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Solution of integral equations via new Z-contraction mapping in G_b-metric spaces

A. A. Mebawondu[1] orcid.org/0000-0001-6646-5335
C. Izuchukwu[2] orcid.org/0000-0002-8262-8605
K. O. Oyewole[3] orcid.org/0000-0002-4451-2126
O. T. Mewomo[4] orcid.org/0000-0003-0389-7469

University of KwaZulu-Natal, School of Mathematics, Statistics and Computer Science, DST-NRF CoE-MaSS, Durban, South Africa.
[1] 216028272@stu.ukzn.ac.za; [2] izuchukwuc@ukzn.ac.za ; [3] 217079141@stu.ukzn.ac.za
[4] University of KwaZulu-Natal, School of Mathematics, Statistics and Computer Science, DST, Durban, South Africa.
mewomoo@ukzn.ac.za

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Abstract:

We introduce a new type of (α, β)-admissibility and (α, β)-Z-contraction mappings in the frame work of G_b-metric spaces. Using these concepts, fixed point results for (α, β)-Z-contraction mappings in the frame work of complete Gb-metric spaces are established. As an application, we discuss the existence of solution for integral equation of the form: $x(t) = g(t) + \int_0^1 K(t, s, u(s))ds$, $t \in [0, 1]$, O. T. Mewomowhere $K : [0, 1]\times[0, 1]\times R \to R$ and $g : [0, 1] \to R$ are continuous functions. The results obtained in this paper generalize, unify and improve the results of Liu et al., [17], Antonio-Francisco et al. [23], Khojasteh et al. [15], Kumar et al. [16] and others in this direction.

Keywords: (α, β)-Z_F –contraction; (α, β)-admissible type B mapping; Fixed point; G_b-metric space.

MSC (2020): 47H09, 47H10, 49J20, 49J40.

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1. Introduction and Preliminaries

The theory of fixed point plays an important role in nonlinear functional analysis and is known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. Banach [5] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, many authors have generalized this result by considering classes of nonlinear mappings which are more general than contraction mappings and in other classical and important spaces (see [1, 19, 20, 24] and the references therein). Also, over the years, several iterative schemes have been developed for solving fixed point problems for nonlinear operators in different spaces, (see [2, 9, 10, 11, 12, 13, 14, 18, 27, 28, 29] and the references therein).
Samet et al. [25] introduced the notion of α-admissible mapping and obtain some fixed point results for this class of mappings.

Definition 1.1. *[25] Let $\alpha : X \times X \to [0,\infty)$ be a function. We say that a self mapping $T : X \to X$ is α-admissible if for all $x, y \in X$,*

$$\alpha(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1.$$

Definition 1.2. *[25] Let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is a triangular α-admissible if*

1. *T is α-admissible and*
2. *$\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1 \Rightarrow \alpha(x,z) \geq 1$ for all $x, y, z \in X$.*

Theorem 1.3. *[25] Let (X,d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions hold:*

1. *for all $x, y \in X$, we have $\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y))$, where $\psi : [0,\infty) \to [0,\infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
3. *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point.

In [7] Chandok extend and improve the concept of α-admissible by introducing the notion of (α, β)-admissible mapping and obtained some fixed point theorems.

Definition 1.4. *[7] Let X be a nonempty set and $\alpha, \beta : X \times X \to [0, \infty)$ be functions. We say that a self mapping $T : X \to X$ is (α, β)-admissible if for all $x, y \in X$, $\alpha(x, y) \geq 1$ and $\beta(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$ and $\beta(Tx, Ty) \geq 1$.*

In 2014, Ansari [4] introduced the notion of C-class function, he proved some fixed point results using the concept of C-class function and also established that the C-class function is a generalization of a whole lot of contractive conditions.

Definition 1.5. *[4] A mapping $F : [0, \infty)^2 \to \mathbf{R}$ is called a C-class function if it is continuous and the following axioms hold:*

1. *$F(s, t) \leq s$ for all $s, t \in [0, \infty)$;*
2. *$F(s, t) = s$ implies either $s = 0$ or $t = 0$.*

Example 1.6. *The following functions $F : [0, \infty)^2 \to \mathbf{R}$ defined for all $s, t \in [0, \infty)$ by*

1. *$F(s, t) = s - t$, $F(s, t) = s$ implies $t = 0$;*
2. *$F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s$ implies $s = 0$;*
3. *$F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ is a continuous function, $F(s, t) = s$ implies $s = 0$.*

For details about C-class function see [4].

In 2015, Khojasteh et al. [15] introduced the notion of $\mathcal{Z}$-contraction which generalizes the well-known Banach contraction and a host of other contractive conditions. They gave the following definition for $\mathcal{Z}$ as follows.

Definition 1.7. *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbf{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:*
$\zeta(i)$ $\zeta(0, 0) = 0$;
$\zeta(ii)$ $\zeta(t, s) < s - t$, for all $t, s > 0$;
$\zeta(iii)$ If $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$, then $\lim\limits_{n\to\infty} \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Example 1.8. *Suppose* $\zeta_i : [0,\infty)^2 \to [0,\infty), i = 1,2,3,4$ *is defined as*

1. $\zeta_1(t,s) = s - \phi(s) - t$ *for all* $t,s \in [0,\infty)$, *where* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\phi(t) = 0$ *if and only if* $t = 0$.
2. $\zeta_2(t,s) = \eta(s) - t$ *for all* $t,s \in [0,\infty)$, *where* $\eta : [0,\infty) \to [0,\infty)$ *is an upper semicontinuous mapping such that* $\eta(t) < t$ *for all* $t > 0$ $\eta(t) = 0$ *if and only if* $t = 0$.
3. $\zeta_3(t,s) = \lambda s - t$ *for all* $t,s \in [0,\infty)$, *where* $0 < \lambda < 1$.
4. $\zeta_4(t,s) = \frac{s}{s+1} - t$ *for all* $t,s \in [0,\infty)$.

