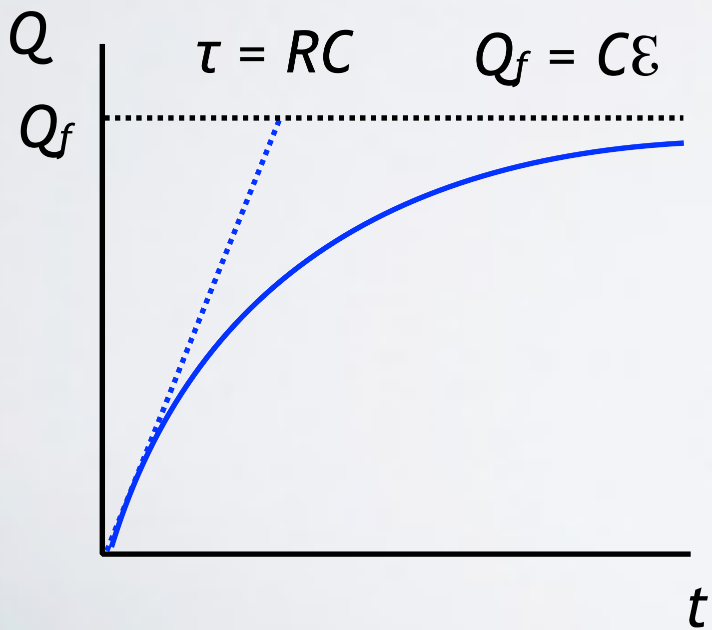
$$I = \frac{\mathcal{E}}{R}\exp^{-\frac{t}{RC}} = I_0 \exp^{-\frac{t}{\tau}}$$

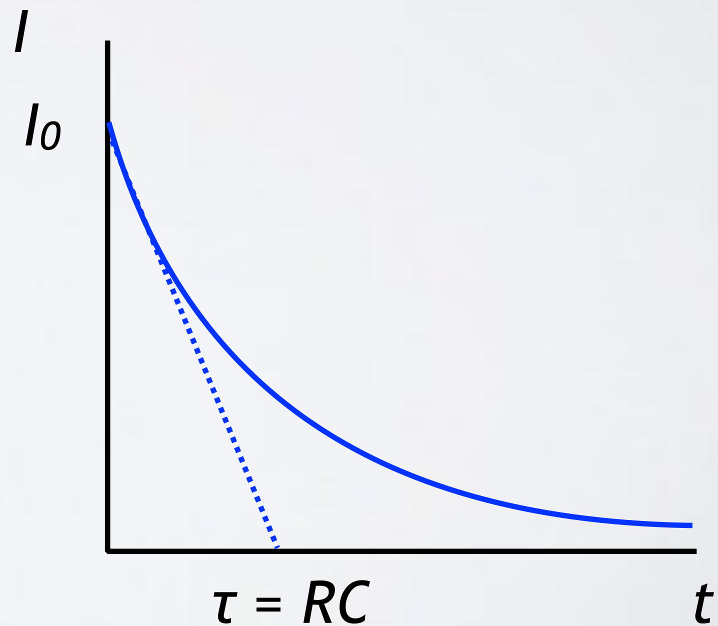
Eq 25-41

RC circuits

Charging a capacitor:







Summary

Energy in electric circuits

$$P = IV \qquad P = I^2 R = \frac{V^2}{R}$$

$$P = \frac{\Delta Q \mathcal{E}}{\Delta t} = I\mathcal{E}$$

Resistors

$$I = \frac{\mathcal{E}}{R + r} \qquad V_a - V_b = \mathcal{E} - Ir$$

$$E_{\text{stored}} = Q\mathcal{E}$$

Kirchoff's Laws

$$R_{eq} = R_1 + R_2 \qquad \frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

RC circuits

$$Q(t) = Q_0 \exp^{-\frac{t}{\tau}} \qquad \tau = RC$$

$$I = I_0 \exp^{-\frac{t}{\tau}}$$

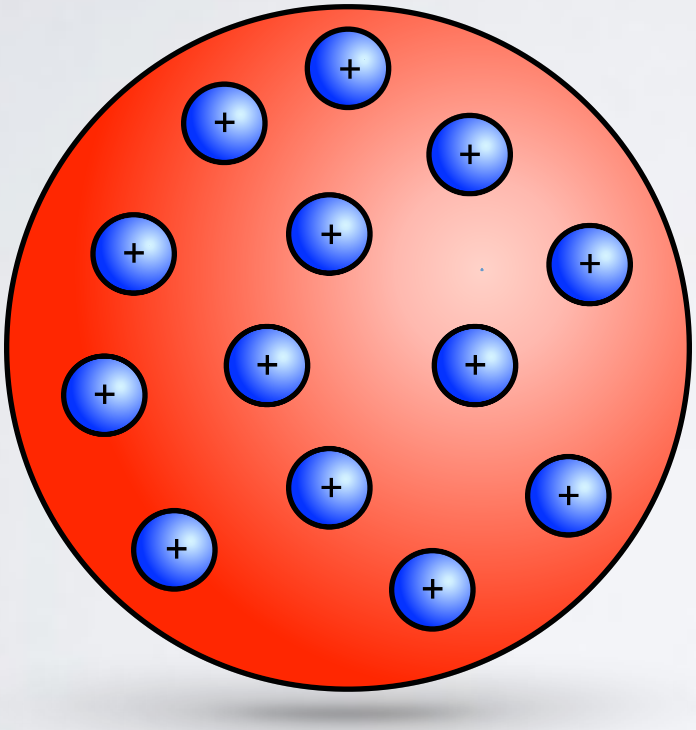
$$Q(t) = C\mathcal{E}\left(1 - \exp^{-\frac{t}{\tau}}\right)$$

Next up…

Follow-up lecture

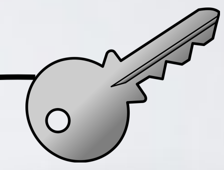
Capacitance

If it's loaded with charge Q, it acquires a uniform potential *V*



Its *capacitance C* is defined as:

$$C = \frac{Q}{V}$$

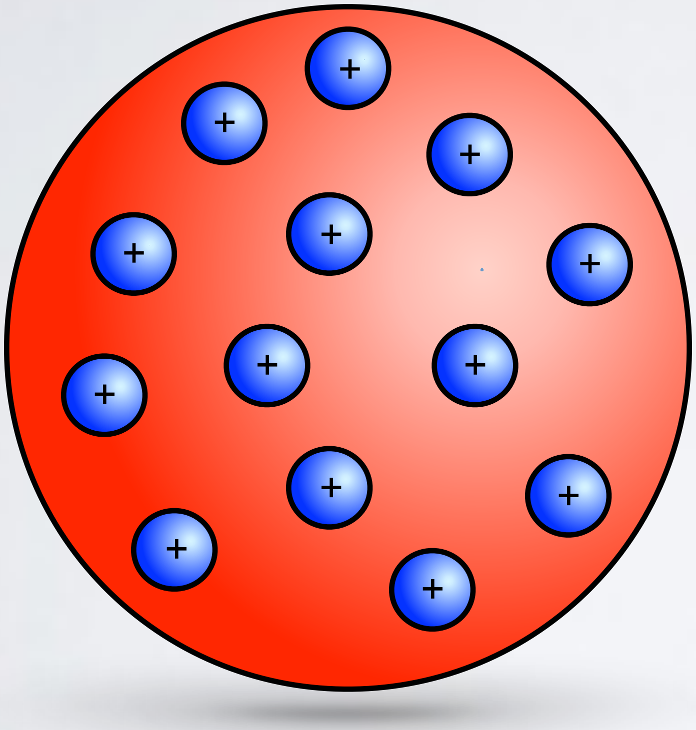


Eq 24-1

This is constant for a given conductor's geometry

Capacitance

If it’s loaded with charge Q, it acquires a uniform potential *V*



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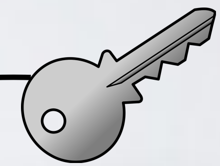
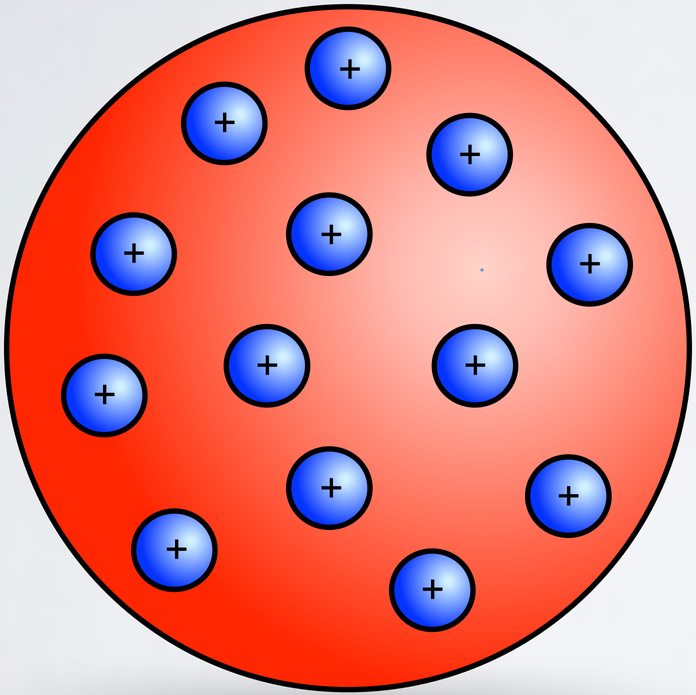


Eq 24-1

Note that it is changed by both the charge <u>and</u> the potential; moving nearby charges changes the capacitance!

Capacitance

If it's loaded with charge Q, it acquires a uniform potential *V*



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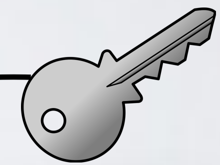


Eq 24-1

Its unit is the Farad (F):
1 Farad = 1 Coulomb per Volt
(this is a large unit!)

