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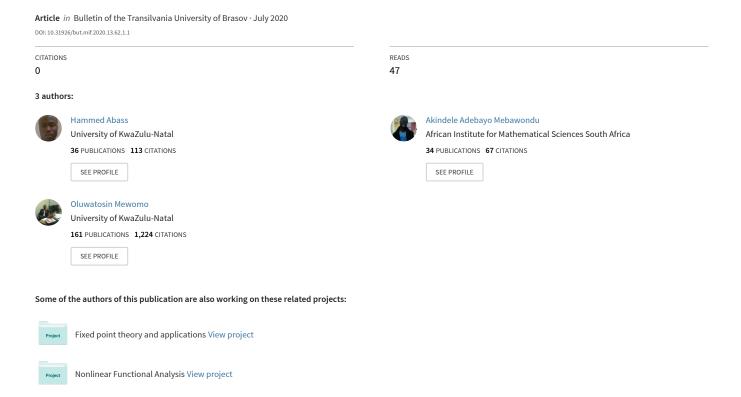
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A DIFFERENT APPROACH TO APPROXIMATING SOLUTIONS OF MONOTONE YOSHIDA VARIATIONAL INCLUSION PROBLEM IN A BANACH SPACE

Hammed Anuoluwapo ABASS[1], Akindele Adebayo MEBAWONDU[2] and Oluwatosin T. MEWOMO[*,3] [4]

Abstract

In this paper, we introduce an iterative algorithm for approximating a common solution of monotone yosida variational inclusion problem in the framework of p-uniformly convex and uniformly smooth Banach spaces. Using our iterative algorithm, we state and prove a strong convergence theorem for approximating a common solution of the aforementioned problem. We also consider an infinite family of Bregman quasi-nonexpansive mapping and prove its strong convergence result. Our result extends and complements some related results in literature.

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Key words: monotone Yosida variational inclusion problem, fixed point problem, Bregman quasi-nonexpansive.

1 Introduction

Let E be a real Banach space with norm $||.||$ and E^* be the dual space of E. Let $K(E) := \{x \in E : ||x|| = 1\}$ denote the unit sphere of E. The modulus of convexity is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf \big\{1 - \frac{||x + y||}{2} : x, y \in K(E), \ ||x - y|| \geq \epsilon \big\}.$$

The space E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let $p > 1$, then E is said to be p-uniformly convex (or to have a modulus of convexity

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of power type p) if there exists $c_p > 0$ such that $\delta_E(\epsilon) \geq c_p\epsilon^p$ for all $\epsilon \in (0, 2]$. Note that every p-uniformly convex space is uniformly convex. The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ := [0, \infty) \to \mathbb{R}^+$ defined by

$$\rho_E(\tau) = \sup\Big\{\frac{||x + \tau y + ||x - \tau y||}{2} - 1 : x, y \in K(E)\Big\}.$$

The space E is said to be uniformly smooth if $\frac{\rho_E(\tau)}{\tau} \to 0$ as $\tau \to 0$. Let $q > 1$, then a Banach space E is said to be q-uniformly smooth if there exists $\kappa_q > 0$ such that $\rho_E(\tau) \leq \kappa_q\tau^q$ for all $\tau > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth. Moreover, a Banach space E is p-uniformly convex if and only if E^* is q-uniformly smooth, where p and q satisfy $\frac{1}{p} + \frac{1}{q} = 1$, (see [9]).
Let $p > 1$ be a real number, the generalized duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{\overline{x} \in E^* : \langle x, \overline{x}\rangle = ||x||^p, ||\overline{x}|| = ||x||^{p-1}\},$$

where $\langle ., .\rangle$ denotes the duality pairing between elements of E and E^*. In particular, $J_E^p = J_E^2$ is called the normalized duality mapping.
If E is p-uniformly convex and uniformly smooth, then E^* is q-uniformly smooth and uniformly convex. In this case, the generalized duality mapping J_E^p is one-to-one, single-valued and satisfies $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the generalized duality mapping of E^*. Furthermore, if E is uniformly smooth then the duality mapping J_E^p is norm-to-norm uniformly continuous on bounded subsets of E, (see [10] for more details).
Let $f : E \to (-\infty, +\infty]$ be a proper, lower semicontinuous and convex function, then the Frenchel conjugate of f denoted as $f^* : E^* \to (-\infty, +\infty]$ is defined as

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \ x^* \in E^*.$$

Let the domain of f be denoted as $(dom f) = \{x \in E : f(x) < +\infty\}$, hence for any $x \in int(dom f)$ and $y \in E$, we define the right-hand derivative of f at x in the direction y by

$$f^0(x, y) = \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

Definition 1. *[6] Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function $\Delta_f : E \times E \to [0, +\infty)$ defined by*

$$\Delta_f(x, y) := f(y) - f(x) - \langle \nabla f(x), y - x\rangle$$

is called the Bregman distance with respect to f.