Definition 1.9. *Let* (X,d) *be a metric space,* $T : X \to X$ *a mapping and* $\zeta \in \mathcal{Z}$. *Then* T *is called a* $\mathcal{Z}$*-contraction with respect to* ζ, *if the following condition is satisfied*

$$\zeta(d(Tx,Ty), d(x,y)) > 0,$$

for all distinct $x,y \in X$.

Theorem 1.10. *Let* (X,d) *be a complete metric space and* $T : X \to X$ *be a* $\mathcal{Z}$*-contraction with respect to a simulation function* $\zeta \in \mathcal{Z}$. *Then* T *has a unique fixed point* $x^* \in X$ *and for every* $x_0 \in X$, *the Picard sequence* $\{x_n\}$, *where* $x_n = Tx_{n-1}$ *for all* $n \in \mathbf{N}$ *converges to the fixed point of* T.

Antonio-Francisco et al. [23] slightly modify the notion of simulation function in the sense of Khojasteh et al. [15], which further generalize the concept of simulation function introduced by Khojasteh et al. in [15].

Definition 1.11. *Let* $\zeta : [0,\infty) \times [0,\infty) \to \mathbf{R}$ *be a mapping, then* ζ *is called a simulation function if it satisfies the following conditions:*
$\zeta(i)$ $\zeta(0,0) = 0$;
$\zeta(ii)$ $\zeta(t,s) < s - t$, *for all* $t,s > 0$;
$\zeta(iii)$ *if* $\{t_n\},\{s_n\}$ *are sequences in* $(0,\infty)$ *such that* $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ *and* $t_n < s_n$ *for all* $n \in \mathbf{N}$, *then* $\lim_{n\to\infty} \zeta(t_n,s_n) < 0$.

They also presented the following example to establish that every simulation function in the sense of Khojasteh is also a simulation function in their sense, but the conserve is not true.

Example 1.12. *[23] Let $k \in \mathbf{R}$ be such that $k < 1$ and let $\zeta : [0,\infty) \times [0,\infty) \to \mathbf{R}$ be the function defined by*

$$\zeta(t,s) = \begin{cases} 2(s-t) & \text{if} \quad s < t \\ ks - t & \text{if} \quad \text{otherwise.} \end{cases}$$

With the aim of generalizing the notion of simulation functions as introduced by Khojasteh et al., in 2018, Liu et al. [17] generalized the concept of simulation function using the notion of C-class function. They gave the following definition.

Definition 1.13. *A mapping $F : [0,\infty)^2 \to \mathbf{R}$ has the property C_F, if there exists a $C_F \geq 0$ such that*
$\eta_{(i)}$ $F(s,t) > C_F \Rightarrow s > t$;
$\eta_{(ii)}$ $F(t,t) \leq C_F$ *for all* $t \in [0,\infty)$.

Definition 1.14. *A C_F simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbf{R}$ satisfying the following conditions:*
$\phi_{(i)}$ $\zeta(t,s) < F(s,t)$, *for all* $t,s > 0$, *where F is a C-class function;*
$\phi_{(ii)}$ *if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that* $\lim_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n > 0$ *and $t_n < s_n$ for all $n \in \mathbf{N}$, then* $\lim\limits_{n\to\infty} \zeta(t_n, s_n) < C_F$.

Some examples of a C-class functions that have property C_F are as follows:

1. $F(s,t) = s - t, C_F = r, r \in [0,\infty)$;
2. $F(s,t) = \frac{s}{1+kt}, k \geq 1, C_F = \frac{r}{1+k}, r \geq 2$.

Remark 1.15. *It is worth mentioning that every simulation function in the sense of Khojasteh is also a C_F simulation function, but the converse is not true. This claim is easy to see using Example 1.12 with $F(s,t) = s - t$.*

One of the interesting generalization of metric spaces is the concept of b-metric spaces introduced by Czerwik in [8]. He established the Banach contraction principle in this frame work with the fact that b need not be continuous. Thereafter, several results has been extended from metric spaces to b-metric spaces, more so, a lot of results on the fixed point theory of various classes of mappings in the frame work of b-metric spaces has been established by different researchers in this area (see [6, 8] and the references therein). For example in [26], Sintunavarat introduced the concept of α-admissible mapping type S as a generalization of α-admissible mapping [25].

Definition 1.16. *[26] Let X be a nonempty set and $s \geq 1$ be a given real number. Let $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. The mapping T is said to be an α-admissible mapping type S if for all $x, y \in X$*

$$\alpha(x, y) \geq s \Rightarrow \alpha(Tx, Ty) \geq s.$$

Remark 1.17. *Clearly, if $s = 1$, we obtain Definition 1.1.*

Remark 1.18. *We remark that using the idea of Sintunavarat [25], we can also generalize the notion $(\alpha,\ \beta)$-admissible mapping as introduced by Chandok in [7].*

Mustafa and Sims [21], introduced the concept of generalized metric space ($G-$metric) to generalize the concept of D-metric spaces and correct some slips up in the notion of D-metric spaces. They established some fixed point theorems in the frame work of complete G-metric spaces.