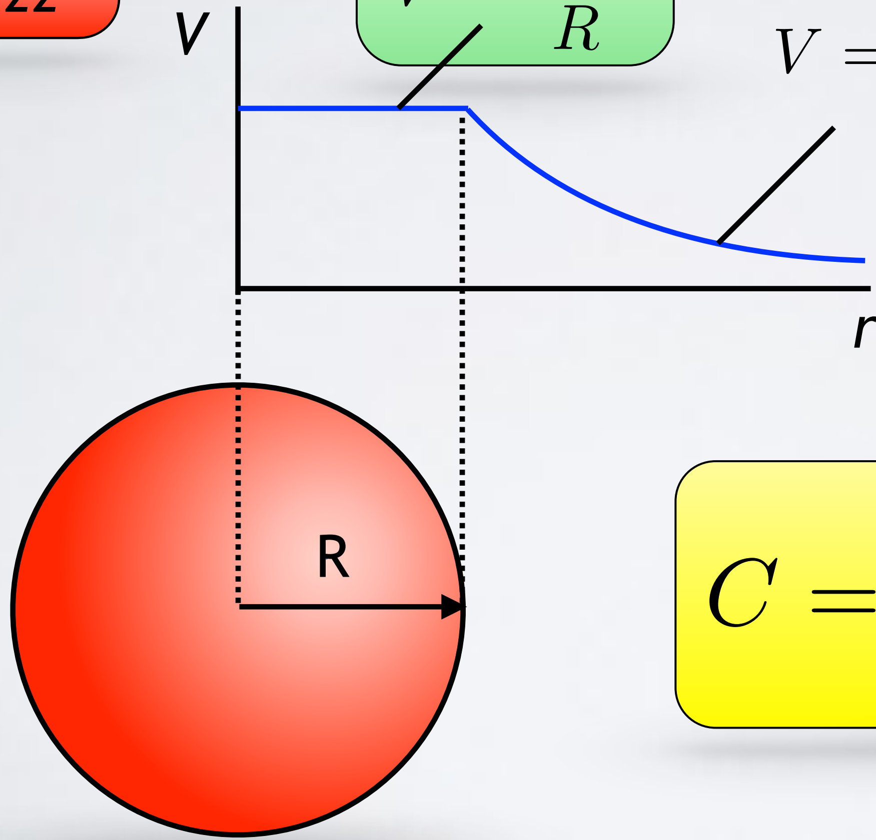
Capacitance

Let's look at the capacitance of a spherical conductor of radius *R*

Eq 23-22

$$V = \frac{kQ}{R}$$

$$V = \frac{kQ}{r}$$



$$C = \frac{Q}{V} = \frac{QR}{kQ}$$

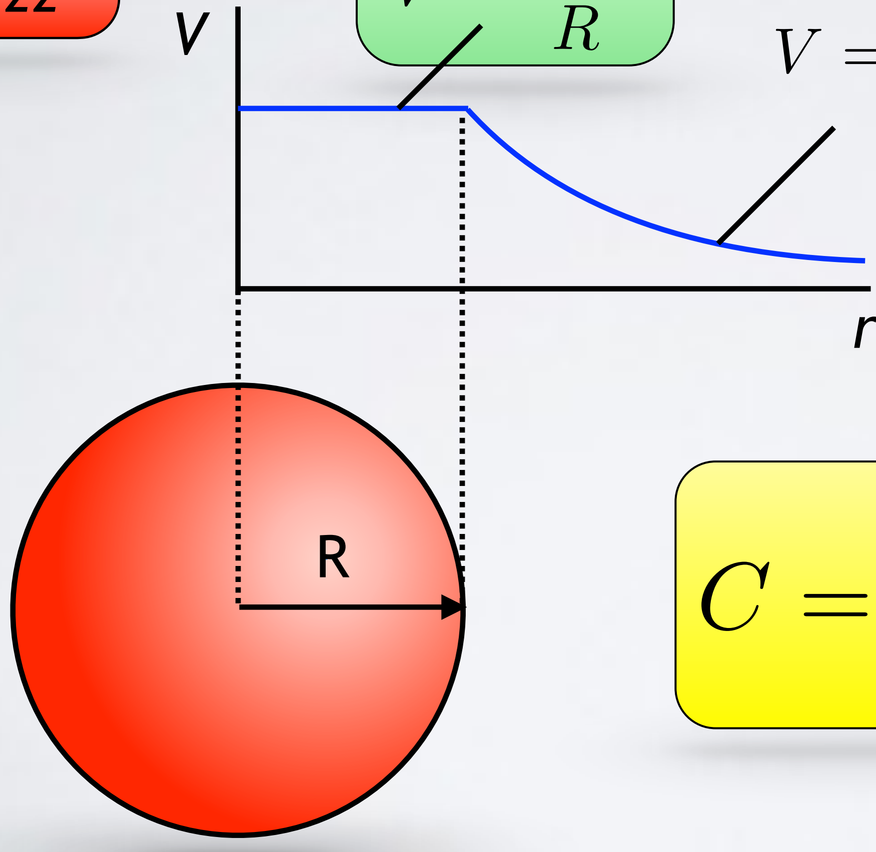
Capacitance

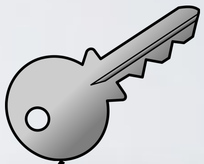
Let's look at the capacitance of a spherical conductor of radius *R*

Eq 23-22

$$V = \frac{kQ}{R}$$

$$V = \frac{kQ}{r}$$



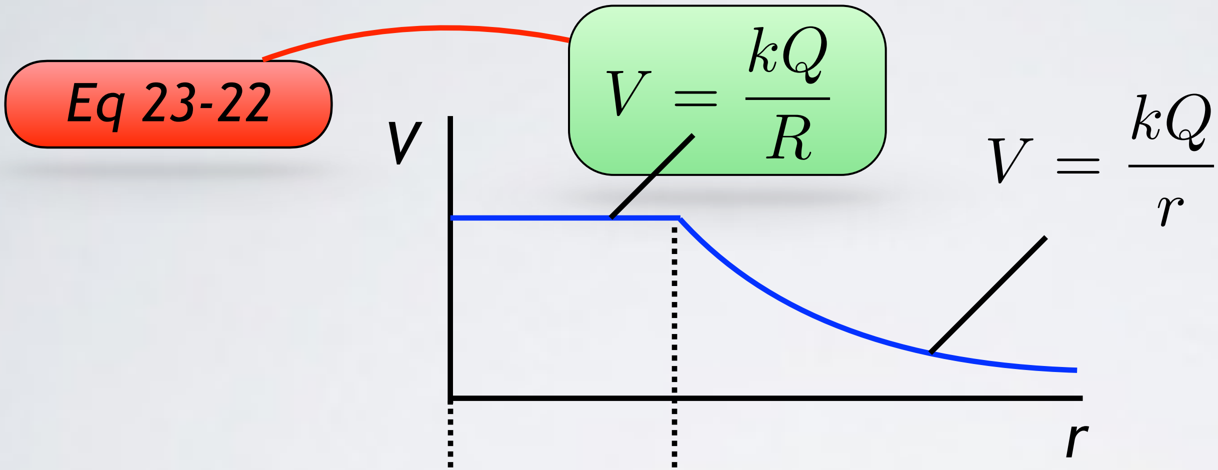


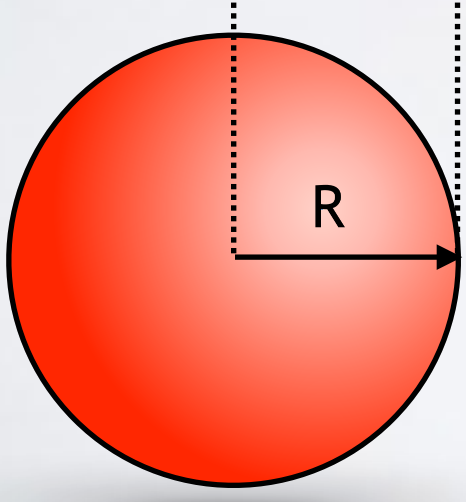
$$C = \frac{R}{k} = 4\pi\epsilon_0 R$$

Eq 24.2

Capacitance

Let's look at the capacitance of a spherical conductor of radius *R*





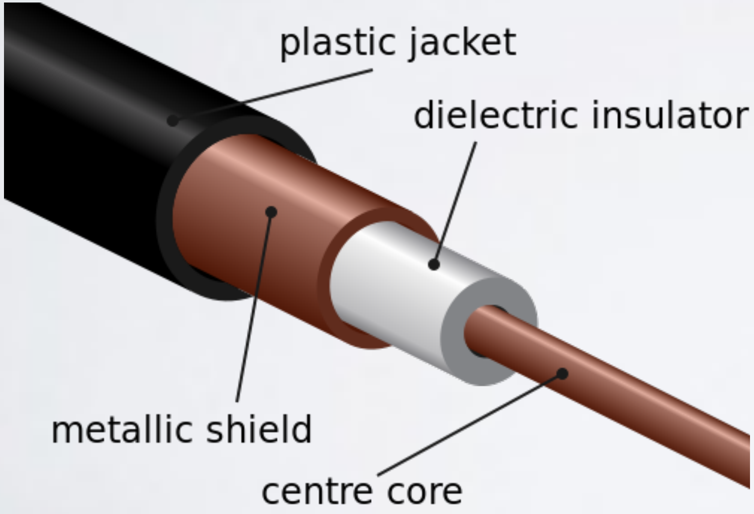
$$C = 4\pi \times 8.85 \times 10^{-12} \times 1\mathrm{m}$$

$$\simeq 10^{-10}\ \mathrm{F}$$

Cylindrical capacitor

Before we begin: why would we care about the properties of a cylindrical capacitor?

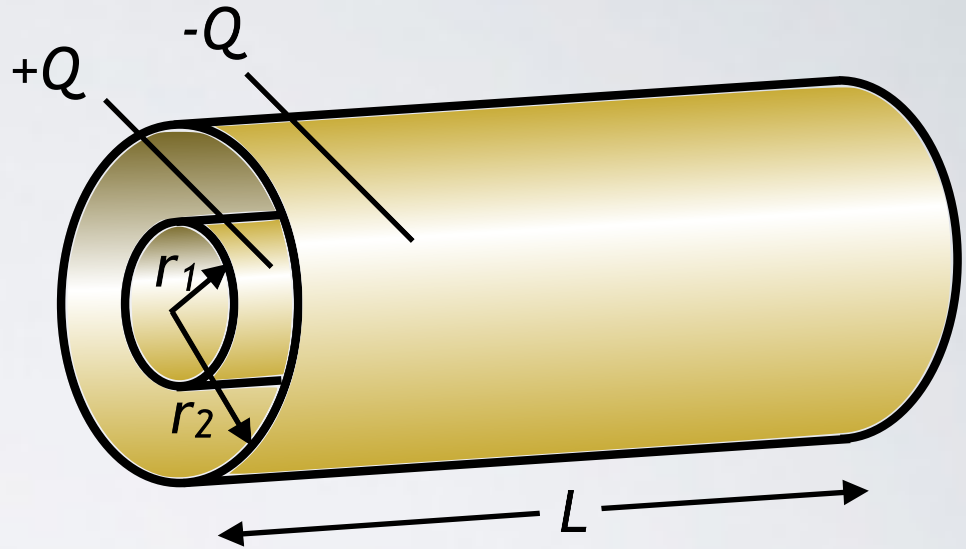
(HINT: have you ever watched TV?)



