A REVIEW OF F-CONTRACTION MAPPING ON METRIC SPACES

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Submitted to the Department of Computer Science and Mathematics

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In Partial Fulfilment of the Award of Bachelor of Science Degree in Mathematics

In the Department of Computer Science and Mathematics

AUGUST, 2021

CERTIFICATION

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DEDICATION

The project is dedicated to Almighty God. And to my wonderful parents,Mr and Mrs Adeyemi for their love, support and encouragement.

ACKNOWLEDGEMENTS

First and foremost, I give thanks to God, without whom I could have never achieved this goal.

My deepest appreciation goes to my parents Prince and Mrs. Adeyemi for their financial support in my education, their words of encouragement and support. May the Lord continue to provide all their needs. I also extend my appreciation to my sisters, Anuoluwapo and Sarah who also encouraged me in this project and for their love and support.

My sincere gratitude goes to my supervisor Dr. Adewole for his maximum support to me, for successfully writing the project. Also to other lecturers in the Department of Mathematics Dr. Mebawondu, Dr. Onifade, Mr. Okone, Mr. Ebo, Dr. Kasali, Mr. Balogun, Miss Anabel, Dr. Igiri for their constant support throughout my project.

My appreciation also goes to my fellow colleagues in my department and other friends apart from my department, Adeoti Faith, Olaiya Ore and Obe Israel, Ibeauwchi Innocent, Olaitan Philip, Joshua Daniel, who were good friends and work partners and to all my friends.

And i will like to thank my best friend Sangosanya Temitope for her good support in this project.

Thanks to all who supported me, Dr Young, Dr Abiala, Pastor Olumide Adesina, Miss Fowobi and the non-teaching staff, for their guidiance and always supply me with information to avoid mistakes throughout my project.

Abstract

It is well-known that the Banach contraction principle has been extended, generalised and improved by researchers in this area of mathematics. In particular, one of the generalisation of the Banach contraction principle is the F-contraction mapping. In this project, we review the concept of F-contraction and Suzuki F-contraction in the frame work of complete metric spaces. We also gave some applications and examples to validate the applicability of our main results.

Table of Contents

Certification ii

Dedication iii

Acknowledgements iv

Abstract v

1 Introduction 1

1.1 Background of Study 1

1.2 Statement of Problem 10

1.3 Objectives of Study 11

1.4 Research Methodology 11

1.5 Scope and limitation of Study 11

1.6 Significance of Study 12

1.7 Operational definition of terms 12

2 Literature Review 16

3 F- Suzuki Contraction Mapping On Metric Spaces 27

3.1 F-contraction 27

3.2 F- Suzuki Contraction 40

3.3 Applications and Examples 47

3.3.1 Application to Integral Equation 47
3.3.2 Application to Second Order Differential Equation 50
3.3.3 Examples 52

4 Conclusion **54**

Chapter 1

Introduction

1.1 Background of Study

Mathematics is an important branch of scientific knowledge that has many applications in solving real life problems for humanity. Mathematics can further be divided into different branches that have one quality according to theory application. One of the important branches is Functional Analysis, which is applicable to many areas, such as solving the problem of linear and non-linear partial differential equations, integral equations, nonlinear matrix equations and so on. In addition functional analysis has a lot of applications in numerical analysis, game theory, and optimisation problems. Combining analysis with geometry is a valuable aspect in the form of functional analysis.

An important and useful tool in functional analysis is the fixed point theory. Analytical fixed point theory and topological fixed point theory are the mainly two areas of fixed point theory. This enhances the importance and significance of functional analysis since it is widespread in the solution of different types of linear and non-linear problems. Fixed point theory is an area of research which has many applications in various direction of science, such as Economics, Computer Science, Optimisation Theory, Variational Inequalities, Engineering and so on. The fixed point theory is an essential area of study in pure and applied mathematics. Optimisation problem such as minimisation problems, equilibrium problems are effectively solved using the fixed point method. Now we define a fixed point problem and give examples.

Definition 1.1. (Fixed point) Let X be any arbitrary space, a point $x \in X$ is called a fixed point of a mapping $T : X \to X$ if

$$Tx = x, \tag{1.1.1}$$

that is, a point $x \in X$ which remains invariant under the action of the mapping T.
A trivial mapping with a fixed point is the identity mapping, that is $I : X \to X$ defined as $Tx = x$, for all $x \in X$.

Example 1.2.

1. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^2 + x - 1$. It is easy to see that $x = 1$ and $x = -1$ are the fixed points of T, because $T(1) = 1$ and $T(-1) = -1$.
2. Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(x) = x^2 + x + 3$. It is also easy to see that T has no fixed point.

In fixed point theory, other spaces of study other than metric spaces have been studied by different authors. We have various generalised metric spaces such as partial metric

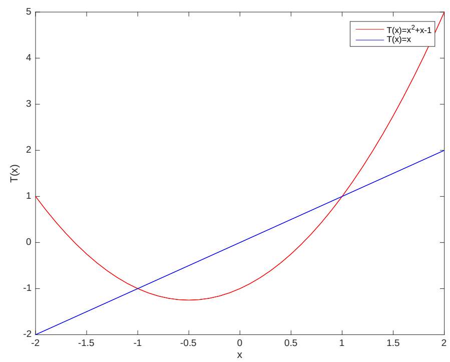

Figure 1.1: A Fixed Point Graph of $T(x) = x^2 + x - 1$ and $T(x) = x$.

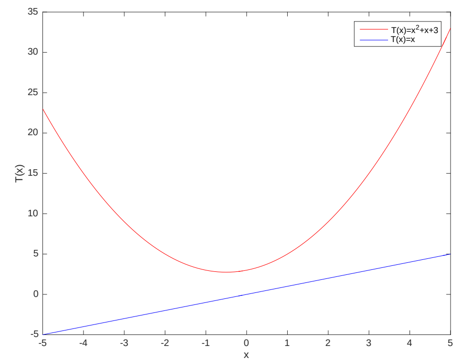

Figure 1.2: A Fixed Point Graph of $T(x) = x^2 + x + 3$ and $T(x) = x$.

spaces, b-metric spaces, partial b-metric spaces, S- metric spaces, cone metric spaces, fuzzy metric spaces, G-metric spaces, G_b-metric spaces, extended b metric spaces and so on.

Poincare (1886) was the first person who worked on fixed point theory, afterward the equation $f(a) = a$ was taken into consideration by Brouwer. This means that the Brouwer fixed point theorem named after L.E.J.(Bertus) Brouwer was inspired by Henri Poincaré. However, the fixed point theorem for a sphere asserts that any continuous mapping, the sphere unto itself has a fixed point. The extension of Brouwer fixed point theorem is the Kakutani fixed point theorem (Lassonde, 1990). The Brouwer's theorem is about continuous point-to-point function and the Kakutani theorem dealt with set valued function, i.e point-to-set function.

The Banach fixed point theorem is the most cited and applied fixed point result in the field of nonlinear analysis. Since it makes use of the iterative scheme, it can easily be implemented on a computer system to find the fixed point of a contractive mapping. It produces approximations of any required accuracy. Due to its simplicity and generality, the Banach fixed point theorem has become a very potent tool in solving existence and uniqueness problems in different areas of mathematical sciences, such as nonlinear Volterra integral equations, dynamical programming, nonlinear integro-differential equations, game theory, numerical approximations, random, ordinary and partial differential equations and so on. The importance of the Banach contraction principle cannot be overemphasised in the study of fixed point theory and its applications. Due to its fruitful applications, many researchers have extended, generalised and improved the well celebrated Banach's fixed point theorem by considering classes of nonlinear mappings and classical spaces which are more general than the class of a contraction mappings and metric spaces (see (Latif, 2014; Ilić et al., 2011; Meyers, 1964; Jleli et al., 2014; Merryfield and Stein Jr, 2002; Sadiq Basha, 2010) and the references therein).

Theorem 1.3. *(**Contraction Mapping Principle**) (Suzuki, 2008): Let X be a complete metric space, $c \in [0,1)$ and let $T : X \to X$ be a mapping such that for $a, b \in X$*

$$d(Ta, Tb) \leq cd(a,b)$$

then T has a unique fixed point $p \in X$ for each $a \in X$, $\lim_{n\to\infty} T^n a = p$.

That is, every contraction mapping on a complete metric space always has a unique fixed point. The contraction mapping principle is also a Banach contraction principle or Banach fixed point theorem (Pata, 2019). The iterative contraction principle that was used in the proof of Banach Contraction was introduced by Picard.

Example 1.4. Let $X = \mathbb{R}$ with the usual metric,

$$d(x,y) = |x-y|$$

where $(\mathbb{R}, d)$ is the metric space.
Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \frac{x}{a} + b$. It is easy to see that f is a contraction if $a > 1$. To see this, observe that

$$\begin{aligned} d(fx, fy) &= |fx - fy| \\ &= \left| \frac{x}{a} + b - \left(\frac{y}{a} + b \right) \right| \\ &= \left| \frac{x}{a} - \frac{y}{a} \right| \\ &= \frac{1}{a}|x-y|. \end{aligned}$$

This is a contraction for $a > 1$ and we claim that $x = \frac{ab}{a-1}$ is the fixed point of f

Proof of Claim:

$$
\begin{aligned}
f\left(\frac{ab}{a-1}\right) &= \frac{ab}{a-1} \\
&= \frac{\frac{ab}{a-1}}{a}+b \\
&= \left(\frac{ab}{a-1}\times\frac{1}{a}\right)+b \\
&= \frac{b}{a-1}+b \\
&= \frac{b+b(a-1)}{a-1} \\
&= \frac{ab}{a-1}.
\end{aligned}
$$

Hence, our claim is justified.