It is well-known that the Bregman distance Δ_f does not satisfy the properties of a metric because Δ_f fail to satisfy the symmetric and triangular inequality property. Moreover, it is well known that the duality mapping J_E^p is the sub-differential of

the functional $f_p(.) = \frac{1}{p}||.||^p$ for $p > 1$, see [8]. Then, the Bregman distance Δ_p is defined with respect to f_p as follows:

$$\Delta_p(x, y) = \frac{1}{p}||y||^p - \frac{1}{p}||x||^p - \langle J_E^p x, y - x \rangle$$

$$= \frac{1}{q}||x||^p - \langle J_E^p x, y \rangle + \frac{1}{p}||y||^p$$

$$= \frac{1}{q}||x||^p - \frac{1}{q}||y||^p - \langle J_E^p x - J_E^p y, y \rangle. \tag{1}$$

Let $T : C \to int(dom f)$ be a mapping, a point $x \in C$ is called a fixed point of T, if for all $x \in C$, $Tx = x$. We denote by $Fix(T)$ the set of all fixed points of T. Moreso, T is said to be Bregman quasi-nonexpansive if

$$Fix(T) \neq \emptyset \text{ and } \Delta_f(p, Tx) \leq \Delta_f(p, x), \ \forall \ x \in C, \ p \in Fix(T);$$

Recall that a metric projection P_C from E onto C satisfies the following property:

$$||x - P_C x|| \leq \inf_{y \in C} ||x - y||, \ \forall \ x \in E.$$

It is well known that $P_C x$ is the unique minimizer of the norm distance. Moreover, $P_C x$ is characterized by the following properties:

$$\langle J_E^p(x - P_C x), y - P_C x \rangle \leq 0, \ \forall \ y \in C. \tag{2}$$

The Bregman projection from E onto C denoted by Π_C also satisfies the property

$$\Delta_p(x, \Pi_C(x)) = \inf_{y \in C} \Delta_p(x, y), \ \forall \ x \in E. \tag{3}$$

Also, if C is a nonempty, closed and convex subset of a p-uniformly convex and uniformly smooth Banach space E and $x \in E$. Then the following assertions hold: see [9]
(i) $z = \Pi_C x$ if and only if

$$\langle J_E^p(x) - J_E^p(z), \ y - z \rangle \leq 0, \ \forall \ y \in C; \tag{4}$$

(ii)

$$\Delta_p(\Pi_C x, y) + \Delta_p(x, \Pi_C x) \leq \Delta_p(x, y), \ \forall \ y \in C. \tag{5}$$

Let $B : E \to 2^{E^*}$ be a multivalued mapping, then the domain of B is defined by $dom B := \{x \in E : Bx \neq \emptyset\}$ and the graph of B is given as $G(B) := \{(x, x^*) \in E \times E^* : x^* \in Bx\}$. A multivalued mapping B is said to be monotone if $\langle x^* - y^*, x - y \rangle \geq 0$ where $(x, x^*), (y, y^*) \in G(B)$ B is said to be maximal monotone if its graph is not contained in the graph of any other monotone operator on E. It is well-known that if B is maximal monotone, then $B^{-1}(0) = \{\overline{x} \in E : 0 \in B(\overline{x})\}$

is closed and convex. A single-valued mapping $A : H \to H$ is called α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha ||Tx - Ty||^2, \forall \ x, y \in H.$$

Let E be a real Banach space and $B : E \to 2^{E^*}$ be a maximal monotone mapping. The Variational Inclusion Problem VIP is to find $x^* \in E$ such that

$$0 \in B(x^*) \tag{6}$$

Many authors have considered the VIP and its split type in both Hilbert and Banach spaces. For instance, Izuchukwu et. al. [13] considered the following Split Variational Inclusion Problem (SVIP): find $x^* \in E_1$ such that

$$0 \in B_1(x^*) \tag{7}$$

and

$$y^* = Ax^* \in E_2 \text{ such that } 0 \in B_2(y^*). \tag{8}$$

where $B_1 : E_1 \to 2^{E_1^*}$ and $B_2 : E_2 \to 2^{E_2^*}$ are maximal monotone mappings of Banach spaces E_1 and E_2 respectively, $A : E_1 \to E_2$ being a bounded linear operator. They proposed a viscosity type iterative algorithm and proved a strong convergence theorem for approximating solutions of SVIP (7)-(8) and the fixed point problem for a multivalued quasi-nonexpansive mapping between a Hilbert and a Banach spaces.