Co-axial cable

Cylindrical capacitor

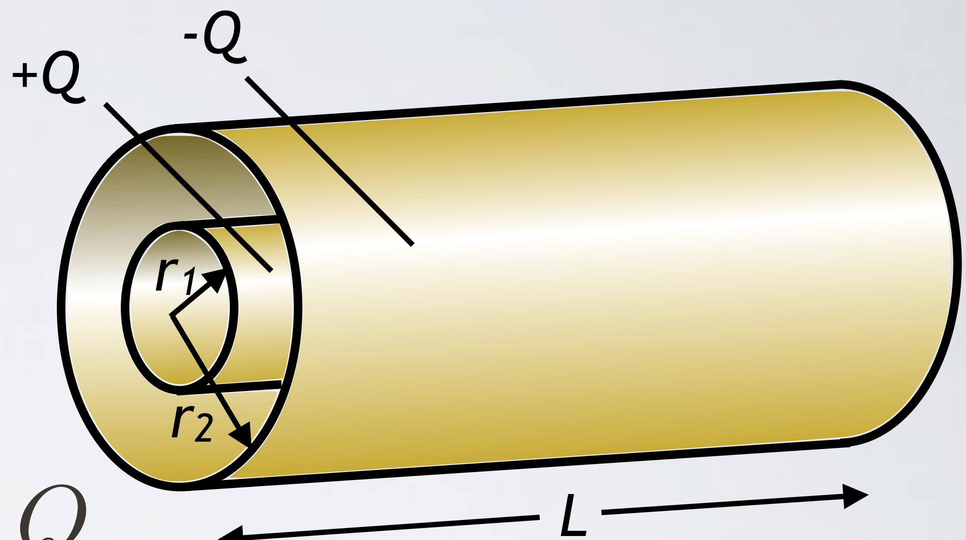
What is the capacitance?



Approach:

- Use Gauss' Law to find the electric field E
- Calculate the potential difference V
- Compute the capacitance from $C=Q/V$

Cylindrical capacitor



What is the capacitance?

Use Gauss' Law to find the electric field *E*

$$\Phi_E = \oint_S E_n \cdot dA = \frac{Q}{\epsilon_0}$$

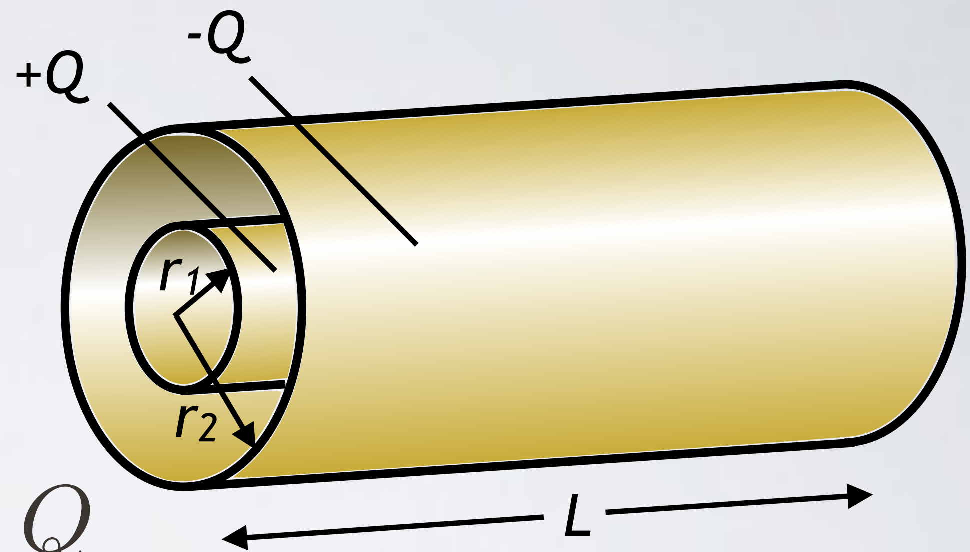
Eq 22-16

Integral of the electric field threading a suitably-chosen Gaussian surface

Cylindrical capacitor

What is the capacitance?

Use Gauss' Law to find the electric field *E*



$$\Phi_E = \oint_S E_n \cdot dA = \frac{Q}{\epsilon_0}$$

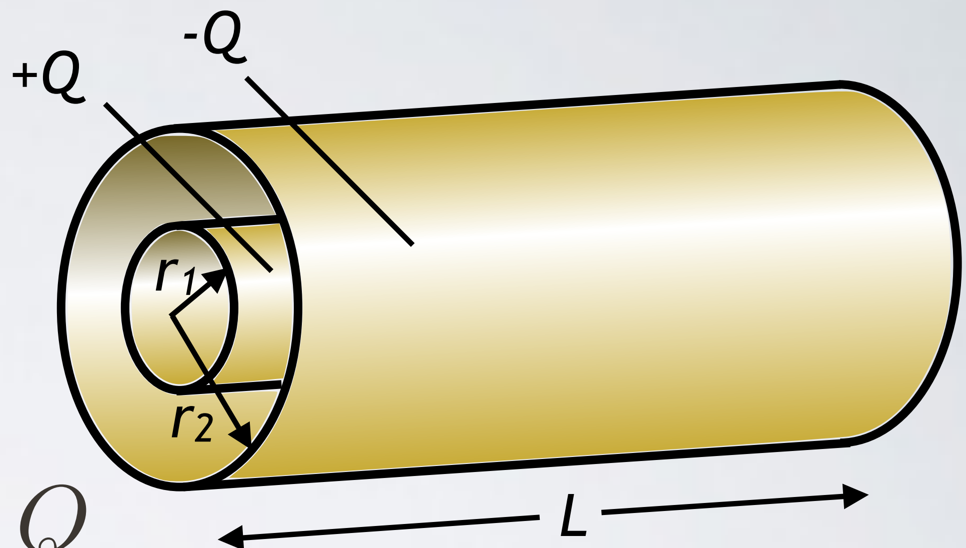
$$E\, 2\pi r L = \frac{Q}{\epsilon_0}$$

Use a cylindrical Gaussian surface, as the geometry is cylindrical

Cylindrical capacitor

What is the capacitance?

Use Gauss' Law to find the electric field E



$$\Phi_E = \oint_S E_n \cdot dA = \frac{Q}{\epsilon_0}$$

$$E\, 2\pi r L = \frac{Q}{\epsilon_0}$$

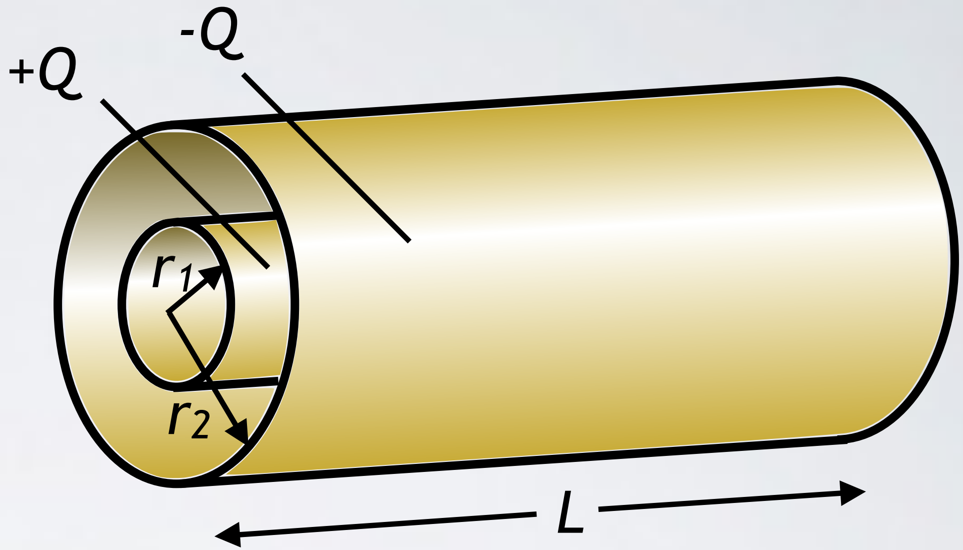
So

$$E = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{r}$$

Cylindrical capacitor

What is the capacitance?

Calculate the potential difference *V*



$$V = \int_{r_1}^{r_2} E(r)\, dr$$

Eq 23-2

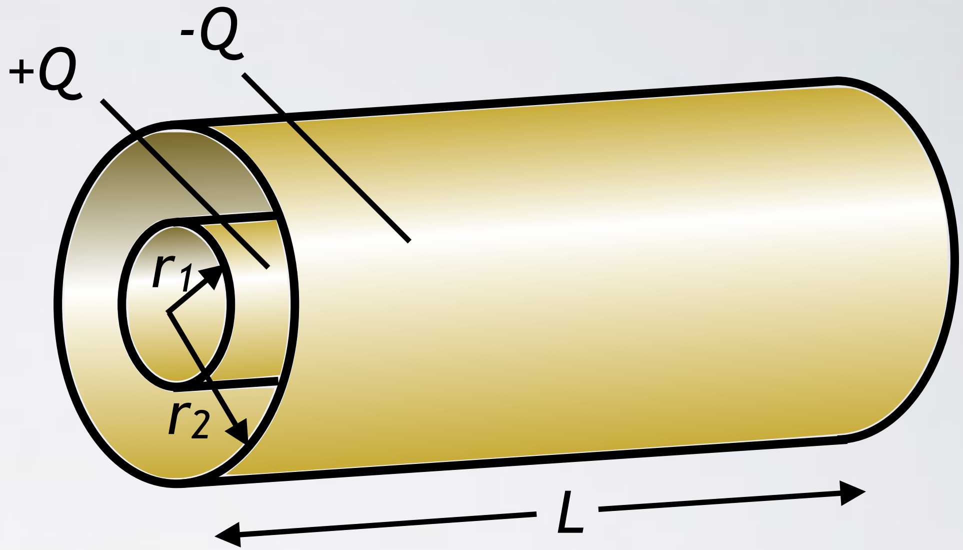
$$V = \frac{Q}{2\pi\epsilon_0 L}\int_{r_1}^{r_2} \frac{1}{r}\, dr = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{r_2}{r_1}\right)$$

Cylindrical capacitor

What is the capacitance?