Definition 1.19. *Let X be a nonempty set and $G : X \times X \times X \to \mathbf{R}^+$ be a function satisfying the following properties*

1. *$G(x, y, z) = 0$ if and only if $x = y = z$,*
2. *$0 < G(x, x, y)$ for all $x, y \in X$ with $x = y$,*
3. *$G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z = y$,*
4. *$G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all the three variables),*
5. *$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.*

Then, the function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Motivated by the concept of b-metric and G-metric spaces [8, 21], Aghajani et al. in [3], introduced the notion of generalized b-metric space (G_b − metric spaces), presented some properties of G_b-metric spaces and prove some coupled coincidence fixed point theorems for (ψ, φ)-weakly con- tractive mappings in the frame work of partially ordered G_b-metric spaces. Thereafter, several results and applications has been extended from metric spaces, b-metric spaces and G-metric spaces to G_b-metric spaces, more so, a lot of results on the fixed point theory of various classes of mappings in the frame work of G_b-metric spaces has been established by different researchers in this area (see [16] and the references therein). The notion of G_b-metric spaces generalize, improves and unifies results in metric spaces, b-metric and G-metric.

Definition 1.20. *[3] Let X be a nonempty set, $s \geq 1$ a given real number and $G_b : X \times X \times X \to \mathbf{R}^+$ a function satisfying the following properties:*

1. *$G_b(x, y, z) = 0$ if and only if $x = y = z$,*
2. *$0 < G_b(x, x, y)$ for all $x, y \in X$ with $x = y$,*
3. *$G_b(x, x, y) \leq G_b(x, y, z)$ for all $x, y, z \in X$ with $z = y$,*
4. *$G_b(x, y, z) = G_b(p\{x, z, y\})$, where p is a permutation of x, y, z (symmetry),*
5. *$G_b(x, y, z) \leq bG_b(x, a, a) + bG_b(a, y, z)$ for all $x, y, z, a \in X$.*

Then, the function G_b is called a generalized b-metric and the pair (X, G_b) is called a generalized b-metric space (G_b − metric space).

Example 1.21. *Let $X = \mathbf{R}$ and $d(x, y) = |x - y|^2$. It is well known that (X, d) is a b-metric space with $b = 2$. Let $G_b(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, it is easy to see that (X, G_b) is not G_b-metric space. However, if we define $G_b(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, then (X, G_b) is a G_b-metric space.*

Definition 1.22. *[3] A G_b-metric space is said to be symmetric if $G_b(x, y, y) = G_b(y, x, x)$ for all $x, y \in X$.*

Proposition 1.23. *[3] Let X be a G_b-metric space. Then for each $x, y, z, a \in X$, it follows that*

1. *$G_b(x, y, z) = 0 \Rightarrow x = y = z$,*
2. *$G_b(x, y, z) \leq bG_b(x, x, y) + bG_b(x, x, z)$,*
3. *$G_b(x, y, y) \leq 2bG_b(y, x, x)$,*
4. *$G_b(x, y, z) \leq bG_b(x, a, z) + bG_b(a, y, z)$.*

Definition 1.24. *[3] Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be;*

1. *G_b-Cauchy if for each $\epsilon > 0$ there exists a positive integer n_0 such that for all $m, n, l \geq n_0, G_b(x_n, x_m, x_l) < \epsilon$;*
2. *G_b-convergent to a point $x \in X$, if for $\epsilon > 0$ there exists a positive integer n_0 such that for all $m, n \geq n_0, G_b(x_n, x_m, x) < \epsilon$. That is $\lim_{n,m\to\infty} G_b(x_n, x_m, x) = 0$. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.*

Definition 1.25. *[3] A G_b-metric space is called G_b-complete, if every G_b-Cauchy sequence is G_b-convergent in X.*

Proposition 1.26. *[3] Let (X, G_b) be a G_b-metric space. The following statements are equivalent*

1. *$\{x_n\}$ is G_b-convergent to x;*
2. *$G_b(x_n, x_n, x) \to 0$ as $n \to \infty$;*
3. *$G_b(x_n, x, x) \to 0$ as $n \to \infty$;*
4. *$G_b(x_n, x_m, x) \to 0$ as $m, n \to \infty$.*

Proposition 1.27. *[3] Let (X, G_b) be a G_b-metric space. The following statements are equivalent:*

1. *$\{x_n\}$ is G_b-Cauchy sequence.*
2. *$G_b(x_m, x_n, x_n) \to 0$ as $n, m \to \infty$.*

Very recently, Kumar et al. [16] introduced the concept of $\mathcal{Z}$-contraction with respect to ζ in the frame work of G-metric spaces. They establish some fixed point results and gave an example to support their main result.

Definition 1.28. *Let (X, G) be a G-metric space, $T : X \to X$ a mapping and $\zeta \in \mathcal{Z}$. Then T is called a $\mathcal{Z}$-contraction with respect to ζ, if the following condition is satisfied*

$$\zeta(G(Tx, Ty, Tz), G(x, y, z)) > 0,$$

for all distinct $x, y \in X$.