Furthermore, Ogbuisi and Mewomo [24] also considered the SVIP and proved the following strong convergence theorem for the approximation solution of SVIP (7)-(8) in the framework of p-uniformly convex Banach spaces which are uniformly smooth as follows:

Theorem 1. *Let E_1 and E_2 be two p-uniformly smooth convex real Banach spaces which are also uniformly smooth. Let $A : E_1 \to E_2$ be a bounded linear operator and $A^* : E_2^* \to E_1^*$ be adjoint of A. Let T and S be resolvents of multivalued maxiaml monotone mappings $B_1 : E_1 \to 2^{E_1^*}$ and $B_2 : E_2 \to 2^{E_2^*}$ respectively. Suppose that SVIP (7)-(8) has a nonempty solution set Ω and that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. For a fixed $u \in E_1$, let the sequence $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be generated iteratively by $x_0 \in E_1$,*

$$\begin{cases} y_n = J_{E^*}^q[J_E^p x_n - t_n A^* J_{E_2}^p (I - S) A x_n]; \\ x_{n+1} = J_{E_1^*}[\alpha_n J_E^p u + (1 - \alpha_n)(\beta_n J_{E_1}^p y_n + (1 - \beta_n) J_{E_1}^p T y_n)]; \end{cases}$$

with conditions:
(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $0 < a \leq \beta_n \leq d < 1$;
(iii) $0 < t \leq t_n \leq k \leq \left(\frac{q}{C_q ||A||^q} \right)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega u$.

Based on the problem stated in (6), Moudafi [22] introduced a split Monotone Variational Inclusion Problem MVIP in the framework of Hilbert spaces as follows: find $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*); \tag{9}$$

and

$$y^* = Ax^* \in E_2 \text{ such that } 0 \in f_2(y^*) + B_2(y^*); \tag{10}$$

where $f_1 : H_1 \to H_1$, $f_2 : H_2 \to H_2$ are inverse strongly monotone mappings of real Hilbert spaces H_1 and H_2 and $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ being maximal monotone mappings, where $A : H_1 \to H_2$ is a bounded linear operator.

It can be deduced from SMVIP (9)- (10) that the Monotone Variational Inclusion Problem MVIP is to find $x^* \in H$ such that

$$0 \in f(x^*) + B(x^*). \tag{11}$$

Zarantonello [32] and Minty [19] introduced the notion of monotone operators, and since then many other researchers have shown significant interest because of the firm relation it has with the following evolution equation.

$$\begin{cases} \frac{dx}{dt} + A(x) = 0; \\ x(0) = x; \end{cases} \tag{12}$$

which is the model of many physical problems of practical applications. If function A in (12) is not continuous, then it will be very difficult to solve these types of models. To solve these problem, Yosida introduced a natural step which is to find a sequence of lipschitz functions that approximate A in some sense. It is well known that two quite useful single-valued lipschitz continuous operators can be associated with monotone operators, namely its resolvent operator and its Yosida approximation operator. The Yosida approximation operators are useful for approximating solutions of VIP using resolvent operators, (see [7]).

Very recently, Ahmad et. al. [4] introduced the following Yosida approximation inclusion problem which is to find $x \in X$ such that

$$0 \in J_{M,\lambda}^{\mathcal{H}(.,.)}(x) + M(x), \lambda > 0, \tag{13}$$

where X is a smooth Banach space, M being an $\mathcal{H}(.,.)$- accretive operator with respect to A and B, $A, B : X \to X$ being single-valued mappings, $\mathcal{H}(A, B)$ be α-strongly accretive with respect to A, β-relaxed accretive with respect to B with $\alpha > \beta$ and $J_{M,\lambda}^{\mathcal{H}(.,.)}(x)$ being the generalized Yosida approximation operation defined by

$$J_{M,\lambda}^{\mathcal{H}(.,.)}(u) = \frac{1}{\lambda}\left[I - R_{M,\lambda}^{\mathcal{H}(.,.)}\right](u), \ \forall \ u \in X, \tag{14}$$

where I is the identity mapping on X and $R_{M,\lambda}^{\mathcal{H}\,(.,.)}$ is the resolvent operator associated with the mappings $\mathcal{H}(.,.)$ and M. It was shown in [4] that the resolvent operator

$$R_{M,\lambda}^{\mathcal{H}\,(.,.)}(u) = [\mathcal{H}(A,B) + \lambda M]^{-1}(u), \ \forall \ u \in X, \ \lambda > 0$$

and the generalized Yosida approximation operation in (13) are connected by the following relation

$$\lambda J_{M,\lambda}^{\mathcal{H}\,(.,.)}(x) \in [\lambda M + \mathcal{H}(A,B) - I](R_{M,\lambda}^{\mathcal{H}\,(.,.)}(x)).$$