Compute the capacitance from *C=Q/V*

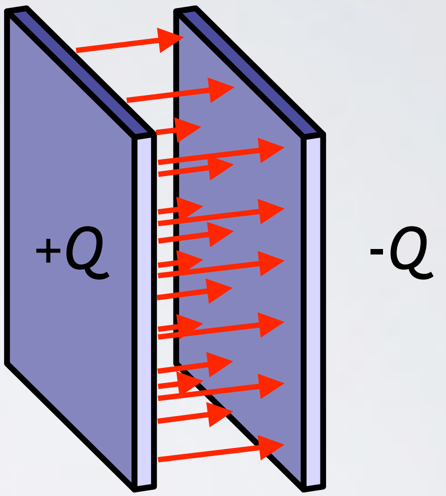
$$C = \frac{Q}{V}$$



$$C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{r_2}{r_1}\right)}$$

Parallel plate capacitor

This is the 'classic' form of a capacitor. What is its capacitance?

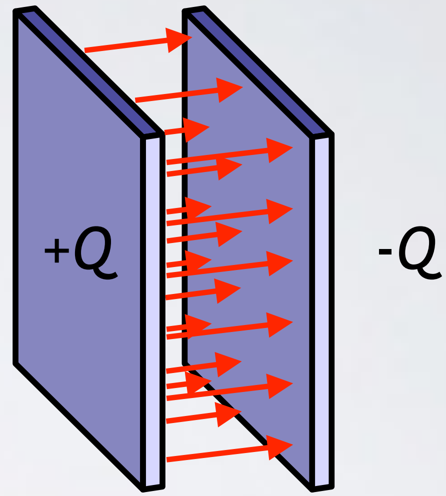


Approach:

- Use Gauss' Law to find the electric field E
- Calculate the potential difference V
- Compute the capacitance from $C=Q/V$

Parallel plate capacitor

This is the 'classic' form of a capacitor. The potential *V* is calculated using Gauss' Law:



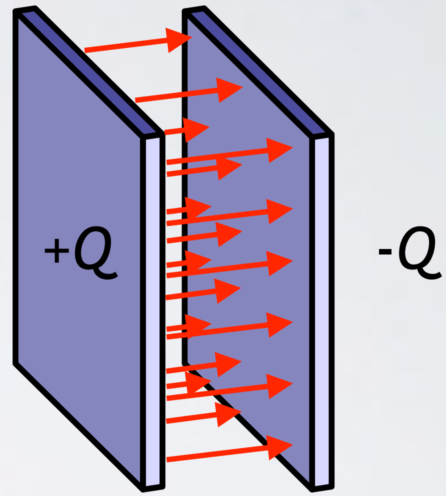
$$\Phi_E = \oint_S E_n \cdot dA = \frac{Q}{\epsilon_0}$$

Eq 22-16

Integral of the electric field threading a suitably-chosen Gaussian surface

Parallel plate capacitor

This is the 'classic' form of a capacitor. The potential *V* is calculated using Gauss' Law:



$$\Phi_E = \oint_S E_n \cdot dA = \frac{Q}{\epsilon_0}$$

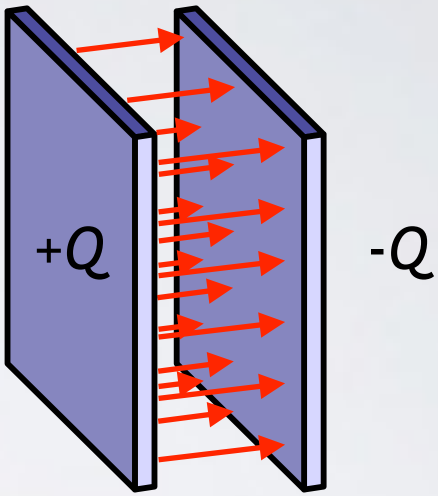
so

$$E = \frac{Q}{\epsilon_0 A}$$

Eq 22-21

Parallel plate capacitor

This is the 'classic' form of a capacitor. The potential *V* is calculated using Gauss' Law:

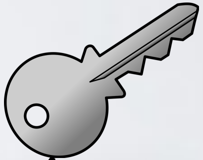


$$V = Ed = \frac{Qd}{\epsilon_0 A}$$

Eq 23-2

So its capacitance is

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

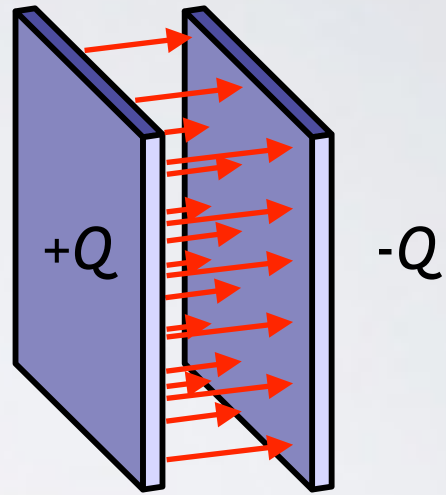


Eq 24-6

Parallel plate capacitor

Calculating the capacitance

$$C = \frac{\epsilon_0 A}{d}$$



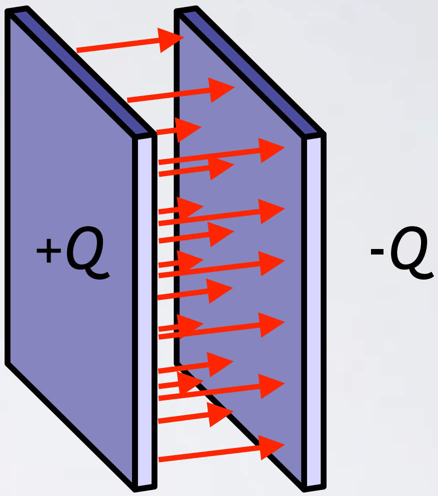
If we e.g. had a capacitor with two 2 x 2 m plates 1mm apart:

$$C = 8.85 \times 10^{-12} \frac{4}{0.001} \simeq 3 \times 10^{-8}\ \mathrm{F}$$

Capacitance

Worked example:

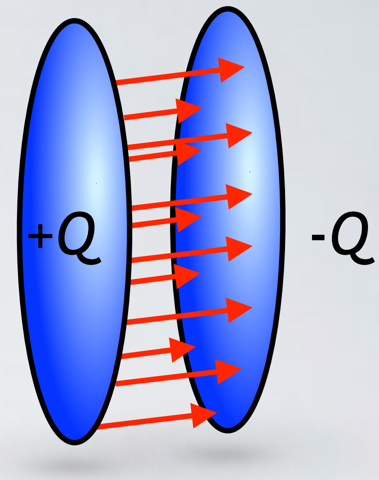
The charge on one plate of a capacitor is +30 μC and the charge on the other plate is −30 μC. The potential difference between the plates is 400 V. What is the capacitance of the capacitor?



$$C = \frac{Q}{V}$$

$$C = \frac{30 \times 10^{-6}}{400} = \underline{7.5 \times 10^{-8}\,\mathrm{F} = 75\,\mathrm{nF}}$$

Capacitance



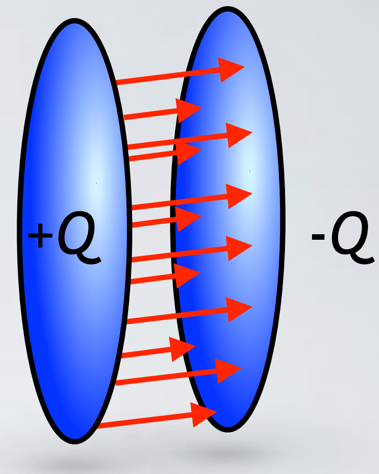
Worked example:

An electric field of $2{\times}10^4$ Vm^{-1} exists between the circular plates of a a parallel plate capacitor that has plate separation of 2 mm. (*a*) What is the potential difference across the capacitor plates? (*b*) What is the plate radius required if the positively charged plate is to have a charge of 10 μC.

$$V = Ed$$

$$V = 2 \times 10^4 \times 2 \times 10^{-3} = \underline{40\,\mathrm{V}}$$

Capacitance



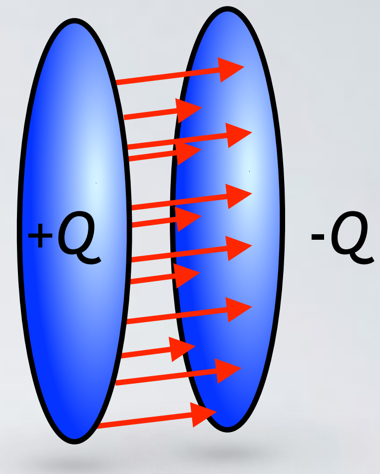
Worked example:

An electric field of 2×10^4 Vm^{-1} exists between the circular plates of a a parallel plate capacitor that has plate separation of 2 mm. (*a*) What is the potential difference across the capacitor plates? (*b*) What is the plate radius required if the positively charged plate is to have a charge of 10 μC.

$$V = Ed = \frac{Qd}{\epsilon_0 A} \Rightarrow A = \frac{Qd}{\epsilon_0 V}$$

$$r = \left(\frac{Qd}{\pi\epsilon_0 V}\right)^{\frac{1}{2}} = \left(\frac{10 \times 10^{-6} \times 2 \times 10^{-3}}{\pi\epsilon_0 \times 40}\right)^{\frac{1}{2}}$$

Capacitance



Worked example:

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$$V = Ed = \frac{Qd}{\epsilon_0 A} \Rightarrow A = \frac{Qd}{\epsilon_0 V}$$

$$r = \left(\frac{Qd}{\pi \epsilon_0 V} \right)^{\frac{1}{2}} = \underline{4.2\,\mathrm{m}}$$