Example 1.5. We consider the Euclidean metric space $(\mathbb{R}^2, d)$. The function given as $f : \mathbb{R}^2 \to \mathbb{R}^2$, we have

$$f(x,y) = \left(\frac{x}{a}+b, \frac{y}{c}+d\right)$$

is a contraction for $a > 1$ and $c > 1$. Using the same approach from the previous example, we claim that $x = \frac{ab}{a-1}$, $x = \frac{cd}{c-1}$ is the fixed point of f.

Proof of Claim:

$$
\begin{aligned}
f\left(\frac{ab}{a-1},\frac{cd}{c-1}\right) &= \left(\frac{ab}{a-1},\frac{cd}{c-1}\right)\\
&= \left(\frac{\frac{ab}{a-1}}{a}+b,\frac{\frac{cd}{c-1}}{c}+d\right)\\
&= \left(\left(\frac{ab}{a-1}\times\frac{1}{a}\right)+b,\left(\frac{cd}{c-1}\times\frac{1}{c}\right)+d\right)\\
&= \left(\frac{b}{a-1}+b,\frac{d}{c-1}+d\right)\\
&= \left(\frac{b+b(a-1)}{a-1},\frac{d+d(c-1)}{c-1}\right)\\
&= \left(\frac{ab}{a-1},\frac{cd}{c-1}\right).
\end{aligned}
$$

Hence, our claim is justified.

Some other types of the contractive mapping is the Kannan (1968). Kannan (1968, 1969) established fixed point theorems that are independent of the Banach contraction principle. It is well-known that the Banach contraction mappings are necessarily continuous on its domain, but the class of Kannan mappings need not to be continuous. In addition, it was established by Subrahmanyam (1975) that Kannan's fixed point theorem characterises metric completeness, meanwhile, Connell (1959) gave an example to show that the Banach contraction principle does not characterise metric completeness.

Theorem 1.6. *(Kannan, 1968): Let X be a metric space and let $T : X \to X$ be a mapping such that there exists $0 < c < \frac{1}{2}$ for all $x,y \in X$,*

$$d(Tx,Ty) \leq c[d(x,Tx)+d(y,Ty)] \text{ for all } x,y \in X$$

Then, T has a unique fixed point $p \in X$, and for any $x \in X$ the sequence $\{x_n\}$ of iterates $\{T^n x\}$ defined by

$$x_{n+1} = Tx^n$$

converges to p for and $x_0 \in X$.

Many authors worked on different generalised metric spaces and proved the Banach contraction principles. The fixed point theorems are obtained by extending the contraction conditions and then generalising the Banach Contraction Principle. Some generalisation of contraction mapping principle are obtained by considering contraction conditions which do not only include only the $d(x,y)$ on the right hand side, but also the x and y under the mapping T, where we have $d(Tx,y), d(Tx,Tx), d(x,Ty), d(y,Ty)$. Ordinary fixed point theorems in this class are:

a) Chatterjea's(1972) theorem (Rhoades, 1977), where the operator T is defined by;

$$d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)] \quad \text{such that} \quad 0 < c < \frac{1}{2} \quad \text{is satisfied.}$$

b) Zamfirescu'a (1972) theorem (Rhoades, 1977), where the operator T is defined by:

$$d(Tx,Ty) \leq \delta d(x,y) + 2\delta d(x,Tx) \quad satifying,$$
$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\} \quad \text{such that} \quad \delta[0,1)$$

c) Ciric's(1974) theorem (Rhoades, 1977), we then define the operator T as,

$$d(Tx,Ty) \leq p.\max\{d(x,y),d(x,Tx),d(y,Ty),d(y,Tx),d(x,Ty)\}$$

satisfying $(0,\frac{1}{2})$ is considered then we have,

d) Rhodes' (1974) theorem (Rhoades, 1977), where the operator T is defined by;

$$d(Tx,Ty) \leq p.\max\left\{d(x,y),d(x,Ty),d(y,Ty),\frac{d(y,Tx)+d(x,Ty)}{2}\right\}$$

which satisfies $(0,\frac{1}{2})$.

There are various extensions in this field, so it is impossible to discuss all of these generalisations. However, we'll shortly discuss some of them.

The generalisation of metric space named as Partial metric space was introduced by Matthews (1994) in which he proved the Banach contraction on this space (partial metric space). Another space is the cone space which was introduced by Huang and Zhang (2007). We have other fixed point results which were proved by different authors in this space. Bakhtin (1989), for the first time, he introduced the idea of b-metric space, then because of it's importance, b-metric space is used for generalising contraction mapping and proving some new results. Czerwik (1993) created different result for b-metric space. For different purposes, the results were further extended and generalised in single and multivalued mapping. Khamsi and Hussain (2010) proved some new results. Many contraction conditions have been established after Banach Contraction principle, but we are going to base on the one we are using in our work.

The F-contraction was introduced by Wardowski (2012) and also proved a fixed point theorem concerning F-contractions in different way as done by many authors. Wardowski generalised the Banach contraction principle in many form, which we are going to show in our result. Secelean (2013) established some fixed point results for the F-contraction mappings using iterative method. In addition, Kumam and Piri (2014) generalised the fixed point theory of Wardwoski for F-suzuki contraction by applying some weaker conditions on the self map of a complete metric space. Abbas et al. (2013) extended the Wardwoski and established new fixed point theorems using the the F- contraction mapping. Batra et al. also contributed on the explanation of the graph using the idea of F-contraction. Cosentino and Vetro (2014) proved some new results for self contraction mappings, on a complete metric space. Also, on the complete metric space Vetro proved some important result using F-contraction.

In this project, we review the work of Kumam and Piri (2014) and explicitly establish the fixed point results obtained in this paper. In addition, we give some other examples and applications to establish the fixed point results obtained in this paper in the framework of complete metric spaces.

1.2 Statement of Problem

The notion of F-contraction as introduced by Wardowski (2012) has proven to be a great tool for generalising the Banach contraction theorem and it has been well used to establish existence and uniqueness of solution of differential and integral equations. Due to its fruitful applications, researchers in this area have generalised the concept.

The problem this work will deal with is to give an explicit analysis of what seemingly appeared unclear in the paper of Kumam and Piri (2014) and also provide some examples and applications to establish the applicability of the F-contraction.

1.3 Objectives of Study

The aim of this project is to investigate fixed point theory on metric spaces and its application proposed by Kumam and Piri (2014) and

a.) make explicit the ambiguities in the prove of Kumam and Piri (2014)

b.) discuss, through proof, the F-contraction introduced by Kumam and Piri (2014)

c.) give some examples and applications of the results.

1.4 Research Methodology

The F-contraction has been discussed in the paper ”Some fixed point theorem concerning F-contraction in complete metric spaces” in Kumam and Piri (2014). We seek to explore the explanation in this work.

1.5 Scope and limitation of Study

The scope of this work is all about the F-contraction introduced by Kumam and Piri (2014). This work is limited to establishing the fixed point results for this mapping and proving some examples and application.

1.6 Significance of Study

The significance of F-contraction can not be over emphasised in the study of fixed point theory and its application. Aside from generalising the well-known Banach contraction principle, the F-contraction theorem is then used to show the existence and uniqueness of ordinary and partial differential equations, integral equations, nonlinear matrix equations, partial differential equations and so on.

1.7 Operational definition of terms

Vector Spaces: Let V be a non-empty set **u,v,w** are vectors. Then V is called a vector space if there are exist two algebraic operation, vector addition and scalar multiplication such that the following condition are satisfied.

1. Closure property: $\mathbf{u}+\mathbf{v} \in V$ $\forall$**u,v** $\in X$.
2. Commutative property: $\mathbf{u}+\mathbf{v} = \mathbf{u}+\mathbf{v}$.
3. Associativity property: $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ $\forall$**u,v,w** $\in X$.
4. Zero vector: Let $\mathbf{u} \in X$ then there exist a zero vector given by **0**, such that $\mathbf{u}.\mathbf{0} = \mathbf{0}.\mathbf{u}$.
5. Additive inverse: Let $\mathbf{u} \in X$ then there exist a unique vector given by **-u**, such that **-u**$+\mathbf{u} = \mathbf{u}+(-\mathbf{u})$.
6. Closure property:Closure property: $\mathbf{u}.\mathbf{v} \in V$ $\forall$**u,v** $\in X$
7. Associativity property: $(\mathbf{u}.\mathbf{v}).\mathbf{w} = \mathbf{u}.(\mathbf{v}.\mathbf{w})$ $\forall$**u,v,w** $\in X$.

8. Distributivity(left): For any scalar c and vectors $\mathbf{u},\mathbf{v} \in X$, $p(\mathbf{u}+\mathbf{v}) = p\mathbf{u}+p\mathbf{v}$.