Theorem 1.29. *Let (X, G) be a complete G-metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a simulation function $\zeta \in \mathcal{Z}$. Then T has a unique fixed point $x^* \in X$ and for every $x_0 \in X$, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbf{N}$ converges to the fixed point of T.*

Motivated by the works of Liu et al. [17], Kumar et al. [16], Khojasteh et al. [15], Antonio-Francisco [23] and the current research interest in this direction, we introduce the notions of b-C_F simulation function, (α, β)- admissible type B mapping and (α, β)-$\mathcal{Z}_F$-contraction mapping with re- spect to ζ in the frame work G_b-metric spaces. Furthermore, we establish some fixed point results for $(\alpha,\ \beta)$-$\mathcal{Z}_F$ -contraction mapping in the frame work of complete G_b-metric spaces and apply our results to establish the existence of solution of an integral equation.

2. Main Results

In this section, we introduce the notion of b-C_F simulation function, (α, β)-admissible type B mapping, triangular (α, β)-admissible type B mapping and (α, β)-$\mathcal{Z}_F$-contraction mapping with respect to ζ in the frame work G_b-metric spaces and established the existence and uniqueness results of the fixed point for this class of mappings in the frame work of a complete G_b-metric spaces.

Definition 2.1. *A b-C_F simulation function is a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbf{R}$ satisfying the following conditions:*
$\zeta_*(i)$ *$\zeta(t, s) < F(s, t)$, for all $t, s > 0$, where F is a C-class function;*
$\zeta_*(ii)$ *if $\{t_n\}, \{s_n\}$ are sequences in $(0, \infty)$ such that $0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq b\ lim_{n\to\infty} t_n < \infty$ and $t_n < s_n$ for all $n \in \mathbf{N}$, then*

$$\lim_{n\to\infty} \zeta(bt_n, s_n) < C_F.$$

Remark 2.2. *It is easy to see that if $b = 1$, we obtain Definition 1.14.*

Remark 2.3. *We remark that Definitions 1.7, 1.11, 1.13, 1.14 and Definition 2.1 are important in the study of fixed point and its applications because they are used to obtain new contractive definitions and for extending, generalizing and unifying existing fixed point results in the literature and hence generalizing the Banach Contraction Principle in different abstract spaces.*

Definition 2.4. *Let X be a nonempty set with $b \geq 1$ a given real number, $T : X \to X$ and $\alpha, \beta : X \times X \times X \to [0, \infty)$ be mappings. Then T is called (α, β)-admissible type B mapping if for all $x, y, z \in X$ with $\alpha(x, y, z) \geq b$ and $\beta(x, y, z) \geq b$ implies $\alpha(Tx, Ty, Tz) \geq b$ and $\beta(Tx, Ty, Tz) \geq b$.*

Remark 2.5. *We remark that if $b = 1$, we obtain Definition 1.4 in the frame work of G-metric spaces.*

Definition 2.6. *Let X be a nonempty set with $b \geq 1$ a given real number, $T : X \to X$ and $\alpha, \beta : X \times X \times X \to [0, \infty)$ be mappings. Then T is called triangular (α, β)-admissible type B mapping if*

1. *T is (α, β)-admissible type B mapping,*
2. *$\alpha(x, a, a) \geq b, \alpha(a, y, z) \geq b$ and $\beta(x, a, a) \geq b, \beta(a, y, z) \geq b$ implies $\alpha(x, y, z) \geq b$ and $\beta(x, y, z) \geq b$,*

for all $x, y, z, a \in X$.

Lemma 2.7. *Let X be a nonempty set with $b \geq 1$ a given real number and T be a triangular (α, β)-admissible type B mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq b$ and $\beta(x_0, Tx_0, Tx_0) \geq b$. Suppose that the sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_n$, then $\alpha(x_m, x_n, x_n) \geq b$ and $\beta(x_m, x_n, x_n) \geq b$ for all $n, m \in \mathbf{N} \cup \{0\}$, with $m < n$.*

Proof. Suppose that T is triangular (α, β)-admissible type B mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq b$ and $\beta(x_0, Tx_0, Tx_0) \geq b$, we then have that $\alpha(x_0, Tx_0, Tx_0) = \alpha(x_0, x_1, x_1) \geq b$ and $\beta(x_0, Tx_0, Tx_0) = \beta(x_0, x_1, x_1) \geq b$, which implies that $\alpha(Tx_0, Tx_1, Tx_1) = \alpha(x_1, x_2, x_2) \geq b$ and $\beta(Tx_0, Tx_1, Tx_1) = \beta(x_1, x_2, x_2) \geq b$. Continuing the process, we obtain that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq b$ and $\beta(x_n, x_{n+1}, x_{n+1}) \geq b$. For all $n, m \in \mathbf{N} \cup \{0\}$ with $m < n$, observe that since $\alpha(x_m, x_{m+1}, x_{m+1}) \geq b, \beta(x_m, x_{m+1}, x_{m+1}) \geq b$ and $\alpha(x_{m+1}, x_{m+2}, x_{m+2}) \geq b, \beta(x_{m+1}, x_{m+2}, x_{m+2}) \geq b$, we obtain $\alpha(x_m, x_{m+2}, x_{m+2}) \geq b, \beta(x_m, x_{m+2}, x_{m+2}) \geq b$. Also, since $\alpha(x_m, x_{m+2}, x_{m+2}) \geq b, \beta(x_m, x_{m+2}, x_{m+2}) \geq b$ and $\alpha(x_{m+2}, x_{m+3}, x_{m+3}) \geq b, \beta(x_{m+2}, x_{m+3}, x_{m+3}) \geq b$, we obtain $\alpha(x_m, x_{m+3}, x_{m+3}) \geq b, \beta(x_m, x_{m+3}, x_{m+3}) \geq b$. Continuing the process, we have that

$$\alpha(x_m, x_n, x_n) \geq b \quad \text{and} \quad \beta(x_m, x_n, x_n) \geq b.$$