Summary

$$U = \frac{1}{2}\sum_{i=0}^{n} q_i V_i$$

- Electrostatic potential energy

$$U = \frac{1}{2}QV$$

- Capacitance

$$C = \frac{Q}{V}$$

  - Spherical capacitors

$$C = \frac{R}{k} = 4\pi\epsilon_0 R$$

  - Cylindrical capacitors
  - Parallel plate capacitors

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d}$$

Some Qs

A2. (a) The potential difference across a capacitor is $V = 10$ V when the charge on one of its plates is $Q = 10^{-6}$ C. What is the charge on this capacitor if the voltage is $V = 1$ V?
Numerical Answer: 10^{-7} C

(b) Calculate the electrostatic potential energy of the capacitor when charge Q is $Q = 2 \times 10^{-3}$ C.
Numerical Answer: 20 J

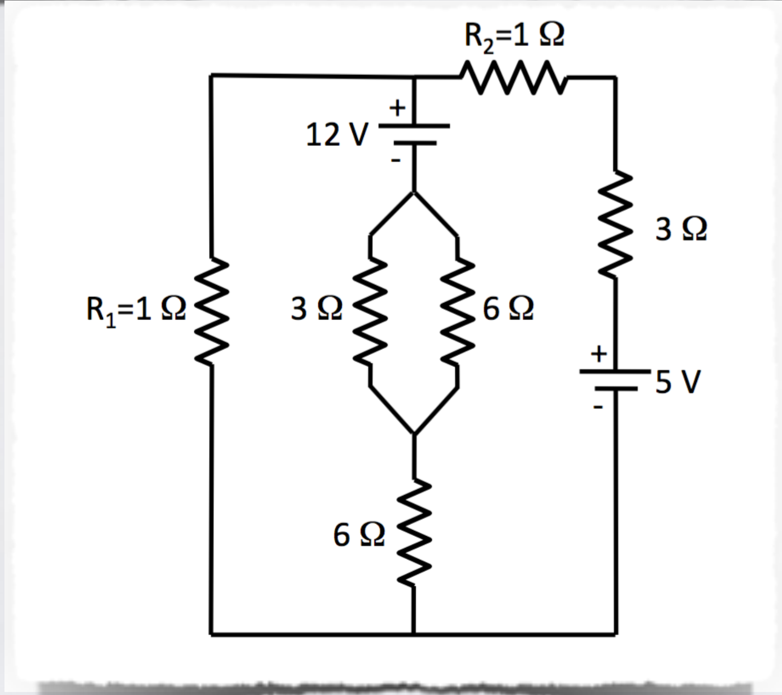
(c) Two resistors with $R_1 = 1\Omega$ and $R_2 = 3\Omega$, are connected in parallel. The circuit is connected to an ideal battery with an output voltage of $V = 3$ V. What is the electrical power released in the circuit?
Numerical Answer: 12 W

Some Qs

B3. (a) State Kirchhoff's rules for electric circuits. [4]

(b) Simplify the circuit shown in Figure 1 by placing equivalent resistors wherever possible for resistors in series or parallel and draw the resulting circuit. [4]

(c) What current flows through resistor R_1? What current flows through resistor R_2? [12]



Some Qs

(b) A cylindrical capacitor is made of a long wire with length L and radius R_1, and a concentric outer cylindrical shell of the same length and radius $R_2 > R_1$. Find the capacitance of the capacitor. **[6]**

B3. (a) A capacitor C discharges through a resistor R. Find the current I in the resistor as a function of time if the initial charge on the capacitor is Q_0 at time $t = 0$. **[6]**

Magnetic Flux

Analogous to electric flux, magnetic flux is defined as

$$\phi = \int_s \vec{B}.\hat{n}\, dA = \int_s B_n dA$$

Eq 28-1

Component of $\vec{B}$ in direction of $\hat{n}$

where

$$B_n = B\cos\theta$$
$$\phi = BA\cos\theta$$

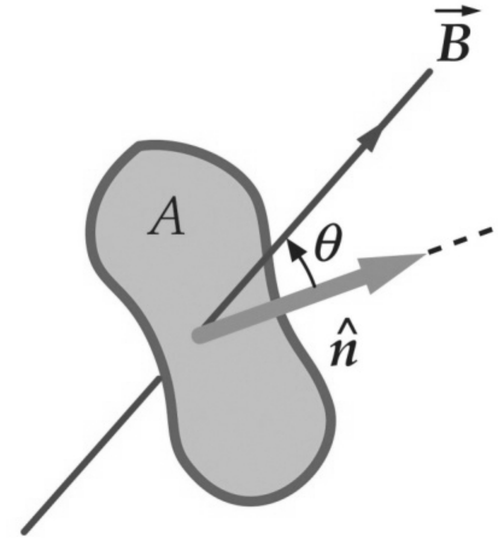


Fig 28-1

If the surface is bounded by a coil of N turns then

$$\phi = NBA\cos\theta$$

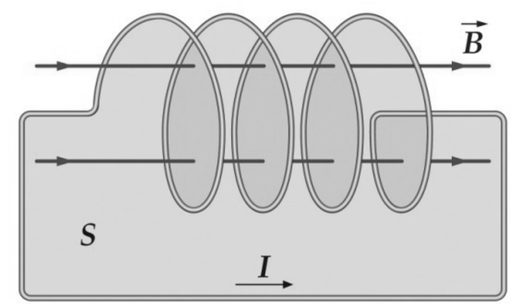


Fig 28-2

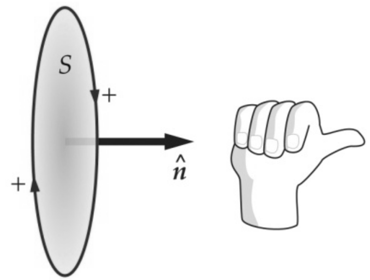
Units of ϕ are Webers (Wb) or Tm^2

Direction of the Induced EMF

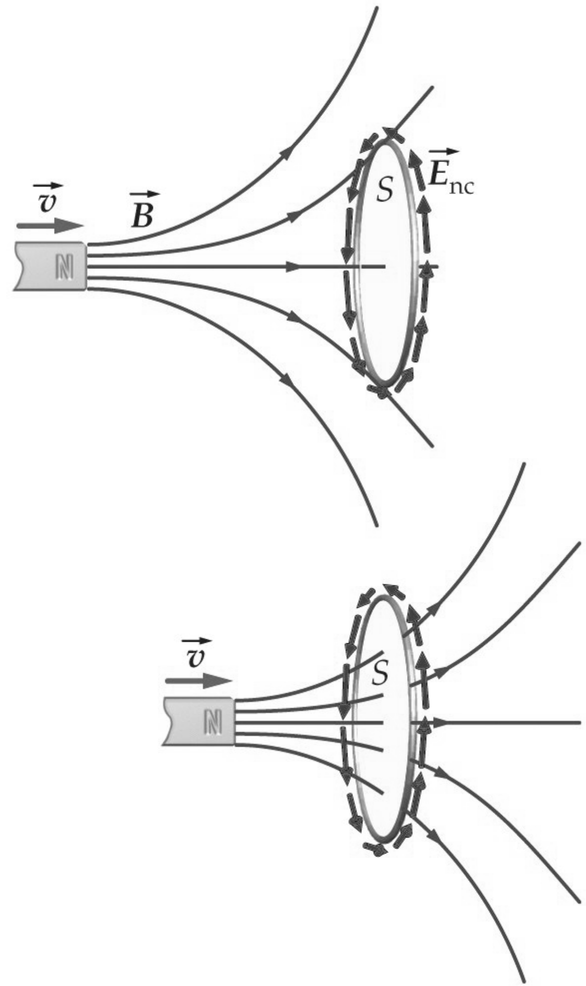
Faraday's Law:

$$\varepsilon = \oint_c \vec{E}.d\vec{l} = -\frac{d}{dt}\int_s \vec{B}.\hat{n}dA = -\frac{d\phi}{dt}$$

Use the right-hand curl rule to determine the *positive* direction for the induced EMF (thumb points in the direction of $\hat{n}$)



$\hat{n}$ can be chosen arbitrarily since the right answer will come out in the maths due to the $B.\hat{n}$ term in Eq 28-5



Induced EMF in a Circular Coil

Example: EMF Induced in a Circular Coil

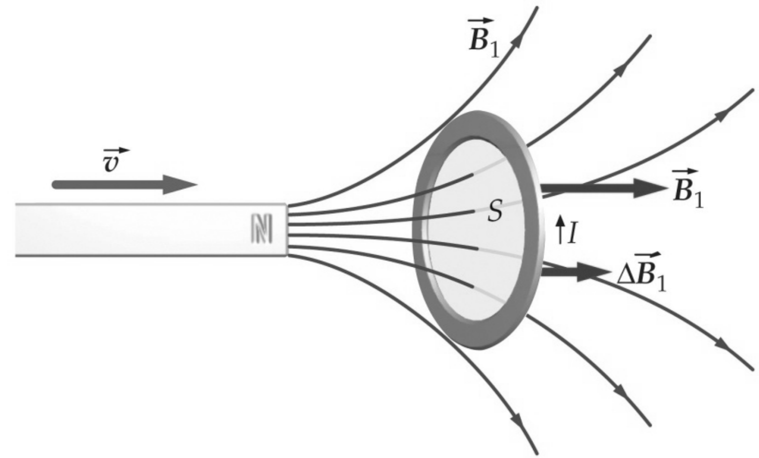
An 80-turn coil of radius 5 cm and resistance 30Ω sits in a region with a uniform magnetic field normal to the plane of the coil.

At what rate must the magnitude of the magnetic field change to produce a current of 4A in the coil?

