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On some fixed points properties and convergence theorems for a Banach operator in hyperbolic spaces

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Abstract

In this paper, we prove some fixed points properties and demiclosedness principle for a Banach operator in uniformly convex hyperbolic spaces. We further propose an iterative scheme for approximating a fixed point of a Banach operator and establish some strong and Δ-convergence theorems for such operator in the frame work of uniformly convex hyperbolic spaces. The results obtained in this paper extend and generalize corresponding results on uniformly convex Banach spaces, CAT(0) spaces and many other results in this direction.

Keywords: Banach operator; uniformly convex hyperbolic spaces; strong and Δ-convergence theorem; Modified Picard Normal S-iteration.
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1. Introduction

Let (X, d) be a metric space and C be a nonempty closed and convex subset of X. A point $x \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if

$$Tx = x. \tag{1.1}$$

The set of all fixed points of T is denoted by $F(T)$.

Many real life problems emanating from different discipline such as physcial science, engineering and management science are modelled into mathematical equations. Over the years mathematicians have been able to express these equations in form of Equation (1.1). However, it became very

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tedious to get an analytic solution to Equation (1.1). Thus, researchers in this area opted for an approximate solutions. In view of this, different researchers came up with different iteration process to approximate Equation (1.1) with suitable nonlinear map in different domain. For example, the Picard iterative process

$$x_{n+1} = Tx_n, \ \forall n \in \mathbb{N}, \tag{1.2}$$

is one of the earliest iterative process used to approximate Equation (1.1). When T is a Banach contraction mapping, Picard iteration converges uniquely to a fixed point of T. Recall that a mapping $T : C \to C$ is said to be a contraction mapping if there exists $k \in (0, 1]$ such that

$$d(Tx, Ty) \leq kd(x, y), \ \forall x, y \in C. \tag{1.3}$$

If $k = 1$ in (1.3), then T is called a nonexpansive mapping. If T is a nonexpansive mapping, the Picard iterative process fails to approximate the fixed point for T even when the existence of the fixed point is guaranteed. To overcome this limitation, researchers in this area developed different iterative processes to approximate fixed points of nonexpansive mappings and other mappings more general than nonexpansive mappings. Among many others, some well known iterative processes are; Mann [26], Ishikawa [16], Krasnosel'skii [24], Agarwal et al. [4], and Noor [27]. There are numerous papers dealing with the approximation of fixed points of nonexpansive mappings, asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings and so on, in uniformly convex Banach spaces and CAT(0) spaces (for example, see [1, 2, 4, 20] and the references therein).

Sahu in [30] introduced the following Normal S-iteration process in Banach space. Let C be a convex subset of a normed space X and $T : C \to C$ be any nonlinear mapping. For each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ x_{n+1} = Ty_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$. It was established that the rate of convergence of the Normal S-iteration process is as fast as the Picard iteration process but faster than other fixed point iteration processes that was in existence then.

In time past, researchers in this area have introduced iterative processes whose rate of convergence are faster than that of the Normal S-iteration. For example, in [18], Kadioglu and Yildirim introduced Picard Normal S-iteration process and they established that the rate of convergence of the Picard Normal S-iteration process is faster than the Normal S-iteration process. The Picard Normal S-iteration process is defined as follows: Let C be a convex subset of a normed space X and $T : C \to C$ be any nonlinear mapping. For each $x_1 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1 - \beta_n)x_n + \beta_n Tx_n, \\ y_n = (1 - \alpha_n)z_n + \alpha_n Tz_n, \\ x_{n+1} = Ty_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$.

Remark 1.1. Clearly, if $\alpha_n = \beta_n = 0$, then the iterative process (1.5) reduces to (1.2) and if $\beta_n = 0$, the iterative process (1.5) reduces to (1.4).

In [17], Jianren and Zhongkai introduced the notion of Banach operator pairs. They gave the following defintion.