9. Distributivity(right):For any scalar p and q and vector $\mathbf{u} \in X$, $(p+q)\mathbf{u} = p\mathbf{u}+q\mathbf{u}$.

10. Identity property: For any $\mathbf{a} \in V$, $1\mathbf{a} = \mathbf{a}$.

Normed space: Let X be a vector space over the scalar field K. A norm on a vector space X is a real-valued function $\|.\|$, where, $\|.\| : X \to [0,\infty)$ such that for any $x,y \in X, \lambda \in K$, the following axioms are satisfied;

1. $\|x\| \geq 0$ (non-negative or real-valued)

2. $\|x\| = 0$ iff $x = 0$

3. $\|\lambda x\| = |\lambda|\|x\|$

4. $\|x+y\| \leq \|x\|+\|y\|$ (Triangle inequalities)

Hence a vector space X with a norm defined on it is called a Normed space.

Metric Space: Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ is metric induced on X such that the following axioms are satisfied for all $x,y,z \in X$.

1. $d(x,y) \geq 0 \; \forall x,y \in X$; (non-negative or real-valued)

2. $d(x,y) = 0$ iff $x = y$;

3. $d(x,y) = d(y,x)$ (symmetry)

4. $d(x,y) \leq d(x,z)+d(z,y)$ (triangle inequality).

Fixed Point: Let $T : X \to X$ be a self mapping on a set X. A point $a_0 \in X$ is a fixed point on the map T if $Ta_0 = a_0$.

Contraction map: Let (X,d) be a metric space. A self mapping $T : X \to X$ satisfying $d(Ta,Tb) \leq cd(a,b)$ for all $a,b \in X$ where $c \in [0,1)$.

Complete metric space: is a space X where every Cauchy sequence converges to a point let say $x \in X$.

Closed set: A subset A of X is said to be closed if its complement which is in X is open, i.e $A^c = X - A$ is open.

Cauchy Sequence: is a sequence $\{a_n\}$ in a metric space (X,d) if $\forall \delta > 0$, there exists a positive number $\mathbb{N}$ which depends on δ such that $d(a_n,a_m) < \delta$ where $m,n > \mathbb{N}$.

Convergent Sequence: A sequence $\{a_n\}$ in a metric space (X,d) is convergent to a point say $a \in X$ if $\lim_{n\to\infty} d(a_n,a) = 0$.

Convex space: A set X is said to be convex if for any two point $a,b \in X$, there exists no points on the line between a and b that are not member of set X.

Continuous Mapping: Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be continuous at a point a_0 if for each $\varepsilon > 0$, there exists $\delta > 0$, such that $d(Ta,Ta_0) \leq \varepsilon \quad satisfying \quad d(a,a_0) < \delta$.

Iteration: This is a repeated process usually with the target of approaching a particular result. It can also be a computational approach in which a cycle of operation is repeated.

Iterative methods: These are methods used to solve a particular mathematical problem numerically.

Throughout this project we denote $\mathbb{R}$ as the set of real numbers, $\mathbb{R}_+$ as the set of all positive real numbers and $\mathbb{N}$ as the set of all natural numbers. We also denote the metric space with a metric d as (X,d) which we can also write in short as X.

The rest of this project is organised as follows: In Chapter 2, we present the review of relevant literature to this work. Chapter 3 is devoted to the review of the theory of F-contraction and F-Suzuki contraction mappings. Chapter 3 also contains some examples and applications to establish the applicability of F-contraction and F-Suzuki contraction maps while the project is concluded in Chapter 4.

Chapter 2

Literature Review

The Banach fixed point theorem is the most cited and applied fixed point result in nonlinear analysis since the contraction condition on the self mapping T is easy to verify, and the only property needed on the metric space is the completeness. Also, since it makes use of iterative scheme, it can easily be implemented on a computer system to find the fixed point of the contractive mapping as it produces approximations of any required accuracy. Due to its simplicity and generality, the Banach fixed point theorem has become a prevalent tool in solving existence problems in different areas of mathematical sciences such as nonlinear Volterra integral equations, dynamical programming, nonlinear integro-differential equations, game theory, numerical approximations, random, ordinary and partial differential equations and so on. The importance of the Banach contraction principle cannot be over emphasised in the study of fixed point theory and its applications. Due to its fruitful applications, many researchers have extended, generalised and improved the well celebrated Banach's fixed point theorem by considering classes of nonlinear mappings and classical spaces which are more general than the class of contraction mappings and metric spaces (see (Latif, 2014; Ilić et al., 2011; Meyers, 1964; Jleli et al., 2014; Merryfield and Stein Jr, 2002; Sadiq Basha, 2010) and the references therein). In particular, Wardowski (2012) introduced a class of mappings called the F-contractions. This class of mappings is defined as follows:

Definition 2.1. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\lambda > 0$ such that for all $x, y \in X$;

$$d(Tx,Ty) > 0 \implies \lambda + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{2.0.1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

(F_1) F is strictly increasing;

(F_2) for all sequences $\{\beta_n\} \subseteq \mathbb{R}^+, \lim_{n\to\infty} \beta_n = 0$ if and only if $\lim_{n\to\infty} F(\beta_n) = -\infty$;

(F_3) there exists $c \in (0,1)$ such that $\lim_{\beta\to 0^+} \beta^c F(\beta) = 0$.

Let $\mathscr{F}$ denote a set of functions that satisfy (F_1), (F_2) and (F_3).

Some well-known examples of the mapping F include;

1. $F(t) = \ln(t)$,

2. $F(t) = \frac{-1}{\sqrt{t}}$,

3. $F(t) = \ln(t^2 + t)$.

Wardowski established the following result:

Theorem 2.2. *(Wardowski, 2012) Let (X,d) be a complete metric space and $T : X \to X$ be an F-contraction. Then, T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.*

Remark 2.3. (Wardowski, 2012) If we suppose that $F(t) = \ln(t)$, then an F-contraction mapping becomes the Banach contraction mapping.

Kumam and Piri (2014) used the continuity condition instead of condition (F_3) and proved the following result:

Definition 2.4. Let X be a complete metric space and $T : X \to X$ be a self map on X. Assume that there exists $\lambda > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \lambda + F(d(Tx,Ty)) \leq F(d(x,y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. Then, T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.

Recently, Hussain and Ahmad (2017) generalised the result of Kumam and Piri (2014) by introducing the notion of a Suzuki-Berinde F-contraction in the framework of a complete metric space. They established the following result.

Theorem 2.5. *(Hussain and Ahmad, 2017) Let X be a complete metric space and $T : X \to X$ be a self-map on X. Assume that there exist $L \geq 0$ and $\lambda > 0$ such that for all $x, y \in X$ with*

$Tx \neq Ty$,

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow$$
$$\lambda + F(d(Tx,Ty)) \leq F(d(x,y)) + L\min\{d(x,Tx), d(x,Ty), d(y,Tx)\},$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. Then, T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.

Secelean (2013) replaced the condition F_2 in the definition of F-contraction with the following condition:

(F_1') $\inf F = -\infty$ or, also by

(F_2') there exists a sequence $\{\beta_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

He proved the following lemma.

Lemma 2.6. *(Secelean, 2013) Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\beta_n\}$ be a sequence of positive integers. Then the following assertion hold:*

1. *if $\lim_{n\to\infty} F(\beta_n) = -\infty$ then $\lim_{n\to\infty} \beta_n = 0$;*
2. *if $\inf F = -\infty$ and $\lim_{n\to\infty} \beta_n = 0$ then $\lim_{n\to\infty} F(\beta_n) = -\infty$.*

Karapinar et al. (2015) introduced the notion of conditionally F-contraction and studied the fixed point theorem of such mappings in the framework of metric-like spaces. They gave the following definitions and results.

Definition 2.7. (Karapinar et al., 2015) Let (X, σ) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (A) if there exist $F \in \mathscr{F}$ and $\lambda > 0$ such that, for all $x, y \in X$ with $\sigma(Tx, Ty) > 0$,

$$\frac{1}{2}\sigma(x, Tx) < \sigma(x, y) \Rightarrow \lambda + F(\sigma(Tx, Ty)) \leq F(M_T(x, y)),$$

where

$$M_T(x, y) = \max\left\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{\sigma(x, Ty) + \sigma(y, Tx)}{4}\right\}.$$

Definition 2.8. (Karapinar et al., 2015) Let (X, σ) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (B) if there exist $F \in \mathscr{F}$ and $\lambda > 0$ such that, for all $x, y \in X$ with $\sigma(Tx, Ty) > 0$,

$$\begin{aligned}&\tfrac{1}{2}\sigma(x, Tx) < \sigma(x, y) \Rightarrow \\ &\qquad\qquad \lambda + F(\sigma(Tx, Ty)) \leq F(\max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\}).\end{aligned}$$

Theorem 2.9. *(Karapinar et al., 2015) Let (X, σ) be a complete metric-like space. If T is a conditionally F-contraction of type (A), then T has a fixed point $x^* \in X$.*

Latif et al. (2015) proved some fixed point results for $\alpha-$*admissible* mapping which satisfies Suzuki type contractive condition in the frame work of b-metric spaces which was introduced by Czerwik (1993). The authors gave some examples to verify the effectiveness and the applicability of the main results. Furthermore, Karapinar et al. (2015) introduced the concept of a generalised F-Suzuki type contraction in b-metric spaces and investigated the existence of fixed point for such mappings. The results presented generalises and improved several results in the literature. The authors made it clear that the notion of b-metric is a real generalisation of usual metric since a b- metric space is a metric space when s=1, and then defined the generalised F-Suzuki type contraction mapping with respect to the b-metric space (X,d) with constant $s \geq 1$. A mapping $T : X \to X$ is called a generalised F-Suzuki type contraction if there exists $\lambda > 0$ such that for all $x,y \in X$ with $x \neq y$ then;

$$\frac{1}{2s}d(x,Tx) < d(x,y) \implies \lambda + F(d(Tx,Ty)) \leq F(d(x,y))$$

where a mapping $F : \mathbb{R}_+ \to \mathbb{R}$ satisfies the following conditions.