In order to study the strong convergence characteristics of the solutions of Yosida inclusion (13), Ahmad et. al. [4] proposed the following iterative algorithm: For $x_0 \in X$, define the sequence $\{x_n\} \subset X$ by the following scheme:

$$x_{n+1} = R_{M_n,\lambda}^{\mathcal{H}\,(.,.)}\big[\mathcal{H}(A,B)x_n - \lambda J_{M_n,\lambda}^{\mathcal{H}\,(.,.)}(x_n)\big].$$

Based on the SVIP and SMVIP discussed above, Rahaman et. al. [25] introduced the Split Monotone Yosida Variational Inclusion Problem (SMYVIP) which is to find a point $x^* \in H_1$ such that

$$0 \in f_1(x^*) + B_1(x^*) - T_{\lambda_1}^{B_1}(x^*), \tag{15}$$

and

$$y^* = Ax^* \in H_2 \text{ solves } 0 \in f_2(y^*) + B_2(y^*) - T_{\lambda_2}^{B_2}(y^*), \tag{16}$$

where $B_i : H_i \to 2^{H_i}$, $i = 1,2$ is a multivalued maximal monotone mapping, $f_i : H_i \to H_i$ is a single-valued mapping, $T_{\lambda_i}^{B_i} = \frac{1}{\lambda_i}(I_i - R_{\lambda_i}^{B_i})$ is the Yosida approximation operator of the mapping B_i, $R_{\lambda_i}^{B_i} = (I_i + \lambda_i B_i)^{-1}$ is the resolvent of the mapping B_i for $\lambda_i > 0$ and I_i is the identity mapping on Hilbert space H_i. Rahaman et. al. [25] presented the following Yosida approximation technique to approximate the solution of SMYVIP (15)-(16).

$$\begin{cases} x_0 \in H_1; \\ u_n = T[x_n + \gamma A^*(S - I)Ax_n]; \\ v_n = \delta_n u_n + \tau g_n(u_n); \\ x_{n+1} = (1 - \alpha_n D)v_n + \alpha_n \beta f(v_n); \end{cases}$$

where H_1 and H_2 are real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator with adjoint A^*, D is a strongly positive bounded linear operator on H_1 with coefficient $\bar{r} > 0$ and $\beta \in (0, \frac{\bar{r}}{k})$, $\{g_n\}$ is a family of k-demicontractive mappings and uniformly convergent for any $x \in K$, where K is any bounded subset of H_1, $f : H_1 \to H_1$ a ξ-contraction mapping, $\tau > 0$ and $\{\alpha_n\}$ and $\{\delta_n\}$ are sequences in $[0,1)$. Furthermore, they proved that the sequence $\{x_n\}$ converges strongly to $x^* \in$ SMYVIP (15)-(16).

Let E be a real Banach space with its dual E^*. It can be deduced from (15)-(16) that the Monotone Yosida Varaiational Inequality Problem MYVIP is nothing but to find a point $x^* \in E$ such that

$$0 \in f(x^*) + B(x^*) - T_\mu^B(x^*), \tag{17}$$

where $B : E \to 2^{E^*}$ is a multivalued maximal monotone mapping, $f : E \to E^*$ is an inverse strongly monotone mapping, $T_\mu^B = \frac{1}{\mu}(I - R_\mu^B)$ is the Yosida approximation operator of the mapping B, $R_\mu^B = (I + \mu B)^{-1}$ is the resolvent of multivalued maximal monotone mapping B for $\mu > 0$ and I is the identity mapping on Hilbert space E.

In this article, we consider the following problems:

(i) Can we introduce the MYVIP considered in [25] to a p-uniformly convex and uniformly smooth Banach space?

(ii) Can we approximate the solution of MYVIP (17) using a different method of proof different from the ones considered in [1, 5, 9, 14, 13, 24]?

Motivated by the works of Izuchukwu [13], Rahaman et. al. [25], Ogbuisi and Mewomo [24], we introduce an Halphern-type iterative method for approximating the solution of MYVIP (17) in the framework of p-uniformly convex and uniformly smooth Banach space. We prove a strong convergence theorem of the aforementioned problem. Our result extends and complements the result of [13], [22], [25] and other related results in literature.

2 Preliminaries

We state some known and useful results which will be needed in the proof of our main theorem. In the sequel, we denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively.