Definition 1.2. The ordered pair (T, S) of two self maps of a metric space (X, d) is called a Banach operator pair, if the set $F(S)$ is T-invariant, namely $T(F(S)) \subseteq F(S)$.

Note that if T and S commute, i.e $T \circ S = S \circ T$, then the ordered pairs (T, S) and (S, T) are Banach operator pairs. However, the converse is not true in general (see [17], Example 1(iii)). If the self maps T and S of X satisfy,

$$d(STx, Tx) \leq kd(Sx, x), \tag{1.6}$$

for all $x \in X$ and $k \in (0, 1)$ then (T, S) is a Banach operator pair. In particular, when $S = T$, then (1.6), becomes

$$d(T^2x, Tx) \leq kd(Tx, x), \tag{1.7}$$

for all $x \in X$. Such T is called a Banach operator (see, [29]). It is well-known that operators which are not continuous, may have more than one fixed points. In [29], the authors gave the following result.

Theorem 1.3. Let X be a complete metric space, $T : X \to X$ a continuous Banach operator. Then T has a fixed point.

More so, beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. It is known in literature that Banach space have been studied extensively. This is because of the fact that Banach spaces always have convex structures. However, metric spaces do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The notion of convex metric spaces was first introduced by Takahashi [33] who studied the fixed point theory for nonexpansive mappings in the settings of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced to hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [13, 22, 28]). Although the class of hyperbolic spaces defined by Kohlenbach [22] is slightly restrictive than the class of hyperbolic spaces introduced in [13], it is however, more general than the class of hyperbolic spaces introduced in [28]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [22]. Some other examples of this class of hyperbolic spaces includes Hadamard manifold, Hilbert ball with the hyperbolic metric, Cartesian products of Hilbert balls and $\mathbb{R}$-trees. The reader should please see [6, 3, 13, 14, 11, 10, 19, 22, 28] for more discussion and examples of hyperbolic spaces.

Motivated by all these facts, we prove some fixed points properties and demiclosedness principle for a Banach operator in uniformly convex hyperbolic spaces. We further propose an iterative scheme called the Modified Picard Normal S-iteration for approximating a fixed point of Banach operator and establish some strong and Δ-convergence theorems for such operator in the frame work of uniformly convex hyperbolic spaces. The results obtained in this paper extend and generalize corresponding results on uniformly convex Banach spaces, CAT(0) spaces and many other results in this direction.

2. Preliminaries

Throughout this paper, we carry out all our study in the frame work of hyperbolic spaces introduced by Kohlenbach in [22].

Definition 2.1. A hyperbolic space (X, d, W) is a metric space (X, d) together with a mapping $W : X^2 \times [0, 1] \to X$ satisfying

1. $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$;
2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$;
3. $W(x, y, \alpha) = W(y, x, 1 - \alpha)$;
4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$;
   for all $w, x, y, z \in X$ and $\alpha, \beta \in [0, 1]$.

Example 2.2. [32] Let X be a real Banach space which is equipped with norm $||\cdot||$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = ||x - y||$$

as a metric on X. Then, we have that (X, d, W) is a hyperbolic space with mapping $W : X^2 \times [0, 1] \to X$ defined by $W(x, y, \alpha) = (1 - \alpha)x + \alpha y$.

Example 2.3. [9] Consider the open unit ball in a complex domain $\mathbb{C}$ w.r.t. the Poincare metric (also called 'Poincare distance')

$$d_B(x, y) := argtanh\Big|\frac{x - y}{1 - xy}\Big| = argtanh\Big(1 - \sigma(x, y)\Big)^{\frac{1}{2}}$$

where

$$\sigma(x, y) := \frac{(1 - |x|^2)(1 - |y|^2)}{|1 - xy|^2} \quad \text{for all } x, y \ \in B.$$

We note here that the above example can be extended from $\mathbb{C}$ to general complex Hilbert spaces $(H, \langle\cdot\rangle)$ as follows.