F_1.) F is strictly increasing, that is $\forall\ \alpha, \beta \in \mathbb{R}_+$, such that $\alpha < \beta$, then $F(\alpha) < F(\beta)$;

F_2.) For each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, the $\lim_{n\to\infty} \alpha_n = 0$ if and only if the $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Then they proved that T has a unique fixed point $x \in X$; that is $T(x) = x$.

Budhia et al. (2016) introduced two new concepts of an α-type almost F-contraction and α-type F-Suzuki contraction and proved some fixed point theorems for such mappings in a complete metric space. The authors gave some examples and the application to a nonlinear fractional differential equation which are given to illustrate the usefulness of the new theory. Afterward, Budhia et al. (2016) gave the definition of α-type almost F-contraction as a mapping $T : X \to X$ and $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ be a symmetric function. We say the mapping T is said to be α-type almost F-contraction if there exist

$F \in \mathscr{F}$ and $\lambda > 0$ and $L \geq 0$ then $x, y \in X$ such that;

$$d(Tx, Ty) > 0 \implies \lambda + \alpha(x,y)F(d(Tx,Ty)) \leq F(d(x,y) + Ld(y,Tx))$$
$$d(Tx, Ty) > 0 \implies \lambda + \alpha(x,y)F(d(Tx,Ty)) \leq F(d(x,y) + Ld(x,Ty))$$

also the definition of the F-Suzuki is given as a mapping $T : X \to X$ and $\alpha : X \times X \to \{-\infty\} \cup (0,\infty)$ be a symmetric function. The mapping T is said to be α-type almost F-Suzuki contraction if there exist $F \in \mathscr{F}$ and $\lambda > 0$ then $x, y \in X$ with $Tx \neq Ty$ such that;

$$\frac{1}{2}d(x,Tx) < d(x,y) \implies \lambda + \alpha(x,y)F(d(Tx,Ty)) \leq F(d(x,y)$$

Let $\mathscr{F}$ denote a set of functions that satisfy (F_1), (F_2) and (F_3). The proof of the first and second result is that the map of the α-type almost F-contraction and α-type F-Suzuki contraction respectively have a unique fixed point. In addition, Piri and Kumam (2016) established some new fixed point theorems for generalised F-Suzuki contraction mappings in complete b-metric spaces and extended the fixed point results of Wardowski, Wardowski and Dung, Dung and Hang and Piri and Kumam (Wardowski, 2012; Wardowski and Van Dung, 2014; Van Dung and Le Hang, 2015; Kumam and Piri, 2014) respectively by introducing a generalised F-Suzuki contraction in b-metric spaces. The authors gave the contraction conditions for self-mapping T on a metric space (X,d) which contains at most five values $d(x,y), d(x,Tx), d(y,Ty), d(y,Tx)$. Also adding recently four new values $d(T^2x,x), d(T^2x,Tx), d(T^2x,y), d(T^2x,Ty)$ which is used in defining the new generalised quasi-contraction condition (Kumam et al., 2015). The authors defined the F-weak contraction with respect to a complete metric space. Let mapping $T : X \to X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ then the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.

Arooj (2017) introduced the notion F-contraction in b-metric space. He also stated that F- contraction plays an important role in the extension and generalisation of Banach Contraction principle. In this way, he also proved some fixed point results in complete b-metric space using the F-contraction mapping. Then he further extended the fixed point results for b- metric space using F-Suzuki contractions that is the generalisation of the work of Wardowski's result in F-contraction.

Chandok et al. (2018) compared the works written by many authors, and also introduced the general concept of a generalised F-Suzuki type contraction mappings in b-metric spaces and to established some fixed point theorems with respect to the b-metric spaces. The main result unifies, complements, and generalises previous researches in the field. They gave the following definition, a mapping $T : X \to X$ is called a generalised F-Suzuki type contraction if there exist $F \in \mathscr{F}$ such that for all $x, y \in X$ with $s \geq 1, x \neq y$ then;

$$\frac{1}{2s}d(x,Tx) < d(x,y) \implies F(s^{\varepsilon}d(Tx,Ty)) \leq F(M_T(x,y)) - \phi(M_T(x,y))$$

where $\phi \in \varphi$ and $\varepsilon > 1$ is a constant and

$$\begin{aligned} M_T(x,y) = \max\Bigg\{ & d(x,y), d(T^2x,y), \frac{d(Tx,y)+d(x,Ty)}{2s}, \\ & \frac{d(T^2x,x)+d(T^2x,Ty)}{2s}, d(T^2x,Ty)+d(T^2x,Tx), \\ & d(T^2x,Ty)+d(Tx,x), d(Tx,y)+d(y,Ty)\Bigg\}. \end{aligned}$$

Clearly, if $\varepsilon = 5$, then the definition above is reduced to definition given by Kumam and Piri (2014) . This implies that Chandok definition of F-Suzuki is a generalisation of Kumam-Piri's definition and many others in the literature.

Lashkaripour et al. (2019) introduced a new extension of F-Suzuki contraction mappings called generalised F-Suzuki contraction. The authors established the existence and uniqueness theorem for the new type of F-contraction under a weaker condition and also provide an affirmative answer to the open question raised by Rhoades (1977) regarding the existence of a contractive definition which is good enough to generate a fixed point but does not force the mapping to be continuous at the fixed point. The authors also provided some examples to show that their main theorem is a generalisation of the previous result. At the end, they gave an application to the boundary value problem of a non-linear fractional differential equation for their results. In addition, Mehravaran et al. (2019) proposed the concept of a dislocated Sb-metric space, followed by the new concepts of generalised F-contraction and generalised F-Suzuki-contraction in the context of dislocated Sb-metric spaces. In complete dislocated S_b-metric spaces, the authors proved several fixed point theorems involving this contractions. They also provided several instances to demonstrate the effectiveness and applicability of the method.

Gili'c et al. (2020) gave some proofs of recent main results in the context of generalised F-Suzuki contraction mappings in b-metric spaces. They used a new approach to prove that a Picard sequence is b-Cauchy. The results then generalises and improved many known results in the existing space. The authors further introduced some new contractive condition which are provided to illustrate the usability of the obtained theoretical result. Saleem et al. (2020) also introduced the generalised Suzuki type F-contraction fuzzy mappings and also proved the existence of fixed fuzzy points for such mappings with respect to complete metric space. Saleem et al. (2020) gave the generalised Suzuki type F-contraction ordered fuzzy mapping. The authors gave examples to show the validity of the results which is followed by different remarks about the comparison of the obtained results with the existing result in the paper and some applications of the results to the domain of the words. Futhermore, Sisodiya and Bhargav (2020) proved a new fixed point theorem for generalised F-Suzuki-rational-type multivalued contraction with respect to the complete metric space. The authors gave the result in this paper as the extension of the Banach contraction principle, Suzuki contraction theorem (Suzuki, 2008) and Wardowski fixed point theorem (Wardowski, 2012) and Piri and Kumam (Kumam and Piri, 2014). They also gave the definition of F- contraction as in (Wardowski, 2012), and F-Suzuki as in (Kumam and Piri, 2014).

Beg et al. (2021) introduced the concept of generalised orthogonal F-Suzuki contraction mapping and proved some fixed point theorems on orthogonal b-metric spaces (Czerwik, 1993). Their results generalise and extend some well-known existing results. As the product of the result, the authors show the existence of a unique solution of the first-order ordinary differential equation as an application of their result. Furthermore, Mani et al. (2021) introduced the concept of generalised orthogonal F-contraction and orthogonal F-Suzuki contraction mappings and proved few fixed point theorems for a self-mapping in an orthogonal metric space. The proved result generalises and extend some of the known results in the field. The authors then gave an example to support their result which are presented in the paper.

Furthermore, as applications of the main result, they applied their main results to show the existence of a unique solution of the first-order ordinary differential equation. In addition, Farajzadeh et al. (2021) gave the notions of the generalised F-contraction, simulation function, and admissible function are introduced to introduce the most recent generalisation of the Banach contraction. They examined the presence and uniqueness of fixed points for the newly designed contraction self-mapping on the entire metric spaces.Then the result of the paper can be seen as a refinement of the primary result presented in the references.