Lemma 1. *[8] Let E be a Banach space and $x, y \in E$. If E is q-uniformly smooth, then there exists $C_q > 0$ such that*

$$||x - y||^q \leq ||x||^q - q\langle J_q^E(x), y\rangle + C_q||y||^q.$$

Lemma 2. *[16] Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $z, x_k \in E$ $(k = 1, 2, ..., N)$ and $\alpha_k \in (0, 1)$ with $\sum_{k=1}^{N} \alpha_k = 1$. Then, we have*

$$\Delta_p(J_q^{E^*}(\sum_{k=1}^{N} \alpha_k J_p^E(x_k)), z) \leq \sum_{k=1}^{N} \alpha_k \Delta_p(x_k, z) - \alpha_i \alpha_j g_r^*(||J_p^E(x_i) - J_p^E(x_j)||),$$

for all $i, j \in 1, 2, ..., N$ and $g_r^* : \mathbb{R}^+ \to \mathbb{R}^+$ being a strictly increasing function such that $g_r^*(0) = 0$.

Lemma 3. *[28] Let E be a real p-uniformly convex and uniformly smooth Banach space. Let $V_p : E^* \times E \to [0, +\infty)$ be defined by*

$$V_p(x^*, x) = \frac{1}{q}||x^*||^q - \langle x^*, x \rangle + \frac{1}{p}||x||^p, \ \forall \ x \in E, x^* \in E.$$

Then, the following assertions hold:
(i) V_p is nonnegative and convex in the first variable.
(ii) $\Delta_p(J_q^{E^*}(x^*), x) = V_p(x^*, x), \ \forall \ x \in E, \ x^* \in E.$
(iii) $V_p(x^*, x) + \langle y^*, J_q^{E^*}(x^*) - x \rangle \leq V_p(x^* + y^*, x), \forall \ x \in E, \ x^*, y^* \in E.$

Lemma 4. *[9] Let E be a real p-uniformly convex and uniformly smooth Banach space. Suppose that $\{x_n\}$ and $\{y_n\}$ are bounded sequences in E. Then the following assertions are equivalent:*
(i) $\lim_{n\to\infty} \Delta_p(x_n, y_n) = 0$;
(ii) $\lim_{n\to\infty} ||x_n - y_n|| = 0.$

Lemma 5. *[26] Let $f : E \to (-\infty, +\infty]$ be uniformly Frechet differentiable and bounded on bounded subsets of E. Then, f is uniformly continuous on bounded subsets of E and $\bigtriangledown f$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 6. *[23] Let $\{a_n\}$ be a sequence of non-negative number such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n r_n;$$

where $\{r_n\}$ is a sequence of real numbers bounded from above and $\{\lambda_n\} \subset [0, 1]$ satisfies $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then, the following conditions holds:

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} r_n.$$

3   Main Result

Lemma 7. *Let E be a p-uniformly convex and uniformly smooth Banach space E with E^* its dual. Let $B : E \to 2^E$ be a multivalued maximal monotone mapping with nonempty value and $f : E \to E$ be an inverse strongly monotone mapping. Assume that $\Omega \neq \emptyset$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $(0, 1)$. Let the sequence $\{x_n\}$ be generated iteratively by*

$$\begin{cases} x_0 \in E, \\ y_n = J_{E^*}^q[(1 - \alpha_n)J_E^p x_n + \alpha_n J_E^p u]; \\ z_n = J_{E^*}^q[(1 - \beta_n)J_E^p y_n + \beta_n J_E^p R_\mu^B(I + \mu(T_\mu^B - f))y_n]; \\ x_{n+1} = J_{E^*}^q[(1 - \gamma_n)J_E^p y_n + \gamma_n J_E^p z_n]; \end{cases} \tag{18}$$

then, $\{x_n\}$ is bounded.

Proof. Let $\overline{x} \in \Omega$ and $K = R_\mu^B[I + \mu(T_\mu^B - f)]$. Then, we have from (18) and the fact that K is nonexpansive (see [25]), we have that

$$
\begin{aligned}
\Delta_p(x_{n+1}, \overline{x}) &= \Delta_p[J_{E^*}^q(1 - \gamma_n)J_E^p y_n + \gamma_n J_E^p z_n, \overline{x}] \\
&\leq (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n \Delta_p(z_n, \overline{x}) \\
&\leq (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n[(1 - \beta_n)J_E^p y_n + \beta_n J_E^p K y_n, \overline{x}] \\
&\leq (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n[(1 - \beta_n)\Delta_p(y_n, \overline{x}) + \beta_n \Delta_p(K y_n, \overline{x})] \\
&\leq (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n[(1 - \beta_n)\Delta_p(y_n, \overline{x}) + \beta_n \Delta_p(y_n, \overline{x})] \\
&= (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n \Delta_p(y_n, \overline{x}) \\
&= \Delta_p(y_n, \overline{x}) \\
&= \Delta_p[J_{E^*}^q(1 - \alpha_n)J_E^p x_n + \alpha_n J_E^p u, \overline{x}] \\
&\leq (1 - \alpha_n)\Delta_p(x_n, \overline{x}) + \alpha_n \Delta_p(u, \overline{x}) \\
&\leq \max\left\{\Delta_p(u, \overline{x}), \ \Delta_p(x_n, \overline{x})\right\} \\
&\ \ \vdots \\
&\leq \left\{\Delta_p(u, \overline{x}), \ \Delta_p(x_1, \overline{x})\right\}.
\end{aligned}
\tag{19}
$$