Lenz's Law

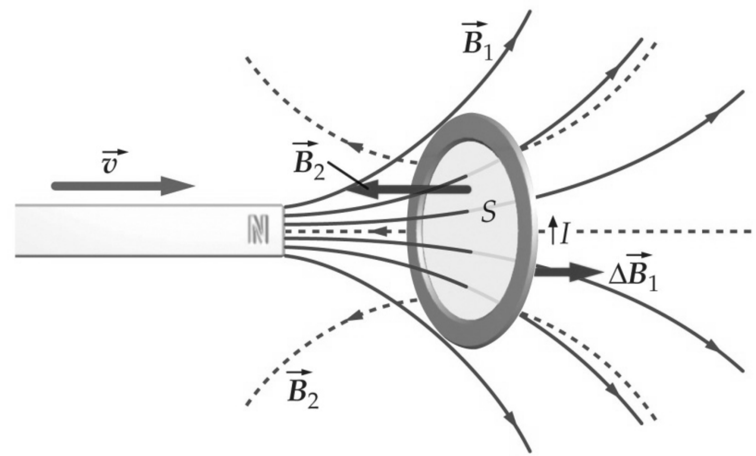
Defines the direction of the EMF induced in Faraday's Law

The induced EMF is in a direction which will tend to oppose the change which is causing it



Moving bar magnet towards a conducting loop induces an EMF

The EMF in the loop has an associated magnetic field which is in the direction opposing the magnetic field change

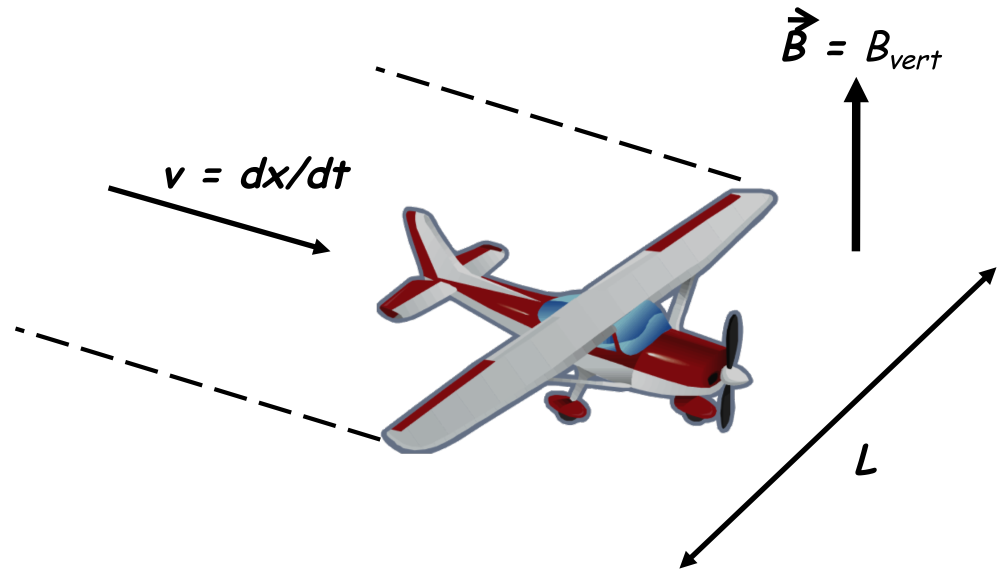


When a magnetic flux through a surface changes, the magnetic field due to any induced current produces a flux of its own through the same surface and in opposition to the change
(Alternative form of Lenz's Law)

Motional EMF

When a conductor cuts through magnetic flux an EMF is induced across it

Eg. Between the wing-tips of a plane moving through the Earth's magnetic field.



Faraday's law can be used to show

$$\varepsilon = -BLv$$

So magnitude of ε given by

$$|\varepsilon| = BLv$$

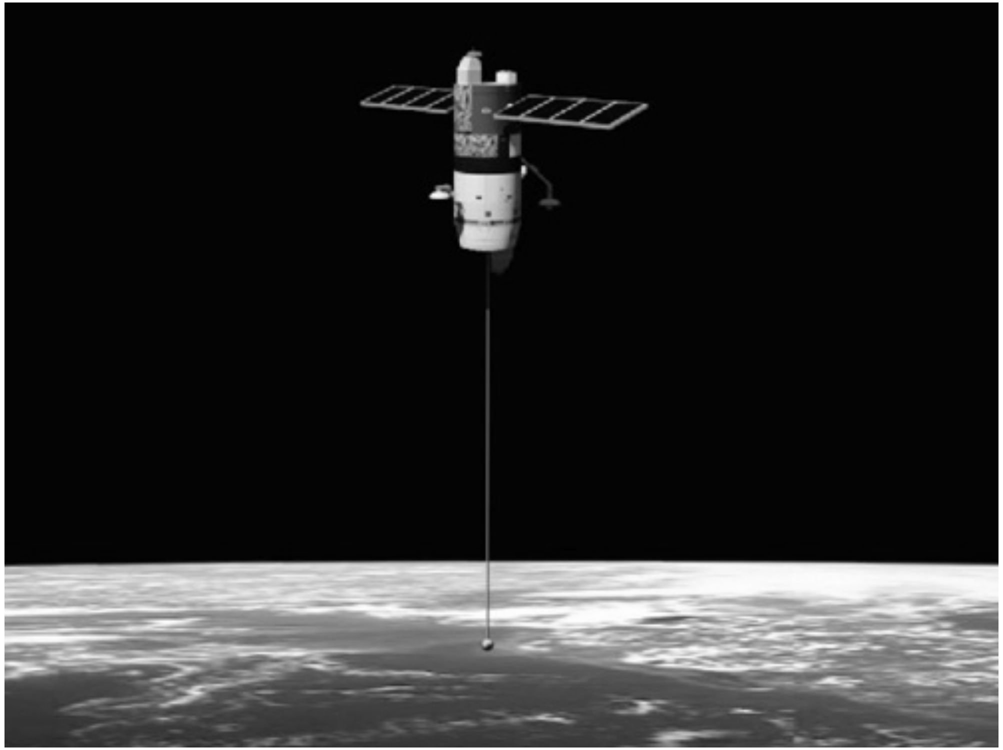
Typically, for this example, induced EMF would be ~0.5 V

Motional EMF: Electrodynamic Tethers in Space

- Spacecraft suspends a long conducting cable underneath it (L ~1 km)

- This cuts through magnetic flux in the Earth's B field and generates an EMF providing power

$$|\varepsilon| = BLv$$



Motional EMF: Electrodynamic Tethers in Space

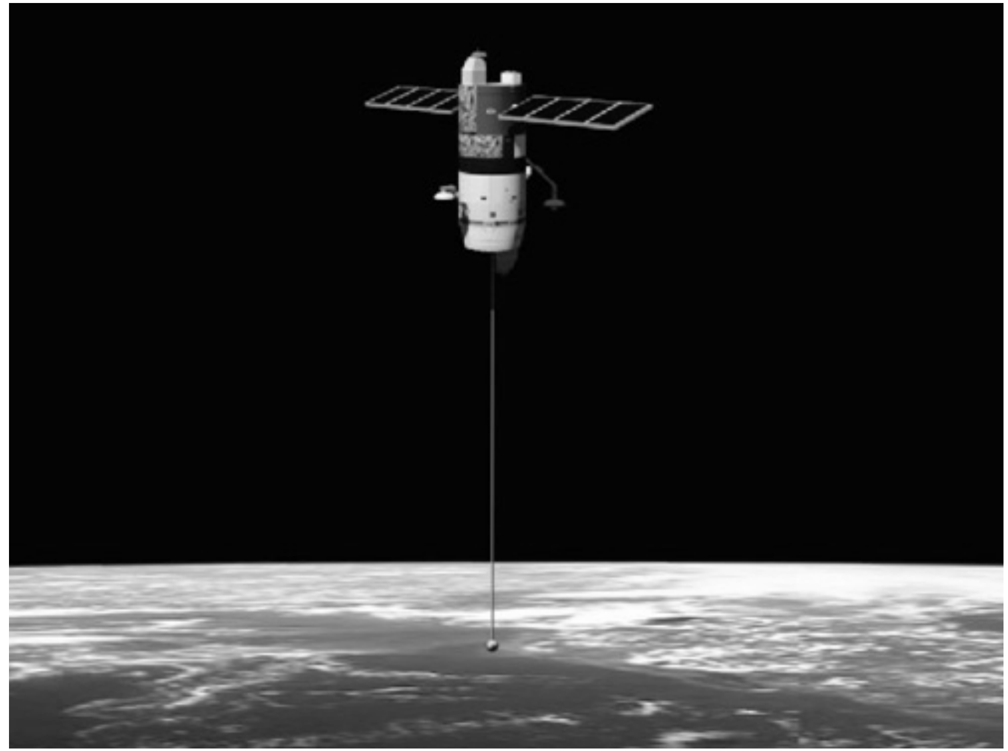
- Spacecraft suspends a long conducting cable underneath it (L ~1 km)

- This cuts through magnetic flux in the Earth's B field and generates an EMF providing power

$$|\varepsilon| = BLv$$

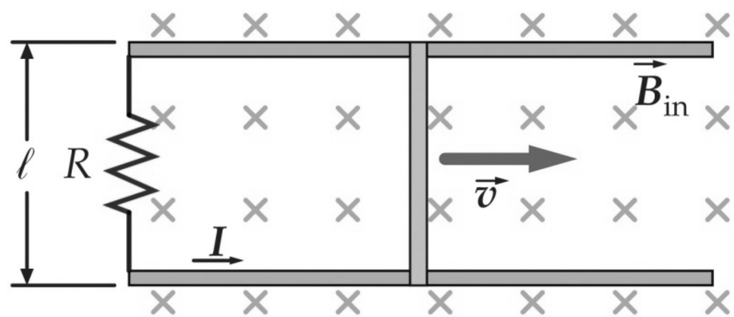
- What is the typical EMF induced for a spacecraft at 300 km? Use a realistic value for ***B*** at this height.

- First person to email the correct answer to Darren.Wright@le.ac.uk wins a chocolate bar!



Example: Magnetic Drag

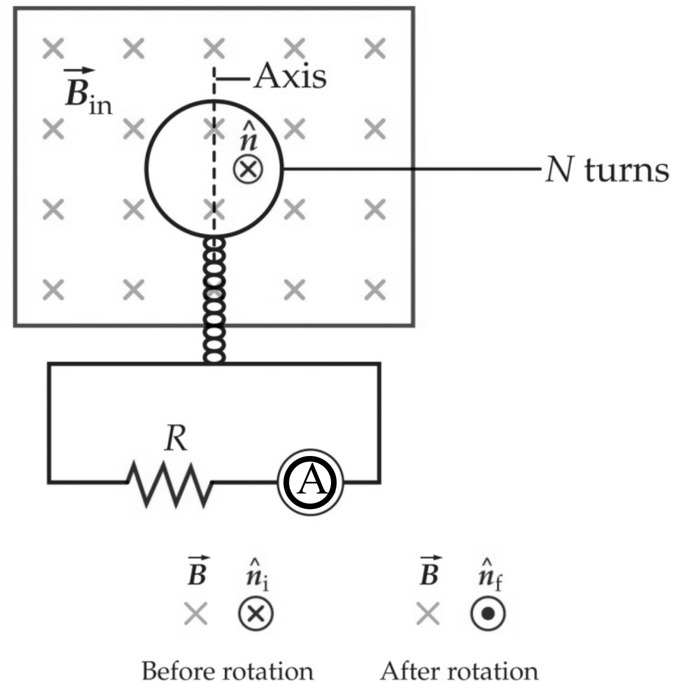
A rod of mass m slides on frictionless conducting rails in a region of static uniform magnetic field $\mathbf{B}$ directed into the page. An external agent is pushing the rod, maintaining its motion to the right at constant speed v_0. At time t=0 the force stops pushing and the rod continues forward, with an initial velocity v_0, being slowed by the magnetic force. Find the speed v of the rod as a function of time.