Chapter 3

F- Suzuki Contraction Mapping On Metric Spaces

In this chapter, our aim is to prove the concepts regarding the F-contraction mappings and F-Suzuki contraction mappings in a complete metric space which were considered and defined by Wardowski (2012). We will also like to see the results of fixed point for the F-Suzuki contractions that is the generalisation of Wardowski's worked result in F-contraction.

3.1 F-contraction

Wardowski (2012) introduced a new concept of contraction and prove a fixed point theorem which generalises Banach contraction principle in the known result from the literature. He also gave some examples which showed the validity of the results. He introduced a new type of contraction which is the F-contraction and proved a new fixed point theorem concerning F-contraction. We Recall the definition of F-contraction as in Definition 2.1, where the symbol $\mathscr{F}$ is used for a set containing all those function that satisfy (F_1),(F_2) and (F_3).

Remark 3.1. We can deduce that from (F_1) in the definition of F-contraction and (2.0.1), that any F-contraction must be continuous.

Wardowski also stated the modified version of the Banach contraction principle in Theorem 2.2.

Theorem 3.2 (F-contraction mapping)**.** *(Kumam and Piri, 2014) Let (X,d) be a complete metric space and the mapping $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for $x \in X$ the sequence $\{T^m x\}_{m\in\mathbb{N}}$ converges to x^*.*

Proof. In order to demonstrate that T has a unique fixed point, consider the case where arbitrary $x^* \in X$ is the fixed point of T. We then take a sequence $\{x_m\}_{m\in\mathbb{N}} \subset X$ as

$$x_{m+1} = Tx_m, \quad m = 0,1,2,3...$$

We consider that,

$$\beta_m = d(x_{m+1}, x_m) \quad m = 0,1,2,3...$$

If there exist $m_0 \in \mathbb{N}$ for which $x_{m_0+1} = x_{m_0}$, then

$$Tx_{m_0} = x_{m_0}$$

we have the theorem to be proved.

Suppose that $x_{m+1} \neq x_m$ then for every $m \in \mathbb{N}$ then,

$$\beta_m > 0 \qquad \forall m \in \mathbb{N}$$

In terms of (2.0.1), for $\lambda > 0$ we observe that

$$\begin{aligned}
\lambda + F(d(Tx_{m+1}, Tx_m)) &\leq F(d(x_{m+1}, x_m)) \\
F(d(Tx_{m+1}, Tx_m)) &\leq F(d(x_{m+1}, x_m)) - \lambda \\
&= F(d(Tx_m, Tx_{m-1})) - \lambda \\
&\leq F(d(x_m, x_{m-1})) - 2\lambda \\
&= F(d(Tx_{m-1}, Tx_{m-2})) - 2\lambda \\
&\leq F(d(x_{m-1}, x_{m-2})) - 3\lambda \\
&\vdots \\
&\leq F(d(x_1, x_0)) - m\lambda \qquad (3.1.1)
\end{aligned}$$

Then we have from (3.1.1),

$$F(\beta_m) \leq F(\beta_{m-1}) - \lambda \leq F(\beta_{m-2}) - 2\lambda \leq .... \leq F(\beta_0) - m\lambda \qquad (3.1.2)$$

We then take the $\lim_{m\to\infty}$, and have

$$\lim_{m\to\infty} F(\beta_m) = -\infty$$

From condition (F_2) in Definition 2.1 we have,

$$\lim_{m\to\infty} \beta_m = 0 \tag{3.1.3}$$

where the sequence $\{\beta_m\}_{m\in\mathbb{N}}$ is a positive number.

From condition (F_3) in Definition 2.1, there exists $c \in (0,1)$ such that

$$\lim_{m\to\infty} \beta_m^c F(\beta_m) = 0 \tag{3.1.4}$$

We multiply (3.1.2) through by β_m^c, so we have

$$\beta_m^c(F(\beta_m)) \leq \beta_m^c(F(\beta_{m-1}) - \lambda) \leq \beta_m^c(F(\beta_{m-2}) - 2\lambda) \leq .... \leq \beta_m^c(F(\beta_0) - m\lambda)$$

and then we get,

$$\beta_m^c(F(\beta_m)) \leq \beta_m^c(F(\beta_0) - m\lambda) \tag{3.1.5}$$

Subtracting $\beta_m^c(F(\beta_0)$ from (3.1.5), we derive

$$\beta_m^c(F(\beta_m) - F(\beta_0)) \leq \beta_m^c(F(\beta_0) - m\lambda) - \beta_m^c F(\beta_0) = -m\beta_m^c\lambda \tag{3.1.6}$$

Let $m \to \infty$ in (3.1.6), and using (3.1.3) and (3.1.4), we obtain

$$\lim_{m\to\infty} m\beta_m^c = 0 \tag{3.1.7}$$

Now from (3.1.7), we observe that there exists $m_1 \in \mathbb{N}$ such that,

$$m\beta_m^c \leq 1 \qquad \forall m \geq m_1 \tag{3.1.8}$$

and consequently we have from (3.1.8) that,

$$\beta_m \leq \frac{1}{\sqrt[c]{m}} = \frac{1}{m^{\frac{1}{c}}}, \quad \forall m \geq m_1 \tag{3.1.9}$$

We now want to prove that the sequence $\{x_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence. Let us consider $a, b \in \mathbb{N}$ such that $a > b \geq m_1$. From the definition of metric, and from (3.1.9), by triangle inequality, we have

$$d(x_a, x_b) \leq d(x_a, x_{a-1}) + d(x_{a-1}, x_{a-2}) + \ldots + d(x_{b+1}, x_b)$$

Recall that,

$$\beta_m = d(x_{m+1}, x_m) \quad m = 0, 1, 2, 3\ldots$$

$$d(x_a, x_b) \leq \beta_{a-1} + \beta_{a-2} + \ldots + \beta_b < \sum_{m=1}^{\infty} \beta_m \leq \sum_{m=1}^{\infty} \frac{1}{\sqrt[c]{m}} = \sum_{m=1}^{\infty} \frac{1}{m^{\frac{1}{c}}} \tag{3.1.10}$$

From the convergence of the series $\sum_{m=1}^{\infty} \frac{1}{m^{\frac{1}{c}}}$ and from (3.1.10), it is obvious that $\{x_m\}_{m\in\mathbb{N}}$ is a Cauchy sequence.

Since X is complete, there exists $x_0 \in X$ such that $\lim_{m\to\infty} x_m = x_0$. Now from continuity of T, we have that

$$d(Tx_0, x_0) = \lim_{m\to\infty} d(Tx_m, x_m) = \lim_{m\to\infty} d(x_{m+1}, x_m) = 0$$

This implies that,

$$\begin{aligned} d(Tx_0, x_0) &= 0 \\ Tx_0 - x_0 &= 0, \\ Tx_0 &= x_0 \end{aligned}$$

Therefore, x_0 is a fixed point of T.

To prove the Uniqueness, we assume that x_1, x_2 be two fixed points in the metric space X, where $Tx_1 = x_1 \neq Tx_2 = x_2$. By definition of F-contraction we then have

$$\begin{aligned} \lambda + F(d(Tx_1, Tx_2)) &\leq F(d(x_1, x_2)) \\ \lambda &\leq F(d(x_1, x_2)) - F(d(Tx_1, Tx_2)) = 0 \end{aligned} \tag{3.1.11}$$

This is a contradiction, since $\lambda > 0$, therefore our assumption is wrong.

Hence T has a unique fixed point, which completes the proof. □

Lemma 3.3. *If $\{\gamma_c\}_{c\in\mathbb{N}}$ is a bounded sequence of real numbers with the same limit d for all convergent sub-sequences, then it is convergent and* $\lim_{c\to\infty} \gamma_c = d$

Lemma 3.4. *Let an increasing mapping be given as $F : \mathbb{R}_+ \to \mathbb{R}$ and the sequence $\{\gamma_c\}_{c\in\mathbb{N}}$ of positive real number. Then the following condition holds:*

a.) $\lim_{c\to\infty} F(\gamma_c) = -\infty$, *then* $\lim_{c\to\infty}\gamma_c = 0$;

b.) if $infF = -\infty$, *and* $\lim_{c\to\infty}\gamma_c = 0$, *then* $\lim_{c\to\infty} F(\gamma_c) = -\infty$

Proof. We first need to observe that $\{\gamma_c\}_{c\in\mathbb{N}}$ is bounded.

Actually, if a sequence is not bounded above, then there exists a subsequence $\{\gamma_{c(k)}\}_{k\in\mathbb{N}}$ such that $\lim_{c\to\infty} F(\gamma_{c(k)}) = \infty$,

There is a $k(\delta) \in \mathbb{N}$, such that for every $\delta > 0$, we have $\gamma_{c(k)} \geq \delta$ for all $k \geq k(\delta)$ Then by (F_1) in Definition 2.1

$$F(\delta) \leq F(\gamma_{c(k)})$$

which implies

$$F(\delta) \leq \lim_{k\to\infty} F(\gamma_{c(k)}) = -\infty$$

This is a contradiction. Hence $\{\gamma_c\}_{c\in\mathbb{N}}$ is bounded, and therefore has a convergent sub-sequence.