Therefore, $\{x_n\}$ is bounded and consequently, $\{y_n\}$ and $\{z_n\}$ are all bounded. $\qquad\square$

Theorem 2. *Let E be a p-uniformly convex and uniformly smooth Banach space with E^* its dual. Let $B : E \to 2^E$ be a multivalued maximal monotone with nonempty value and $f : E \to E$ be an inverse strongly monotone mapping. Assume that $\Omega \neq \emptyset$, then the sequence $\{x_n\}$ is generated by (18), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:*
(i) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(ii) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$. Then $\{x_n\}$ converges strongly to $x^* \in \Omega$.

Proof. Let $\overline{x} = \Pi_\Omega u$, then from Lemma 2, we have that

$$
\begin{aligned}
\Delta_p(z_n, \overline{x}) &= J_{E^*}^q[(1 - \beta_n)J_E^p y_n + \beta_n J_E^p K y_n, \overline{x}] \\
&\leq (1 - \beta_n)\Delta_p(y_n, \overline{x}) + \beta_n \Delta_p(K y_n, \overline{x}) - \beta_n(1 - \beta_n)g_r^*(||J_E^p y_n - J_E^p K y_n||) \\
&\leq (1 - \beta_n)\Delta_p(y_n, \overline{x}) + \beta_n \Delta_p(y_n, \overline{x}) - \beta_n(1 - \beta_n)g_r^*(||J_E^p y_n - J_E^p K y_n||) \\
&= \Delta_p(y_n, \overline{x}) - \beta_n(1 - \beta_n)g_r^*(||J_E^p y_n - J_E^p K y_n||) \\
&\leq \Delta_p(y_n, \overline{x}).
\end{aligned}
\tag{20}
$$

We obtain from (18) that

$$
\begin{aligned}
\Delta_p(z_n, y_n) &= \frac{1}{\gamma_n}\Delta_p(x_{n+1}, y_n) \\
&= \frac{\alpha_n}{\gamma_n}\left(\frac{\Delta_p(x_{n+1}, y_n)}{\gamma_n}\right).
\end{aligned}
\tag{21}
$$

We now have from (18), (19), (20), Lemma 2 and the fact that $J_{E^*}^q$ is a norm-to-norm uniformly continuous on bounded subset of E^* that

$$
\begin{aligned}
\Delta_p(x_{n+1}, \overline{x}) &\leq (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n\Delta_p(z_n, \overline{x}) - \gamma_n(1 - \gamma_n)g_r^*(||J_E^p y_n - J_E^p z_n||) \\
&\leq (1 - \gamma_n)\Delta_p(y_n, \overline{x}) + \gamma_n\Delta_p(y_n, \overline{x}) - \gamma_n(1 - \gamma_n)g_r^*(||J_E^p y_n - J_E^p z_n||) \\
&= \Delta_p(y_n, \overline{x}) - (1 - \gamma_n)\Delta_p(x_{n+1}, y_n) \\
&= \Delta_p(J_{E^*}^q[(1 - \alpha_n)J_E^p x_n + \alpha_n J_E^p u], \overline{x}) \\
&= V_p(\alpha_n J_E^p u + (1 - \alpha_n)J_E^p x_n, \overline{x}) - (1 - \gamma_n)\Delta_p(x_{n+1}, y_n) \\
&\leq V_p(\alpha_n J_E^p u + (1 - \alpha_n)J_E^p x_n - \alpha_n(J_E^p u - J_E^p \overline{x}, \overline{x})) \\
&\quad - \langle -\alpha_n(J_E^p u - J_E^p(\overline{x})), J_{E^*}^p[\alpha_n J_E^p u + (1 - \alpha_n)J_E^p \overline{x}, x_{n+1} - \overline{x}\rangle] \\
&\quad - (1 - \gamma_n)\Delta_p(x_{n+1}, y_n) \\
&\leq \alpha_n V_p(J_E^p \overline{x}, \overline{x}) + (1 - \alpha_n)V_p(J_E^p x_n, \overline{x}) \\
&\quad + \alpha_n\langle J_E^p u - J_E^p \overline{x}, x_{n+1} - \overline{x}\rangle - (1 - \gamma_n)\Delta_p(x_{n+1}, y_n) \\
&= \alpha_n\Delta_p(\overline{x}, \overline{x}) + (1 - \alpha_n)\Delta_p(x_n, \overline{x}) + \alpha_n\langle J_E^p u - J_E^p \overline{x}, x_{n+1} - \overline{x}\rangle \\
&\quad - (1 - \gamma_n)\Delta_p(x_{n+1}, y_n) \\
&\leq (1 - \alpha_n)\Delta_p(x_n, \overline{x}) \\
&\quad - \alpha_n\left[-\langle J_E^p u - J_E^p \overline{x}, x_{n+1} - \overline{x}\rangle + (1 - \gamma_n)\left(\frac{\Delta_p(x_{n+1}, y_n)}{\alpha_n}\right)\right]. \quad (22)
\end{aligned}
$$