Motional EMF: The Electric Generator

[See also Ch 29]

If a conducting loop is rotated in a magnetic field then an alternating current is excited. This is the basic principle of an AC generator.



Faraday's Law:

$$\varepsilon = -\frac{d\phi}{dt}$$

Magnetic flux through a loop with N turns

$$\phi = \int_s \vec{B}.\hat{n}\, dA = BNA\cos\theta = BNA\cos\omega t$$

$$\frac{d\phi}{dt} = -BNA\,\omega\sin\omega t$$

where $\theta=\omega t$; θ is the angle between the magnetic field and the normal of the surface of the loop

Hence

$$\varepsilon = BNA\,\omega\sin\omega t$$

Resulting EMF is oscillatory about 0 Volts with

$\varepsilon_{peak}=BNA\omega$

Magnetic Inductance

We know that a changing magnetic field through a conducting loop induces an EMF as defined by Faraday's law

However, the current flowing in the loop leads to a magnetic field which opposes the external magnetic field (Lenz's Law)

Changing the current in the circuit affects that circuit and leads to a *self-induced EMF*

The magnetic flux through the loop is proportional to the current flowing in it

$$\boxed{\phi \propto I} \quad \text{or} \quad \boxed{\phi = LI}$$

where L is a constant and a property of the circuit called the *self-inductance*

Faraday's Law:

$$\boxed{\varepsilon = -\frac{d\phi}{dt} = -\frac{d(LI)}{dt} = -L\frac{dI}{dt}}$$

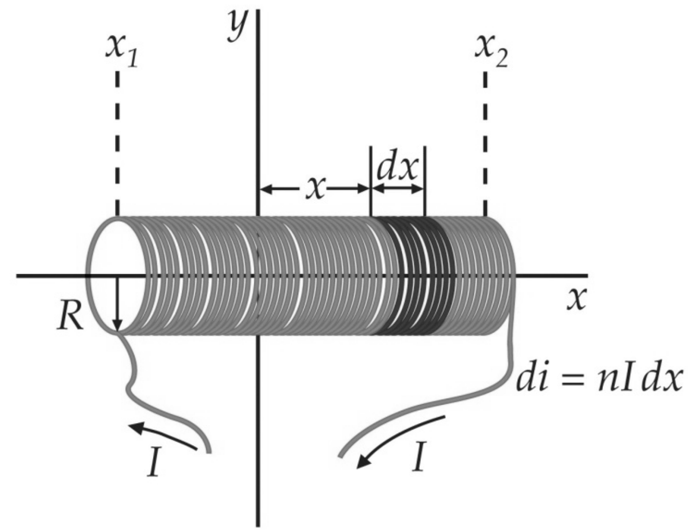
Note: ε is zero for a steady current. Large ε at power on (back EMF)

Units of L: Webers Amp^{-1} or *Henrys*

Self-Inductance in a Solenoid

For a tightly wound solenoid of length *l*, cross-sectional area *A* and *N* turns (or *n* turns per unit length) carrying a current *I*

$$\phi = \frac{\mu_0 N^2 I A}{l} = \mu_0 n^2 I A l$$



Thus the (self-) inductance of the solenoid is

$$L = \frac{\phi}{I} = \frac{\mu_0 N^2 A}{l} = \mu_0 n^2 A l$$

Like capacitance, inductance is dependent only on the geometry of the coil and not the current which is flowing

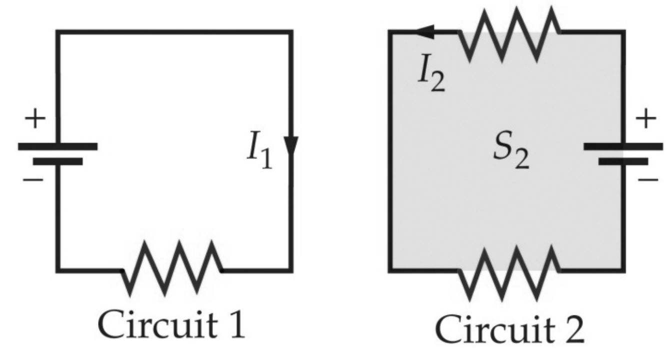
Permeability of free space
$\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}

Note: Real inductors have an internal resistance, *r*, such that the potential difference across the inductor is

$$\Delta V = \varepsilon_{ind} - Ir = -L\frac{dI}{dt} - Ir$$

Mutual Inductance

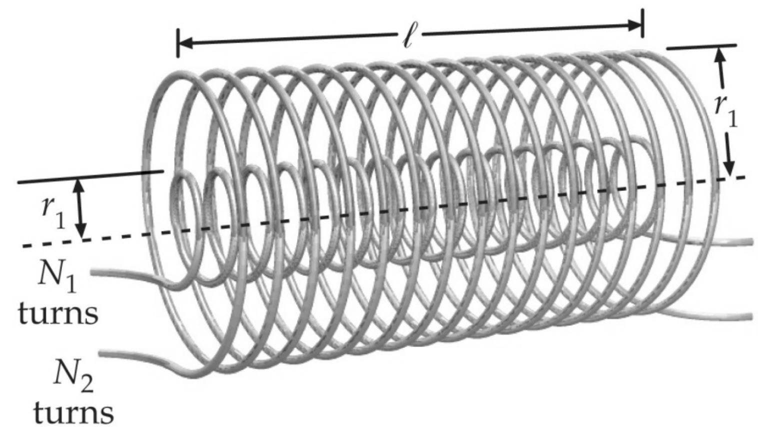
When inductive circuits are placed in close proximity, the magnetic flux through one circuit is now due to the currents flowing through itself AND those in the other circuits



We now consider the mutual inductance, *M*, rather than self-inductance, *L*

The magnetic flux of circuit 1 through circuit 2 is given by

$$\boxed{\phi_{2,1} = M_{2,1} I_1}$$



And the total flux through circuit 2 is

$$\boxed{\phi_2 = \phi_{2,2} + \phi_{2,1}}$$

Mutual Inductance

Consider Fig 28-28. The inner coil carries a current I_1 and within that solenoid the magnetic field magnitude is given by

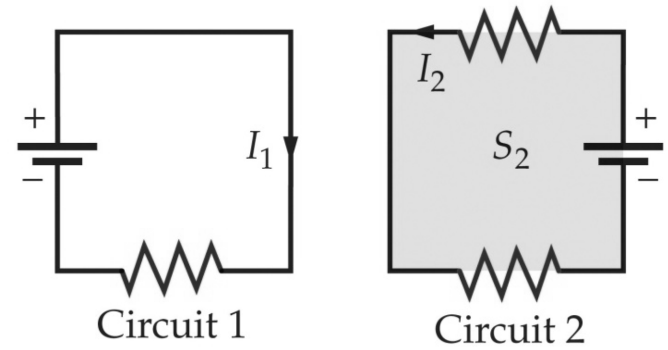
$$B_1 = \mu_0 (N_1 / l) I_1 = \mu_0 n_1 I_1$$

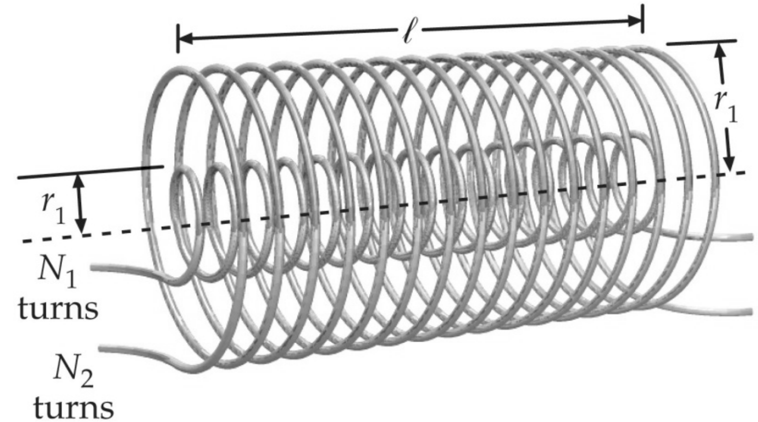
The flux of B_1 through the second (outer) solenoid

$$\phi_{2,1} = N_2 B_1 A = N_2 B_1 . \pi r_1^2 = \mu_0 n_2 n_1 l (\pi r_1^2) I_1$$

(Note that here $A = A_1 = \pi r_1^2$ since B_1 is zero outside the inner coil)

$$M_{2,1} = \frac{\phi_{2,1}}{I_1} = \mu_0 n_2 n_1 l \pi r_1^2$$

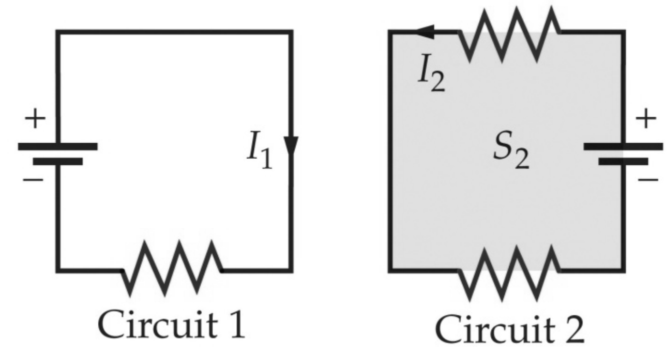




Mutual Inductance

If the geometry of the two circuits is not changing then it can be shown that

$$M_{1,2} = M_{2,1} = M$$

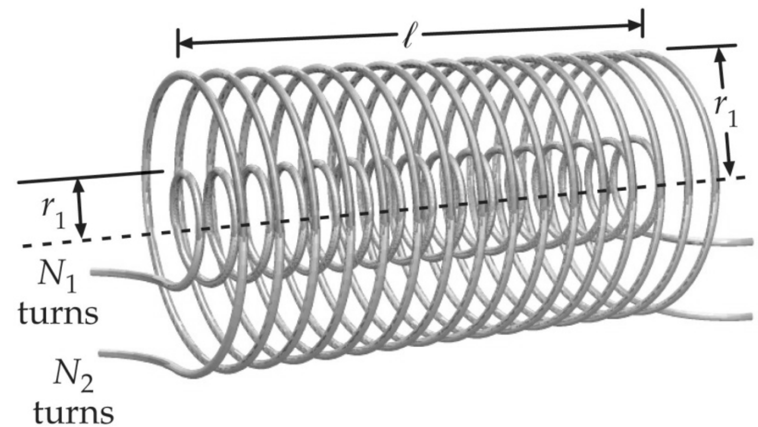


Thus

$$\varepsilon_2 = -M \frac{dI_1}{dt} \quad \text{and} \quad \varepsilon_1 = -M \frac{dI_2}{dt}$$