□

Definition 2.8. *Let (X, G_b) be a G_b-metric space with $b \geq 1$ a given real number, $\alpha, \beta \times X \times X \to [0, \infty)$ be functions and T be a self map on X.*

The mapping T is said to be (α,β)-$\mathcal{Z}_F$-contraction mapping with respect to ζ, if

$$\alpha(x,y,z)\beta(x,y,z)\geq b^2 \Rightarrow \zeta(bG_b(Tx,Ty,Tz),G_b(x,y,z))\geq C_F \tag{2.1}$$

for all distinct $x,y,z\in X$.

Remark 2.9. *If we suppose that $b=1$ and $C_F=0$, we obtain a new type of generalized $\mathcal{Z}$-contraction with respect to ζ,*

$$\alpha(x,y,z)\beta(x,y,z)\geq 1 \Rightarrow \zeta(G(Tx,Ty,Tz),G(x,y,z))\geq 0, \tag{2.2}$$

for all distinct $x,y,z\in X$. It is easy to see that gme is a generalization of Definition 1.28.

Theorem 2.10. *Let (X,G_b) be a G_b-complete metric space and $T:X\to X$ be an (α,β)-$\mathcal{Z}_F$-contraction mapping with respect to ζ. Suppose the following conditions hold:*

1. *T is triangular (α,β)-admissible type B mapping,*
2. *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0,Tx_0)\geq b$ and $\beta(x_0,Tx_0,Tx_0)\geq b$,*
3. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1},x_{n+1})\geq b, \beta(x_n,x_{n+1},x_{n+1})\geq b$ for all $n\geq 0$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x_n,x,x)\geq b$ and $\beta(x_n,x,x)\geq b$.*

Then T has a fixed point.

Proof. To establish the existence of fixed point of T, we divide the proof into four (4) steps.

Step 1: We show that $\lim\limits_{n\to\infty} G_b(x_n,x_{n+1},x_{n+1})=0$.
Let $x_0\in X$ be such that $\alpha(x_0,Tx_0,Tx_0)\geq b$ and $\beta(x_0,Tx_0,Tx_0)\geq b$. We define the sequence $\{x_n\}$ by $x_{n+1}=Tx_n$ for all $n\in\mathbf{N}\cup\{0\}$. If we suppose that $x_{n+1}=x_n$, for some $n\in\mathbf{N}\cup\{0\}$, we obtain the desired result. Now, suppose that $x_{n+1}=x_n$ for all $n\in\mathbf{N}\cup\{0\}$. From Lemma 2.7, it is easy to see that

$$\alpha(x_n,x_{n+1},x_{n+1})\beta(x_n,x_{n+1},x_{n+1})\geq b^2$$

for all $n\in\mathbf{N}\cup\{0\}$. Using $\zeta_*(i),\eta(i)$ and from (2.1), we have that

$$\begin{aligned} C_F \leq \zeta \quad & (bG_b(Tx_n, Tx_{n+1}, Tx_{n+1}), G_b(x_n, x_{n+1}, x_{n+1})) \\ & = \zeta(bG_b(x_{n+1}, x_{n+2}, x_{n+2}), G_b(x_n, x_{n+1}, x_{n+1})) \\ & < F(G_b(x_n, x_{n+1}, x_{n+1}), bG_b(x_{n+1}, x_{n+2}, x_{n+2})). \end{aligned} \tag{2.3}$$

From (2.3), we obtain

$$F(G_b(x_n, x_{n+1}, x_{n+1}), bG_b(x_{n+1}, x_{n+2}, x_{n+2})) > C_F,$$

which implies that

$$G_b(x_n, x_{n+1}, x_{n+1}) > bG_b(x_{n+1}, x_{n+2}, x_{n+2}).$$

That is

$$bG_b(x_{n+1}, x_{n+2}, x_{n+2}) < G_b(x_n, x_{n+1}, x_{n+1}). \tag{2.4}$$

It is easy to see from (2.4) that the sequence $\{G_b(x_n, x_{n+1}, x_{n+1})\}$ is monotonically decreasing and nonnegative. More so, inductively, we have that $\{G_b(x_n, x_{n+1}, x_{n+1})\}$ is bounded. Therefore, there exists $c \geq 0$ such that

$$\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = c.$$

Suppose that $c > 0$, clearly $\lim\limits_{n\to\infty} G_b(x_{n+1}, x_{n+2}, x_{n+2}) = c$. Since T is an (α, β)-$\mathcal{Z}_F$-contraction mapping with respect to $\zeta \in \mathcal{Z}$ and using $\zeta_*(ii)$, we have

$$C_F \leq \limsup_{n\to\infty} \zeta(bG_b(x_{n+1}, x_{n+2}, x_{n+2}), G_b(x_n, x_{n+1}, x_{n+1})) < C_F.$$