Let the $\lim_{l\to\infty}\gamma_{c(l)} = \beta, for \beta > 0$ and $\{\gamma_{c(l)}\}_{l\in\mathbb{N}}$ be a sub-sequence. For we choose $\delta > 0$ and $\delta > \beta$. Then there exists $l \in \mathbb{N}$ which depends on δ such that for all $l \geq l(\delta)$ we have

$$\gamma_{c(l)} \in (\beta - \delta, \beta + \delta)$$

As a result, as F increases, hence

$$F(\beta - \delta) \leq \lim_{l\to\infty} F(\gamma_{c(l)}) = -\infty$$

This is a contradiction that $F(\beta-\delta)\in\mathbb{R}$.

As $\lim_{l\to\infty}\gamma_{c(l)}=0$, then as a result of Lemma 3.3

$$\lim_{c\to\infty}\gamma_c=0$$

Next we prove the second condition

Suppose that $infF=-\infty$ and $\lim_{c\to\infty}\gamma_c=0$. Choose $\delta>0$ and $\beta>0$ such that $F(\beta)<-\delta$.

Then there is $c_\beta\in\mathbb{N}$ such that $\gamma_c<\beta$ for all $c\geq c_\beta$

which implies that

$$F(\gamma_c)<F(\beta) \qquad \forall \quad c\geq c_\beta$$

Hence,

$$\lim_{c\to\infty}F(\gamma_c)=-\infty$$

$\square$

After proving the above Lemma, Secelean (2013) showed that the condition (F_2) in the above definition can be equivalently written as the following condition;

(F_2') $infF=-\infty$

or by

(F_2'') there exist a sequence $\{\gamma_c\}_{c\in\mathbb{N}}$ of $\mathbb{R}_+$, such that $\lim_{c\to\infty}F(\gamma_c)=-\infty$

Instead of condition (F_3) in Definition 2.1, we used (F_3') to define F-contraction as follows:

(F_3') The symbol F is continuous on $(0,\infty)$

$\mathscr{F}$ is generally used to denote a set of functions that satisfy the conditions (F_1), (F_2') and (F_3')

Theorem 3.5. *Let $T : X \to X$ be a self mapping and (X,d) be a complete metric space. Assume $F \in \mathscr{F}$ and there exist $\lambda > 0$, such that $\forall a,b \in X$,*

$$d(Ta,Tb) > 0 \implies \quad \lambda + F(d(Ta,Tb)) \leq F(d(a,b))$$

holds. The sequence $\{T^c a_0\}_{c\in\mathbb{N}}$ then converges to a unique fixed point $a^* \in X$ of T for every $a_0 \in X$

Proof. We choose a_0 and define a sequence $\{a_c\}_{c=1}^{\infty}$ by

$$a_1 = Ta_0, \quad a_2 = Ta_1 = T^2 a_0, \quad ..., \quad a_{c+1} = Ta_c = T^{c+1} a_0 \quad \forall c \in \mathbb{N}$$

For some $c \in \mathbb{N}$, if $d(a_c, Ta_c) = 0$, then there is nothing to prove.
Suppose that for all, $c \in \mathbb{N}$ we have

$$0 < d(a_c, Ta_c) = d(Ta_{c-1}, Ta_c) \tag{3.1.12}$$

For some $c \in \mathbb{N}$ we get,

$$\begin{aligned} \lambda + F(d(Ta_{c-1}, Ta_c)) &\leq F(d(a_{c-1}, a_c)) \\ F(d(Ta_{c-1}, Ta_c)) &\leq F(d(a_{c-1}, a_c)) - \lambda \end{aligned}$$

Progressing with the same procedure of the previous proof, we have

$$
\begin{aligned}
\lambda + F(d(Ta_{c-1}, Ta_c)) &\leq F(d(a_{c-1}, a_c)) \\
F(d(Ta_{c-1}, Ta_c)) &\leq F(d(a_{c-1}, a_c)) - \lambda \\
&= F(d(Ta_{c-2}, Ta_{c-1})) - \lambda \\
&\leq F(d(a_{c-2}, a_{c-1})) - 2\lambda \\
&= F(d(Ta_{c-3}, Ta_{c-2})) - 2\lambda \\
&\leq F(d(a_{c-3}, a_{c-2})) - 3\lambda \\
&\vdots \\
&\leq F(d(a_1, a_0)) - c\lambda \qquad (3.1.13)
\end{aligned}
$$

Taking the $\lim_{c\to\infty}$ on both sides, we get

$$\lim_{c\to\infty} F(d(a_{c-1}, a_c)) = -\infty \qquad (3.1.14)$$

Using (F_2) from Definition 2.1 we obtain,

$$\lim_{c\to\infty} d(a_c, Ta_c) = 0 \qquad (3.1.15)$$

We now want to show that $\{a_c\}_{c=1}^{\infty}$ is a Cauchy sequence.

Using the contradictory argument, we assume that the sequence $\{m(c)\}_{c\in\mathbb{N}}, \{n(c)\}_{c\in\mathbb{N}}$ of natural numbers, and there exists $\delta > 0$ such that $\forall c \in \mathbb{N}$ we have

$$m(c) > n(c) > c, \quad d(a_{m(c)}, a_{n(c)}) \geq \delta, \quad d(a_{m(c)-1}, a_{n(c)}) < \delta \qquad (3.1.16)$$

then, we obtain

$$\begin{aligned}\delta \leq d(a_{m(c)}, a_{n(c)}) &\leq d(a_{m(c)}, a_{m(c)-1}) + d(a_{m(c)-1}, a_{n(c)}) \\ &\leq d(a_{m(c)}, a_{m(c)-1}) + \delta \\ &= d(a_{m(c)-1}, Ta_{m(c)-1}) + \delta\end{aligned}$$

which implies that

$$\delta \leq d(a_{m(c)}, a_{n(c)}) < d(a_{m(c)-1}, Ta_{m(c)-1}) + \delta \tag{3.1.17}$$

We have the $\lim_{c\to\infty}$ and using (3.1.15) in the expression(3.1.17), we then obtain

$$\lim_{c\to\infty} d(a_{m(c)}, a_{n(c)}) = \delta \tag{3.1.18}$$

On the other hand, as

$$\lim_{c\to\infty} d(a_c, Ta_c) = 0 \tag{3.1.19}$$

There exists $N \in \mathbb{N}$, such that $\forall c \in N$

$$d(a_{m(c)}, Ta_{m(c)}) < \frac{\delta}{4} \quad and \quad d(a_{n(c)}, Ta_{n(c)}) < \frac{\delta}{4} \tag{3.1.20}$$

We now then claim that for all $c \in N$ we obtain

$$d(Ta_{m(c)}, Ta_{n(c)}) = d(a_{m(c)+1}, a_{n(c)+1}) > 0 \tag{3.1.21}$$

Using the contradictory argument, there exists $d \geq \mathbb{N}$ such that

$$d(a_{m(c)+1}, a_{n(c)+1}) = 0 \tag{3.1.22}$$

From (3.1.16), (3.1.20) and (3.1.22) we get

$$\begin{aligned}
\delta &\leq d(a_{m(d)}, a_{n(d)}) \leq d(a_{m(d)}, a_{m(d)+1}) + d(a_{m(d)+1}, a_{n(d)}) \\
&\leq d(a_{m(d)}, a_{m(d)+1}) + d(a_{m(d)+1}, a_{n(d)+1}) + d(a_{n(d)+1}, a_{n(d)}) \\
&= d(a_{m(d)}, Ta_{m(d)}) + d(a_{m(d)+1}, a_{n(d)+1}) + d(Ta_{n(d)}, a_{n(d)}) \\
&= d(a_{m(d)}, Ta_{m(d)}) + d(a_{m(d)+1}, a_{n(d)+1}) + d(a_{n(d)}, Ta_{n(d)}) \quad (symmetry) \\
&< \frac{\delta}{4} + 0 + \frac{\delta}{4} = \frac{\delta}{2}
\end{aligned}$$

This is a contradiction, so there exists no such d. From (3.1.21) and for $\lambda > 0$, the supposition of the theorem that

$$\lambda + d(Ta_{m(c)}, Ta_{n(c)}) \leq d(a_{m(c)}, a_{n(c)}), \quad \forall c \geq \mathbb{N} \tag{3.1.23}$$

Now from (F3$'$), (3.1.18) and (3.1.23), we have

$$\lambda + F(\delta) \leq F(\delta) \tag{3.1.24}$$

This is a contradiction. Therefore our assumption is wrong, hence $\{a_c\}_{c=1}^{\infty}$ is a Cauchy sequence.

Let (X, d) be a complete metric space. Then there exists a sequence $\{a_c\}_{c=1}^{\infty}$ which converges to a point $a \in X$. Since T is continuous we obtain

$$\begin{aligned} d(Ta,a) &= \lim_{c\to\infty} d(Ta_c,a_c) \\ &= \lim_{c\to\infty} d(a_{c+1},a_c) \\ &= d(a',a') \\ &= 0 \end{aligned}$$

Which shows that T has a fixed point.