But if the geometry of the system is changing ie. M=M(t) then

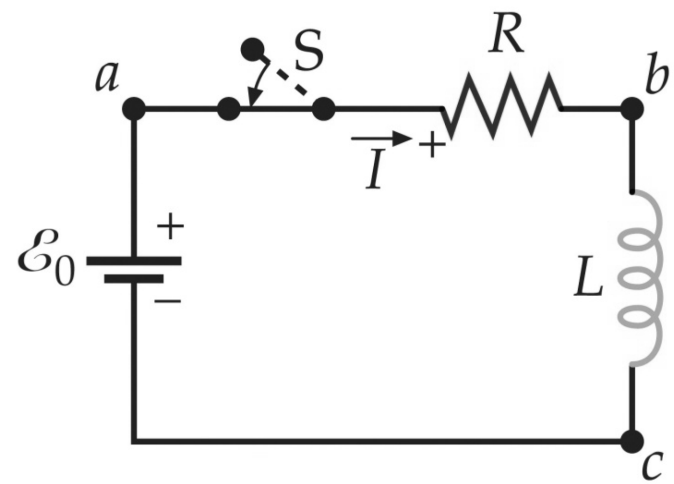
$$\varepsilon_2 = -M \frac{dI_1}{dt} - I_1 \frac{dM}{dt}$$



Magnetic Energy in an Inductor

Consider the potential differences across each component using Kirchoff's loop rule

$$\varepsilon - IR - L\frac{dI}{dt} = 0$$



Multiply by I to derive power

$$\varepsilon I - I^2 R - LI\frac{dI}{dt} = 0$$

Power from battery

Power Dissipated in resistor

Power delivered to inductor

Thus, energy stored in the inductor

$$U_m = P_m t = \int LI\frac{dI}{dt}dt = \int_0^{I_f} LI dI = \frac{1}{2}LI_f^2$$

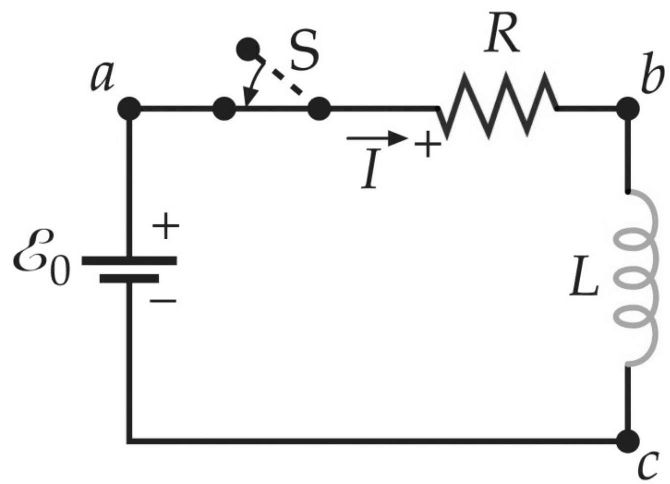
General case: energy stored by an inductor carrying a current I

$$U_m = \frac{1}{2}LI^2$$

Magnetic Energy in an Inductor

In the case of a solenoid we know that

$$B = \mu_0 n I \qquad \text{and} \qquad L = \mu_0 n^2 A l$$



Thus

$$U_m = \frac{1}{2} L I^2 = \frac{1}{2}(\mu_0 n^2 A l)\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}.Al$$

Hence *magnetic energy density* can be written as

$$u_m = \frac{U_m}{Volume} = \frac{U_m}{Al} = \frac{B^2}{2\mu_0}$$

This is one example proving the general result

$$u_m = \frac{B^2}{2\mu_0}$$

Transformers

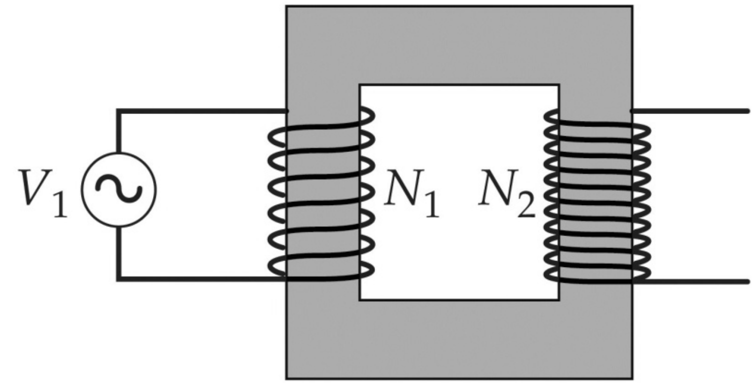
Transformers are devices of great practical importance
- industrial applications

Uses the magnetic linkage (flux) between two mutually inductive circuits to transform voltages

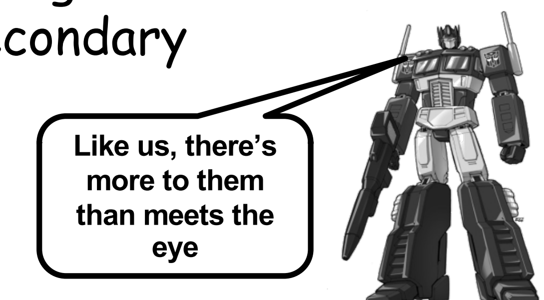
Has the advantage that little power is lost (typically 90-95% efficient)

Circuit 1: Primary
Circuit 2: Secondary

Iron core increases the magnetic field for a given current and guides all the flux created in the primary coil through the secondary



Have to use an alternating current
- generates a solenoidal field in the primary
- time varying field induces an EMF in the secondary



Transformers

In circuit 1:

$$V_1 = N_1 \frac{d\phi_{turn}}{dt}$$

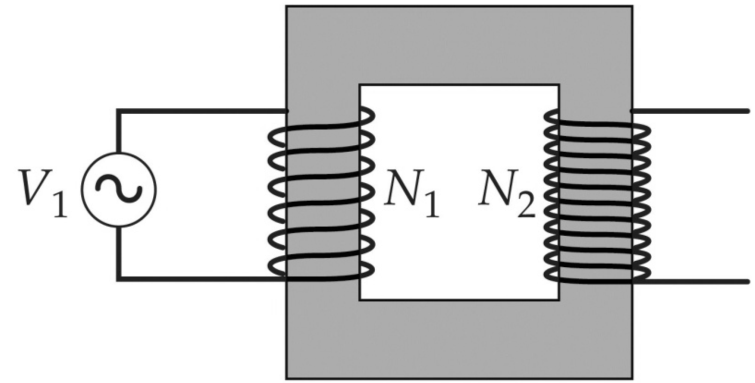
In circuit 2:

$$V_2 = N_2 \frac{d\phi_{turn}}{dt}$$

Where ϕ_{turn} is the flux though *each* turn of the coils

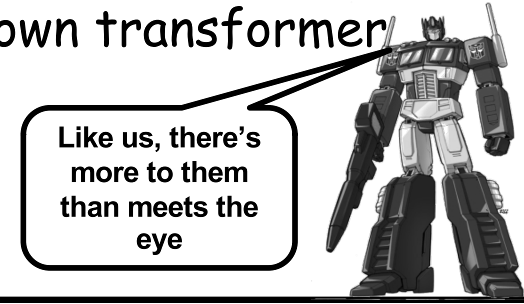
Since ϕ_{turn} is the same through both circuits then

$$V_2 = \frac{N_2}{N_1} V_1$$



If $N_2 > N_1$ then $V_2 > V_1$
- step-up transformer

If $N_2 < N_1$ then $V_2 < V_1$
- step-down transformer



Displacement Current

Ampère's law is defined as

$$\oint_c \vec{B}.d\vec{l} = \mu_0 I$$

For <u>any</u> area S bounded by curve C

Maxwell realised that this breaks down where the current becomes *discontinuous* as is the case for a **capacitor**

S_1 and S_2 are both surfaces bounded by the curve C. Yet current I crosses S_1 but NOT S_2.

S_2 appeared to be a surface which did not obey Ampère's law

Maxwell rewrote Ampère's law in a more generalised form:

$$\oint_c \vec{B}.d\vec{l} = \mu_0 (I + I_d)$$

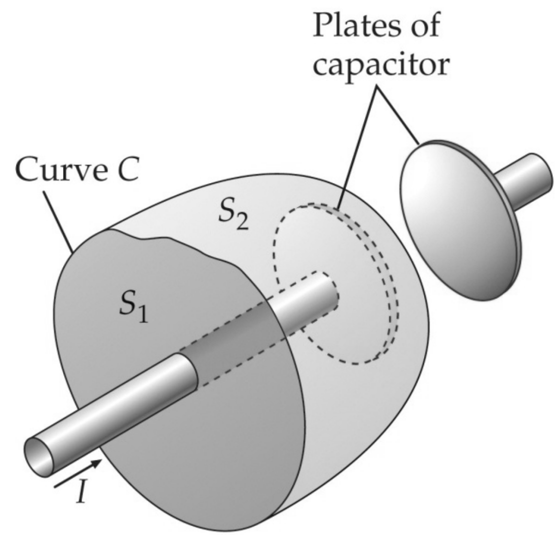


Fig 30-1

I_d is Maxwell's *displacement current*, given by

$$I_d = \varepsilon_0 \frac{d\phi_e}{dt}$$

REFERENCE

Many thank to the University of Leicester