This is a contradiction, thus $c = 0$ and so we have that

$$\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = 0. \tag{2.5}$$

Step 2: We show that $\{x_n\}$ is bounded.
Suppose that $\{x_n\}$ is not a bounded sequence, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for $n_1 = 1$ and for each $k \in \mathbf{N}, n_{k+1}$ is the minimum integer such that

$$G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) > 1 \quad \text{and} \quad G(x_m, x_{n_k}, x_{n_k}) \leq 1 \tag{2.6}$$

for $n_k \leq m \leq n_{k+1} - 1$. Using (2.6) and Proposition 1.23, we have

$$\begin{aligned} 1 < G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \quad & \leq bG(x_{n_{k+1}}, x_{n_{k+1}-1}, x_{n_{k+1}-1}) + bG(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) \\ & \leq 2b^2G(x_{n_{k+1}-1}, x_{n_{k+1}}, x_{n_{k+1}}) + b. \end{aligned}$$

Letting $k \to \infty$ and using lim1, we obtain

$$(2.7)\quad 1 \leq \liminf_{k\to\infty} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \leq \limsup_{k\to\infty} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) \leq b.$$

From (2.4), we deduce that

$$\begin{aligned} bG(x_{n_{k+1}}, x_{n_k}, x_{n_k}) &\leq G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}) \\ &\leq bG(x_{n_{k+1}-1}, x_{n_k}, x_{n_k}) + bG(x_{n_k}, x_{n_k-1}, x_{n_k-1}) \\ &\leq b + 2b^2 G(x_{n_k-1}, x_{n_k}, x_{n_k}). \end{aligned} \tag{2.8}$$

Letting $k \to \infty$, using lim1 and koo, we obtain that

$$\lim_{n\to\infty} G(x_{n_{k+1}}, x_{n_k}, x_{n_k}) = 1 \quad \text{and} \quad \lim_{n\to\infty} G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}) = b.$$

From Lemma 2.7, it is easy to see that $\alpha(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})\beta(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1}) \geq b^2$ and by definition of (α, β)-$\mathcal{Z}_F$-contraction with respect to ζ, and by $\zeta_*(ii)$, we obtain

$$\begin{aligned} C_F &\leq \limsup_{k\to\infty} \zeta(bG(Tx_{n_{k+1}-1}, Tx_{n_k-1}, Tx_{n_k-1}), G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})) \\ &= \limsup_{k\to\infty} \zeta(bG(x_{n_{k+1}}, x_{n_k}, x_{n_k}), G(x_{n_{k+1}-1}, x_{n_k-1}, x_{n_k-1})) < C_F. \end{aligned}$$

This is a contradiction. Thus $\{x_n\}$ is bounded.

Step 3: We show that $\{x_n\}$ is Cauchy.
Suppose that $C_n = \sup\{G_b(x_i, x_j, x_j) : i, j \geq n\}, n \in \mathbf{N}$. Since $\{x_n\}$ is bounded, we have that $C_n < \infty$ for all $n \in \mathbf{N}$, as such $\{C_n\}$ is a positive monotonically decreasing sequence which converges. That is $\lim_{n\to\infty} C_n = C \geq 0$. Suppose that $C > 0$, then by definition of C_n, for every $k \in \mathbf{N}$, we can find n_k, m_k such that $m_k > n_k > k$ and

$$C_n - \frac{1}{K} < G_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq C_k,$$

letting $k \to \infty$, we obtain

$$\lim_{k\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_k}) = C. \tag{2.9}$$

From (2.4) and using the definition of C_n, we deduce that

$$bG_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq G_b(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) \leq C_{k-1}.$$

Letting $k \to \infty$ and using c1, we obtain

$$bC \leq \liminf_{k\to\infty} G_b(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) \leq \limsup_{k\to\infty} G_b(x_{m_k-1}, x_{n_k-1}, x_{n_k-1}) \leq C. \tag{2.10}$$

It is easy to see from Lemma 2.7 that $\alpha(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})\beta(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}}) \geq b^2$, so by definition of (α, β)-$\mathcal{Z}_F$-contraction with respect to ζ and using $\zeta_*(ii)$, we have that

$$\begin{aligned} C_F &\leq \lim_{n\to\infty} \zeta(bG_b(Tx_{m_k-1}, Tx_{n_k-1}, Tx_{n_k-1}), G_b(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})) \\ &= \lim_{n\to\infty} \zeta(bG_b(x_{m_k}, x_{n_k}, x_{n_k}), G_b(x_{m_{k-1}}, x_{n_{k-1}}, x_{n_{k-1}})) < C_F. \end{aligned}$$

This is a contradiction, thus $C = 0$. Hence, $\{x_n\}$ is a Cauchy sequence.