We'll now show T has exactly one fixed point. Assume that a_1, a_2 are two different fixed point of T, with $a_1 \neq a_2$, i.e.

$$Ta_1 = a_1 \neq Ta_2 = a_2$$

Then,

$$d(Ta_1,Ta_2) = d(a_1,a_2) > 0$$

this gives us,

$$\begin{aligned} F(d(a_1,a_2)) &= F(d(Ta_1,Ta_2)) \\ &< \lambda + F(d(Ta_1,Ta_2)) \\ &\leq F(d(a_1,a_2)) \end{aligned}$$

This is a contradiction. As a result, our assumption of two fixed point is incorrect. Hence T

has unique fixed point. $\square$

3.2 F- Suzuki Contraction

We want to discuss the fixed point theorem of F-Suzuki contraction. Recall the definition of F-Suzuki as in contraction in Definition 2.4. We then state the following theorem;

Theorem 3.6 (F-Suzuki Contraction)**.** *(Kumam and Piri, 2014) Let (X,d) be a complete metric space and $T : X \to X$ be a self mapping. The sequence $\{T^c x\}_{c=1}^{\infty}$ converges to a point x^* since T has a unique fixed point.*

Proof. Assume $x_0 \in X$ and consider the sequence $\{x_m\}_{m=1}^{\infty}$ as

$$x_1 = Tx_0, \quad x_2 = Tx_1 = T^2 x_0, \quad \dots \quad x_{m+1} = Tx_m = T^{m+1} x_0 \quad \forall m \in \mathbb{N}. \tag{3.2.1}$$

There is nothing to prove if there exists $m \in \mathbb{N}$ in which $d(x_m, Tx_m)$. We presume $d(x_m, Tx_m) > 0$, for all $m \in \mathbb{N}$

As a result, we have that $\forall m \in \mathbb{N}$

$$\frac{1}{2} d(x_m, Tx_m) < d(x_m, Tx_m)$$

In the case of any $m \in \mathbb{N}$, we obtain

$$\lambda + F(d(Tx_m, T(Tx_m)) \leq F(d(x_m, Tx_m)) \tag{3.2.2}$$

which implies

$$F(d(x_{m+1}, Tx_{m+1})) \leq F(d(x_m, Tx_m)) - \lambda$$

Repeating the same process, we get

$$\begin{aligned} F(d(x_m, Tx_m)) &\leq F(d(x_{m-1}, Tx_{m-1}) - \lambda \\ &\leq F(d(x_{m-2}, Tx_{m-2}) - 2\lambda \\ &\leq F(d(x_{m-3}, Tx_{m-3}) - 3\lambda \\ &\vdots \\ &\leq F(d(x_0, Tx_0) - m\lambda \end{aligned} \tag{3.2.3}$$

Taking the limit as $m \to \infty$ on both sides, we get

$$\lim_{m\to\infty} F(d(x_m, Tx_m)) = -\infty$$

By using (F_2) in Definition 2.1, we have

$$\lim_{m\to\infty} d(x_m, Tx_m) = 0. \tag{3.2.4}$$

We now need to show that $\{x_m\}_{m=1}^{\infty}$ is a Cauchy sequence.

Using the contradictory argument, we assume that the sequence $\{p(m)\}_{m\in\mathbb{N}}, \{q(m)\}_{m\in\mathbb{N}}$

of natural numbers, and there exists $\delta > 0$ such that $\forall m \in \mathbb{N}$ we have

$$p(m) > q(m) > m, \quad d(x_{p(m)}, x_{q(m)}) \geq \delta, \quad d(x_{p(m)-1}, x_{q(m)}) < \delta \tag{3.2.5}$$

then, we obtain

$$\begin{aligned}
\delta \leq d(x_{p(m)}, x_{q(m)}) &\leq d(x_{p(m)}, x_{p(m)-1}) + d(x_{p(m)-1}, x_{q(m)}) \\
&\leq d(x_{p(m)}, x_{p(m)-1}) + \delta \\
&= d(Tx_{p(m)-1}, x_{p(m)-1}) + \delta \\
&= d(x_{p(m)-1}, Tx_{p(m)-1}) + \delta \quad (symmetry)
\end{aligned}$$

which implies that

$$\delta \leq d(x_{p(m)}, x_{q(m)}) < d(x_{p(m)-1}, Tx_{p(m)-1}) + \delta \tag{3.2.6}$$

We have the $\lim_{m\to\infty}$ and using (3.2.4) in the expression (3.2.6), we then obtain

$$\lim_{m\to\infty} d(x_{p(m)}, x_{q(m)}) = \delta \tag{3.2.7}$$

We take an integer $N \in \mathbb{N}$ from (3.2.4) and (3.2.7), such that

$$\frac{1}{2} d(x_{p(m)}, Tx_{p(m)}) < \frac{1}{2}\delta < d(x_{p(m)}, x_{q(m)}) \quad \forall m \geq N$$

Since T is of F-Suzuki type, we have

$$\lambda + F(d(Tx_{p(m)}, Tx_{q(m)})) \leq F(d(x_{p(m)}, x_{q(m)})) \quad \forall m \in N$$

From (3.2.1), we observe that

$$\lambda + F(d(Tx_{p(m)+1}, Tx_{q(m)+1})) \leq F(d(x_{p(m)}, x_{q(m)})) \quad \forall m \in N \tag{3.2.8}$$

From (F_3'), (3.2.4) and (3.2.7) we get

$$\lambda + F(\delta) \leq F(\delta)$$

This is a contradiction. Which implies that the sequence $\{x_m\}_{m=1}^{\infty}$ is a Cauchy sequence. Since (X,d) is complete, hence the sequence $\{x_m\}_{m=1}^{\infty}$ converges to a point $x^* \in X$. That is,

$$\lim_{m \to \infty} d(x_m, x^*) = 0 \tag{3.2.9}$$

For all $m \in \mathbb{N}$, we claim that

$$\frac{1}{2}d(x_m, Tx_m) < d(x_m, x^*) \quad \textit{and} \quad \frac{1}{2}d(Tx_m, T(Tx_m)) < d(Tx_m, x^*) \tag{3.2.10}$$

Assuming that there exists some $q \in \mathbb{N}$ for which

$$\frac{1}{2}d(x_q, Tx_q) \geq d(x_q, x^*) \quad \textit{and} \quad \frac{1}{2}d(Tx_q, T(Tx_q)) < d(Tx_q, x^*) \tag{3.2.11}$$

Hence,

$$\begin{aligned} 2d(x_q,x^*) \leq d(x_q,Tx_q) &\leq d(x_q,x^*)+d(x^*,Tx_q) \\ d(x_q,x^*)+d(x_q,x^*) &\leq d(x_q,x^*)+d(x^*,Tx_q) \end{aligned}$$

It follows from the last inequalities that

$$d(x_q,x^*) \leq d(x^*,Tx_q) \tag{3.2.12}$$

From (3.2.11) and (3.2.12), we have,

$$d(x_q,x^*) \leq d(x^*,Tx_q) \leq \frac{1}{2}d(Tx_q,T(Tx_q)) \tag{3.2.13}$$

We see that,

$$\frac{1}{2}d(x_q,Tx_q) \leq d(x_q,Tx_q)$$

and T, an F-Suzuki contraction. Then, we obtain

$$\lambda + F(d(Tx_q,T(Tx_q)) \leq F(d(x_q,Tx_q))$$

Furthermore, from (F_1) in Definition 2.1, we have

$$d(Tx_q,T(Tx_q)) < d(x_q,Tx_q) \tag{3.2.14}$$

With (3.2.11), (3.2.13) and (3.2.14), we obtain

$$\begin{aligned}
d(Tx_q, T(Tx_q)) &< d(x_q, Tx_q) \\
&\leq d(x_q, x^*) + d(x^*, Tx_q) \\
&\leq \frac{1}{2} d(Tx_q, T(Tx_q)) + \frac{1}{2} d(Tx_q, T(Tx_q)) \\
&= \frac{1}{2} d(Tx_q, T^2 x_q) + \frac{1}{2} d(Tx_q, T^2 x_q) \\
&= d(Tx_q, T^2 x_q) \qquad (3.2.15)
\end{aligned}$$

This is a contradiction, Therefore (3.2.10) is satisfied.

Hence we can say that, either

$$\lambda + F(d(Tx_m, Tx^*)) \leq F(d(x_m, x^*))$$

or

$$\lambda + F(d(T(Tx_m), Tx^*)) \leq F(d(Tx_m, x^*)) = F(d(x_{m+1}, x^*))$$

is satisfied for all $m \in \mathbb{N}$.