Thus, (22) becomes

$$\Delta_p(x_{n+1}, \overline{x}) \leq (1 - \alpha_n)\Delta_p(x_n, \overline{x}) - \alpha_n\Upsilon_n; \quad (23)$$

where

$$\Upsilon_n = -\langle J_E^p u - J_E^p \overline{x}, x_{n+1} - \overline{x}\rangle + \frac{(1 - \gamma_n)}{\alpha_n}\Delta_p(x_{n+1}, y_n). \quad (24)$$

From (18), we have that $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are bounded, thus they are bounded below. It follows from condition (i) of (18) and Lemma 6 that

$$
\begin{aligned}
\limsup_{n\to\infty} \Delta_p(x_n, \overline{x}) &\leq \limsup_{n\to\infty} \Upsilon_n \\
&= -\liminf_{n\to\infty} \Upsilon_n. \quad (25)
\end{aligned}
$$

Therefore, $\lim_{n\to\infty} \Upsilon_n$ exists. Thus, we obtain from (24) that

$$\liminf_{n\to\infty} \Upsilon_n = \liminf_{n\to\infty}\left(\langle J_E^p \overline{x} - J_E^p u, x_{n+1} - \overline{x}\rangle + \frac{(1 - \gamma_n)}{\alpha_n}\Delta_p(x_{n+1}, y_n)\right). \quad (26)$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup x^*$ for some $x^* \in E$, and

$$\liminf_{n\to\infty} \Upsilon_n = \liminf_{j\to\infty}\left(\langle J_E^p \overline{x} - J_E^p u, x_{n_j+1} - \overline{x}\rangle + \frac{(1 - \gamma_{n_j})}{\alpha_{n_j}}\Delta_p(x_{n_j+1}, y_{n_j})\right).$$

Using the fact that $\{x_n\}$ is bounded and $\liminf_{j\to\infty} \Upsilon_n$ exists, we have $\left\{\frac{(1-\gamma_{n_j})}{\alpha_{n_j}}\Delta_p(x_{n_j+1}, y_{n_j})\right\}$ is bounded. Also, by condition (ii), there exists $b \in (0,1)$ such that $\gamma_n \le b < 1$ which implies that $\frac{1}{\alpha_{n_j}}(1-\gamma_{n_j}) \ge \frac{1}{\alpha_{n_j}}(1-b) > 0$. Hence, we have that $\left\{\frac{1}{\alpha_{n_j}}\Delta_p(x_{n_j+1}, y_{n_j})\right\}$ is bounded.

Note from condition (i) and (ii) that there exists $a \in (0,1)$ such that

$$0 < \frac{\alpha_{n_j}}{\gamma_{n_j}} \le \frac{\alpha_{n_j}}{a} \to 0, \ j \to \infty.$$

Therefore, we obtain from (8) that

$$\lim_{j\to\infty} \Delta_p(z_{n_j}, y_{n_j}) = 0, \tag{27}$$

which implies from Lemma 4 that

$$\lim_{j\to\infty} ||z_{n_j} - y_{n_j}|| = 0.$$

From (18) and 27, we obtain that

$$\Delta_p(x_{n_j+1}, y_{n_j}) = \gamma_{n_j}\Delta_p(z_{n_j}, y_{n_j}) \to 0, \ \text{as } j \to \infty. \tag{28}$$

From (18) and (27), we have that

$$\Delta_p(Ky_{n_j}, y_{n_j}) = \frac{1}{\beta_{n_j}}\Delta_p(z_{n_j}, y_{n_j}) \to 0, \ \text{as } j \to \infty. \tag{29}$$

From (18) and condition (i), we have that

$$\Delta_p(y_{n_j}, x_{n_j}) = \alpha_{n_j}\Delta_p(u, x_{n_j}) \to 0, \ \text{as } j \to \infty. \tag{30}$$

Since $x_{n_j} \rightharpoonup x^*$, we obtain from (30) that $y_{n_j} \rightharpoonup x^*$. It then follows (29) that $x^* \in F(K) = \Omega$.