Step 4: Finally, we show the existence of fixed point of T.
Since $\{x_n\}$ is a Cauchy sequence and X is a complete G_b-metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Using condition (3), since $\alpha(x_n, x, x) \geq b$, $\beta(x_n, x, x) \geq b$, we have that $\alpha(x_n, x, x)\beta(x_n, x, x) \geq b^2$, and since T is (α, β)-$\mathcal{Z}_F$-contraction with respect to ζ and using $\eta(i)$, we obtain

$$\begin{aligned} C_F \leq \quad & \zeta(bG_b(Tx_n, Tx, Tx), G_b(x_n, x, x)) \\ & < F(G_b(x_n, x, x), bG_b(Tx_n, Tx, Tx)). \end{aligned}$$

It follows that $F(G_b(x_n, x, x), bG_b(Tx_n, Tx, Tx)) > C_F$, which implies that

$$bG_b(Tx_n, Tx, Tx)) = bG_b(x_{n+1}, Tx, Tx)) < G_b(x_n, x, x)$$

and consequently, we have

$$G_b(x, Tx, Tx) \leq bG_b(x, x_{n+1}, x_{n+1}) + bG_b(x_{n+1}, Tx, Tx) < bG_b(x, x_{n+1}, x_{n+1}) + G_b(x_n, x, x).$$

Letting $n \to \infty$, we obtain that $G_b(x, Tx, Tx) = 0 \Rightarrow x = Tx$. □

Theorem 2.11. *Suppose that the hypothesis of Theorem 2.10 holds and in addition suppose $\alpha(x, y, y) \geq b$ and $\beta(x, y, y) \geq b$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.*

Proof. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x = y$. Using our hypothesis, we have $\alpha(x, y, y)\beta(x, y, y) \geq b^2$. we obtain from (2.1) that

$$\begin{aligned} C_F \leq \; & \zeta(bG_b(Tx,Ty,Ty),G_b(x,y,y)) \\ & < F(G_b(x,y,y),bG_b(Tx,Ty,Ty)) \\ & = F(G_b(x,y,y),bG_b(x,y,y)). \end{aligned}$$

It follows that $F(G_b(x,y,y),bG_b(x,y,y)) > C_F$, which implies that

$$bG_b(x,y,y) < G_b(x,y,y)$$

which is a contradiction, as such, we must have that $G_b(x,y,y)=0 \Rightarrow x = y$. Hence T has a unique fixed point. □

3. Consequences of Main Result

In this section, we present some consequences of our main result.

Corollary 3.1. *Let (X,G_b) be a complete G_b-metric space and $T : X \to X$ be a mapping satisfying*
$\alpha(x,y,z)\beta(x,y,z) \geq b^2 \Rightarrow \zeta(bG_b(Tx,Ty,Tz),G_b(x,y,z)) \geq 0,$
for all distinct $x,y,z \in X$. Suppose the following conditions hold:

1. *T is triangular (α,β)-admissible type B mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,Tx_0) \geq b$ and $\beta(x_0,Tx_0,Tx_0) \geq b$,*
3. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1},x_{n+1}) \geq b, \beta(x_n,x_{n+1},x_{n+1}) \geq b$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x,x) \geq b$ and $\beta(x_n,x,x) \geq b$.*

Then T has a fixed point.

Proof. The result follows from Theorem 2.10. Since by taking $C_F = 0$, and defining $\zeta(t,s) = s - t$, for all $s,t \geq 0$, we obtain

$$\alpha(x,y,z)\beta(x,y,z) \geq b^2 \Rightarrow bG_b(Tx,Ty,Tz) \leq G_b(x,y,z).$$

□

Corollary 3.2. *Let (X,G_b) be a complete G_b-metric space and $T : X \to X$ be a mapping satisfying*
$(x,y,z)\beta(x,y,z) \geq b^2 \Rightarrow \zeta(bG_b(Tx,Ty,Tz),\lambda G_b(x,y,z)) \geq 0,$
where $\lambda \in (0,1)$. Suppose the following conditions hold:

1. *T is triangular (α, β)-admissible type B mapping,*

2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq b$ and $\beta(x_0, Tx_0, Tx_0) \geq b$,*

3. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq b, \beta(x_n, x_{n+1}, x_{n+1}) \geq b$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq b$ and $\beta(x_n, x, x) \geq b$.*

Then T has a fixed point.

Proof. The result follows from Theorem 2.10. Since by taking $C_F = 0$, and defining $\zeta(t, s) = s - t$, for all $s, t \geq 0$, we obtain

$$\alpha(x, y, z)\beta(x, y, z) \geq b^2 \Rightarrow bG_b(Tx, Ty, Tz) \leq \lambda G_b(x, y, z).$$

□

Remark 3.3. *Corollary 3.2 can be seen as a generalization of the well-known Banach contraction priniciple [5] in the frame work of complete G_b-metric spaces.*

Corollary 3.4. *Let (X, G_b) be a complete G_b-metric space and $T : X \to X$ be a mapping satisfying*

$$\begin{aligned} &\alpha(x, y, z)\beta(x, y, z) \geq b^2 \\ &\Rightarrow \zeta(bG_b(Tx, Ty, Tz), G_b(x, y, z) - \psi(G_b(x, y, z))) \geq 0, \end{aligned}$$

where $\psi : \mathbf{R} \to \mathbf{R}$ is a lower semicontinuous function with $\psi^{-1}(0) = (0)$. Suppose the following conditions hold:

1. *T is triangular (α, β)-admissible type B mapping,*

2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq b$ and $\beta(x_0, Tx_0, Tx_0) \geq b$,*

3. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq b, \beta(x_n, x_{n+1}, x_{n+1}) \geq b$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq b$ and $\beta(x_n, x, x) \geq b$.*

Then T has a fixed point.