Actually, from (3.2.9), (F_2') and Lemma 3.4, we obtain

$$\lim_{m \to \infty} F(d(Tx_m, Tx^*)) = -\infty$$

also from (F_2') and Lemma 3.4, we have

$$\lim_{m\to\infty} d(Tx_m, Tx^*) = 0.$$

Which implies that,

$$\begin{aligned} d(x^*, Tx^*) &= \lim_{m\to\infty} d(x_{m+2}, Tx^*) \\ &= \lim_{m\to\infty} d(T^2 x_m, Tx^*) \\ &= 0 \end{aligned}$$

Therefore T has a fixed point x^*

We then want to prove the uniqueness of T

Consider $x^*, y^* \in X$ to be two different fixed points of T which means that $x^* \neq y^*$, then we have

$$Tx^* = x^* \neq y^* = Ty^*$$

It is clear that

$$d(x^*, y^*) > 0$$

Then,

$$\frac{1}{2} d(x^*, Tx^*) = 0$$

so,

$$0 = \frac{1}{2}d(x^*, Tx^*) < d(x^*, y^*)$$

With the definition of the F- Suzuki contraction in Definition 2.4, we obtain

$$\begin{aligned} F(d(x^*, y^*)) &= F(d(Tx^*, Ty^*)) \\ &< \lambda + F(d(Tx^*, Ty^*)) \\ &\leq F(d(x^*, y^*)) \end{aligned}$$

This is a contradiction. As a result, our assumption of two fixed point is incorrect. Hence T has unique fixed point. □

3.3 Applications and Examples

In this section, we presented some applications on the existence of a solution for a non-linear integral equation and differential equation.

3.3.1 Application to Integral Equation

In this section, we present an application to the following integral equation of the form;

$$x(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds \tag{3.3.1}$$

where $M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b], \mathbb{R})$ be the space of all continuous real-valued functions defined on $[a,b]$. We defined $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|$. It is well-known that$(X,d)$ is a complete metric space.

Theorem 3.7. *Let $X = C([a,b], \mathbb{R})$ and $T : X \to X$ be an operator defined by,*

$$Tx(t) = g(t) + \int_a^b M(t,s)K(t,x(s))ds \quad \forall t,s \in [a,b]$$

where g, M and K are defined above. Suppose that the following conditions hold:

1. *There exists a continuous function $\phi : X \to X \to \mathbb{R}^+$ such that*

$$|K(s,x(s)) - K(s,y(s))| \leq \phi(x,y)|x(s) - y(s)| \quad \forall s \in [a,b] \quad \textit{and} \quad x,y \in X.$$

2. *There exists $\lambda > 0$, such that for all $x,y \in X$*

$$\int_a^b M(t,s)\phi(x,y) \leq e^{-\lambda}.$$

Then the integral equation (3.3.1) has a solution.

Proof. Now, observe

$$
\begin{aligned}
|Tx(s)-Ty(s)| &= \left| g(t)+\int_a^b M(t,s)K(t,x(s))ds - g(t) - \int_a^b M(t,s)K(t,y(s))ds \right| \\
&= \left| \int_a^b M(t,s)K(t,x(s))ds - \int_a^b M(t,s)K(t,y(s))ds \right| \\
&= \left| \int_a^b M(t,s)\left[K(t,x(s)) - K(t,y(s))\right] ds \right| \\
&\leq \int_a^b \left| M(t,s)\left[K(t,x(s)) - K(t,y(s))\right] \right| ds \\
&\leq \int_a^b M(t,s)\phi(x,y)\left|x(s)-y(s)\right| ds \\
&\leq \sup_{s\in[a,b]} |x(s)-y(s)| \int_a^b M(t,s)\phi(x,y)ds \\
&\leq d(x,y)e^{-\lambda}
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
d(Tx,Ty) &\leq e^{-\lambda}d(x,y) \\
\implies e^{\lambda}d(Tx,Ty) &\leq d(x,y).
\end{aligned}
$$

Now, suppose that $F(\beta) = ln\beta$, we have that

$$\lambda + In(d(Tx,Ty)) \leq ln(d(x,y))$$

Clearly, all the conditions in Theorem 3.7 are satisfied and so T has a fixed point. Hence, the integral equation (3.3.1) has a solution.

$\square$

3.3.2 Application to Second Order Differential Equation

In this section, we present an application on the existence of solution for the following second order differential equation of the form;

$$x''(t) = -f(t, x(t)) \quad \forall t \in X, \quad x'(0) = x(1) = 0 \tag{3.3.2}$$

where $I = [0,1]$, $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider the space $C(I)$ of continuous function defined on I. It is well-known that $C(I)$ with the metric

$$d(x,y) = \sup_{t \in [0,1]} |x(t) - y(t)|$$

is a complete metric metric space. In addition, it has been established that (3.3.2) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds \tag{3.3.3}$$

for $t \in I$, where G is the Green function defined by

$$G(t,s) = \begin{cases} (1-t)s & if \quad 0 \leq s \leq t \leq 1 \\ (1-s)t & if \quad 0 \leq t \leq s \leq 1. \end{cases}$$

If $x \in C^2(I)$, then $x \in C(I)$ is also solution to (3.3.2) if and only if it is a solution of (3.3.3).

Theorem 3.8. *Let $X = C(I)$ and $T : X \to X$ be the operator given by,*

$$T(x) = \int_0^1 G(t,s) f(s,x(s))ds$$

for all $x \in X$ and $t \in [0,1]$. Furthermore, suppose the following condition hold:

1. *There exists function $\phi, \lambda : I \to [0,\infty)$,such that for all $u, v \in X$, we have*

$$|f(s,u) - f(s,v)| \leq 8\phi(s)|u - v|$$

2. $\sup\limits_{s \in I} \phi(s) = K_1 \leq e^{-\lambda} \quad \forall \lambda > 0$

Then, the second order differential equation (3.3.2) has a solution.

Proof. We have

$$\begin{aligned} |Tx(t) - Ty(t)| &\leq \int_0^1 G(t,s)\,|f(s,x(s)) - f(s,y(s))|\,ds \\ &\leq \int_0^1 G(t,s) 8\phi(s)|x(s) - y(s)|ds \\ &\leq 8K_1\|x - y\| \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \end{aligned} \tag{3.3.4}$$

We obtain that $\int\limits_0^1 G(t,s)ds = \frac{t(1-t)}{2}$ as such, we obtain

$$\sup_{t\in[0,1]} \int\limits_0^1 G(t,s)ds = \frac{1}{8} \tag{3.3.5}$$

Applying (3.3.5) in (3.3.4), we obtain

$$d(Tx,Ty) \leq e^{-\lambda} d(x,y)$$

Then, we have that $e^{\lambda} d(Tx,Ty) \leq d(x,y)$.

Now, suppose that $F(\beta) = ln\beta$, we have

$$\lambda + In(d(Tx,Ty)) \leq ln(d(x,y))$$

Clearly, all the conditions in Theorem 3.8 are satisfied, and so T has a fixed point. Hence, the second order differential equation (3.3.2) has a solution. □

3.3.3 Examples

In this section, we present some examples of F-contraction to show that it generalises some contractive mapping in the literature.

Example 3.9. Let $F : \mathbb{R}_+ \to \mathbb{R}$ be given as $F(\beta) = ln\beta$. It is obvious that F satisfies the following condition $(F_1), (F_2)$ and (F_3) for any $c \in (0,1)$. For each mapping $T : X \to X$

satisfying (2.0.1) is an F-contraction. From the definition of F-contraction such that,

$$\begin{aligned} F(d(Tx,Ty)) &= ln(d(Tx,Ty)) \\ F(d(x,y)) &= ln(d(x,y)) \end{aligned}$$

we obtain,

$$\begin{aligned} \lambda + ln(d(Tx,Ty)) &\leq ln(d(x,y)) \\ e^{\lambda}.d(Tx,Ty) &\leq d(x,y) \\ d(Tx,Ty) &\leq e^{-\lambda}d(x,y) \quad \forall x,y \in X \end{aligned} \tag{3.3.6}$$

It is obvious that for $x,y \in X$ such that $Tx = Ty$ the inequality (3.3.6) also is satisfied. This implies that T is a Banach contraction.

Example 3.10. Given $F(\beta) = ln\beta + \beta$, $\beta > 0$. It is obvious that F satisfies the following condition $(F_1),(F_2)$ and (F_3) for any $c \in (\frac{1}{2},1)$. In this case, each F-contraction T satisfies

$$\begin{aligned} \lambda + ln(d(Tx,Ty)) + d(Tx,Ty) &\leq ln(d(x,y)) + d(x,y) \\ e^{\lambda}.e^{d(Tx,Ty)}d(Tx,Ty) &\leq d(x,y).e^{d(x,y)} \\ \frac{d(Tx,Ty)e^{d(Tx,Ty)}}{d(x,y).e^{d(x,y)}} &\leq e^{-\lambda} \\ \frac{d(Tx,Ty)}{d(x,y)}.e^{d(Tx,Ty)-d(x,y)} &\leq e^{-\lambda}. \end{aligned} \tag{3.3.7}$$

Chapter 4

Conclusion

This project presents a comprehensive review of the F-contraction and Suzuki F-contraction mappings in the frame work of metric spaces. It was established that the notion of F-contraction is a generalisation of the Banach contraction and some other contractive mappings in the literature. We have systematically presented our study, which we structure into 4 chapters. In Chapter 1, we give a brief background of our study, discuss the aims and objectives, limitation of the study, scope of the study, statement of problem and research methodology. In Chapter 2, we gave a detailed literature review on the notion of F-contraction and Suzuki F-contraction and its generalisation. In Chapter 3, which is our main result of this project, the results obtained in this chapter is a detailed review of the work of Piri and Kumam(Kumam and Piri, 2014). In addition, we gave some examples and applications to establish the applicability of the F-contraction and Suzuki-F-contraction.

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