Furthermore, since $x_{n_j} \rightharpoonup x^*$ and the duality map is norm-to-norm uniformly continuous on bounded sets, we obtain from (26),(28) and the Bregman projection property that

$$\liminf_{n\to\infty} \Upsilon_n = \liminf_{j\to\infty} \left(\langle J_E^p \overline{x} - J_E^p u, x_{n_j+1} - \overline{x}\rangle + \frac{(1-\gamma_{n_j})}{\alpha_{n_j}}\Delta_p(x_{n_j+1}, y_{n_j})\right)$$
$$= \langle J_E^p \overline{x} - J_E^p u, x_{n_j+1} - \overline{x}\rangle \ge 0.$$

Hence, we have from (25) that

$$\limsup_{n\to\infty} \Delta_p(x_{n+1}, \overline{x}) \le -\liminf_{n\to\infty} \Upsilon_n \le 0.$$

Therefore, $\lim_{n\to\infty} \Delta_p(x_n, \overline{x}) = 0$ and this implies that $\{x_n\}$ converges strongly to $x^* = P_\Omega u$. $\qquad\square$

Convergence Theorem for a Family of Mapping.
In this section, we apply our main result to a countable family of quasi-nonexpansive
mappings.

Definition 2. *[5] Let C be a subset of a real p-uniformly Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Then $\{T_n\}_{n=1}^{\infty}$ is said to satisfy the AKTT-condition, if for any bounded subset B of C,*

$$\sum_{n=1}^{\infty} \sup\{||J_E^p(T_{n+1}z) - J_E^p(T_n z)||\} < \infty.$$

Lemma 8. *[29] Let C be a nonempty, closed and convex subset of a real p-uniformly Banach space E. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence of mappings of C into E such that $\cap_{n=1}^{\infty} F(T_n) \neq \emptyset$. Suppose that $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition. Suppose that for any bounded subset B of C. Then, there exists the mapping $T : B \to E$ such that*

$$Tx = \lim_{n \to \infty} T_n x, \ \forall \ x \in B,$$

and

$$\limsup_{n \to \infty} ||J_E^p(Tz) - J_E^p(T_n z)|| = 0.$$

In the theorem below, we say that $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition and $\cap_{n=1}^{\infty} F(T_n) = F(T)$.

Theorem 3. *Let E be a p-uniformly convex and uniformly smooth Banach space E with E^* its dual. Let $T : E \to E$ be a finite family Bregman quasi-nonexpansive mapping and assume that $\Omega = \cap_{n=1}^{\infty} F(T_n) = F(T) \neq \emptyset$. Let $\{x_n\}$ be generated iteratively by*

$$\begin{cases} x_0, E, \\ y_n = J_{E^*}^q[(1 - \alpha_n)J_E^p x_n + \alpha_n J_E^p u]; \\ z_n = J_{E^*}^q[(1 - \beta_n)J_E^p y_n + \beta_n J_E^p T y_n]; \\ x_{n+1} = J_{E^*}^q[(1 - \gamma_n)J_E^p y_n + \gamma_n J_E^p z_n]; \end{cases} \tag{31}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1) satisfying the following conditions:
(i) $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1$. In addition, if $(\{T_n\}, T)$ satisfies the AKTT-condition, then the sequence $\{x_n\}$ converges strongly to an element $x^* \in \Pi_{\cap_{n=1}^{\infty} F(T_n)} u$.

Proof. By following the technique in Lemma 7 and Theorem 2, we can prove that $\{x_n\}$ is bounded and $\Delta_p(x_n, T_n x_n) = 0$. Since J_E^p is uniformly continuous on bounded subset of E, then we have that

$$\lim_{n \to \infty} ||J_E^p x_n - J_E^p T_n x_n|| = 0. \tag{32}$$

By Lemma 8, we have that

$$||J_E^p x_n - J_E^p T x_n|| \leq ||J_E^p x_n - J_E^p T_n x_n|| + ||J_E^p T_n x_n - J_E^p T x_n||$$
$$\leq ||J_E^p x_n - J_E^p T_n x_n|| + \sup_{x \in \{x_n\}} ||J_E^p T_n x - J_E^p T x|| \to 0, \text{ as } n \to \infty.$$

Since J_E^q is norm-to-norm uniformly continuous on bounded subset of E^*, it follows that

$$\lim_{n \to \infty} ||x_n - T x_n|| = 0.$$

This completes the proof. □

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