Proof. The result follows from Theorem 2.10. Since by taking $C_F = 0$, and defining $\zeta(t,s) = \lambda s - \psi(s) - t$, for all $s,t \geq 0$, we obtain

$$\alpha(x,y,z)\beta(x,y,z) \geq b^2 \Rightarrow bG(Tx,Ty,Tz) \leq G_b(x,y,z) - \psi(G_b(x,y,z)).$$

□

Remark 3.5. *Corollary 3.4 can be seen as a generalization of Rhoades fixed point result [22] in the frame work of complete G_b-metric spaces.*

Corollary 3.6. *Let (X, G_b) be a complete G_b-metric space and $T : X \to X$ be a mapping satisfying $\alpha(x,y,z)\beta(x,y,z) \geq 1 \Rightarrow \zeta(bG_b(Tx,Ty,Tz), G_b(x,y,z)) \geq 0$, for all distinct $x,y,z \in X$. Suppose the following conditions hold:*

1. *T is triangular (α,β)-admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0,Tx_0,Tx_0) \geq 1$ and $\beta(x_0,Tx_0,Tx_0) \geq 1$,*
3. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1},x_{n+1}) \geq 1, \beta(x_n,x_{n+1},x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n,x,x) \geq 1$ and $\beta(x_n,x,x) \geq 1$.*

Then T has a fixed point.

Proof. The result follows from Theorem 2.10, by taking $C_F = 0$. Since by defining $\zeta(t,s) = s - t$, for all $s,t \geq 0$, we obtain

$$\alpha(x,y,z)\beta(x,y,z) \geq 1 \Rightarrow bG_b(Tx,Ty,Tz) \leq G_b(x,y,z).$$

□

4. Application

In this section, we present an application of Corollary 3.1 to guarantee the existence of solution to an integral equation of the form:

x(t)=g(t)+ $\int_0^1 K(t,s,u(s))ds, \quad t \in [0,1]$. Let $X = C([0,1])$ be the space of real continuous functions defined on $[0,1]$. It is well-known that $C([0,1])$ endowed with the G_b-metric

$$G_b(x,y,z) = \left(\sup_{t\in[0,1]} |x(t)-y(t)| + \sup_{t\in[0,1]} |y(t)-z(t)| + \sup_{t\in[0,1]} |z(t)-x(t)| \right)^2$$

is a complete G_b-metric space with $b = 2$. Define $T : X \to X$ by

$$Tx(t) = g(t) + \int_0^1 K(t, s, u(s))ds, \quad t \in [0, 1].$$

Theorem 4.1. *Suppose that the following hypothesis hold:*

1. $K : [0, 1] \times [0, 1] \times \mathbf{R} \to \mathbf{R}$ *and* $g : [0, 1] \to \mathbf{R}$ *are continuous,*
2. *there exists* $H : [0, 1] \times [0, 1] \to [0, \infty)$ *such that*
$$|K(t, s, u) - K(t, s, v)| \leq H(t, s)|u - v|$$
for all distinct $x, y \in X, t, s \in [0, 1]$ *and* $u, v \in \mathbf{R}$,
3. $\sup_{t\in[0,1]} \int_0^1 H(t, s)ds < \frac{1}{2}$.

Then the integral equation int has a solution $x \in X$.

Proof. We define α and β as follows:
$\alpha, \beta : X \times X \times X \to [0, \infty)$ are defined by

$$\alpha(x, y, z) = \begin{cases} 3 & \text{if} \quad \alpha(x_0, Tx_0, Tx_0) \geq b \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta(x, y, z) = \begin{cases} 2 & \text{if} \quad \alpha(x_0, Tx_0, Tx_0) \geq b \\ 0 & \text{otherwsie.} \end{cases}$$

It is easy to see that for all $x, y \in [0, 1]$, we have $\alpha(x, y, z)\beta(x, y, z) = 6 > 4 = 2^2 = b^2$, as such, we obtain

$$\begin{aligned} 2G_b(Tx, Ty, Ty) &= 2\left(2\sup_{t\in[0,1]} |Tx(t) - Ty(t)|\right)^2 \\ &= 2\left(2\sup_{t\in[0,1]} \left|\int_0^1 K(t, s, x(s)) - K(t, s, y(s))ds\right|\right)^2 \\ &\leq 2\left(2\sup_{t\in[0,1]} \int_0^1 |K(t, s, x(s)) - K(t, s, y(s))|ds\right)^2 \\ &\leq 2\left(2\sup_{t\in[0,1]} \int_0^1 H(t, s)|x(s) - y(s)|\right)^2 ds \\ &\leq 8\sup_{t\in[0,1]} |x(t) - y(t)|^2 \left(\sup_{t\in[0,1]} \int_0^1 H(t, s)ds\right)^2 \\ &= G_b(x, y, y). \end{aligned}$$

Thus Corollary 3.1 is applicable to T which guarantees the existence of the fixed point $x \in X$. Thus, x is the solution of the integral equation int. □

Conclusion

In this paper, we introduce the notion of b-C_F simulation function, (α, β)-admissible type B mapping and (α, β)-$\mathcal{Z}_F$-contraction mapping with respect to ζ in the frame work G_b-metric spaces. Furthermore, we establish some fixed point results for (α, β)-$\mathcal{Z}_F$-contraction mapping in the frame work complete G_b-metric spaces. Finally, we apply our result to the existence of a solution of an Integral equation. The obtained results in this paper generalize, unify and improve the fixed point results of Liu et al., [17], Antonio-Francisco et al. [23], Khojasteh et al. [15], Kumar et al. [16] and other results in this direction in the literature.

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