Let B_H be an open units ball in H. Then

$$\kappa_{B_H}(x, y) := argtanh\Big(1 - \sigma(x, y)\Big)^{\frac{1}{2}},$$

where

$$\sigma(x, y) = \frac{(1 - ||x||^2)(1 - ||x||^2)}{|1 - \langle x, y\rangle|^2} \quad \text{for all } x, y \ \in B_H$$

defines a metric on B_H (also known as the Kobayashi distance). The open unit ball B_H together with this metric is coined as a Hilbert space. Since (B_H, κ_{B_H}) is a unique geodesic space, so one can define W in a similar way for the corresponding hyperbolic space (B_H, κ_{B_H}, W).

Definition 2.4. [32] Let X be a hyperbolic space with a mapping $W : X^2 \times [0, 1] \to X$.

(i) A nonempty subset C of X is said to be convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$.

(ii) X is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided $d(x, z) \leq r, d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$.

(iii) A map $\eta : (0,\infty) \times (0,2] \to (0,1]$ which provides such a $\delta = \eta(r,\epsilon)$ for a given $r > 0$ and $\epsilon \in (0,2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r (for a fixed ϵ).

Definition 2.5. Let C be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in C. For $x \in X$, define a continuous functional $r(\cdot,\{x_n\}) : X \to [0,\infty)$ by

$$r(x,\{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(C,\{x_n\})$ of $\{x_n\}$ with respect to C is given by

$$r(C,\{x_n\}) = \inf\{r(x,\{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic centre of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x,\{x_n\}) = \inf\{r(y,\{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C,\{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then we simply denote them by $r(\{x_n\})$ and $A(\{x_n\})$ respectively.

It is well-known that in uniformly convex Banach spaces and CAT(0) spaces, bounded sequences have unique asymptotic center with respect to closed convex subsets.

Definition 2.6. [21]. A sequence $\{x_n\}$ in X is said to be $\triangle$-converge to $x \in X$, if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim\limits_{n\to\infty} x_n = x$.

Remark 2.7. (see [23]). We note that $\triangle$-convergence coincides with the usual weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.8. [25] Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic centre with respect to any nonempty closed convex subset C of X.

Lemma 2.9. [8] Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{nk}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{n_k}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.

Lemma 2.10. [20] Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x^*) \leq c$, $\limsup_{n\to\infty} d(y_n, x^*) \leq c$ and $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x^*) = c$, for some $c > 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 2.11. Let C be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

Proposition 2.12. [15] Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect to C, then we have the following:

(i) $\{x_n\}$ is bounded.
(ii) The sequence $\{d(x_n, x^*)\}$ is decreasing and converges for all $x^* \in F(T)$.
(iii) $\lim_{n\to\infty} d(x_n, F(T))$ exists.

3. The Main Results.

3.1. Some Fixed Points Properties.

Theorem 3.1. *Let C be a nonempty closed convex subset of a hyperbolic space X. Let $T : C \to C$ be a Banch operator and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.*

Proof . Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$. From (1.7), we have

$$d(x_n, Ty) = d(T^2 x_n, Ty) \leq kd(Tx_n, y) = kd(x_n, y) \leq d(x_n, y).$$

Since $\lim_{n\to\infty} d(x_n, y) = 0$ and $0 \leq d(x_n, Ty) \leq d(x_n, y)$, then we obtain

$$\lim_{n\to\infty} d(x_n, Ty) = 0.$$

By the uniqueness of limit, we have that

$$Ty = y.$$

Hence, $F(T)$ is closed.

Next we show that $F(T)$ is convex. Let $x, y \in F(T)$, by definition of T, we then have

$$d(x, Tz) = d(T^2 x, Tz) \leq kd(Tx, z) = kd(x, z) \leq d(x, z) \tag{3.1}$$

and

$$d(y, Tz) = d(T^2 y, Tz) \leq kd(Ty, z) = kd(y, z) \leq d(y, z). \tag{3.2}$$

For $z = W(x, y, \beta)$, from (3.1) and (3.2), we have

$$\begin{aligned} d(x, y) &\leq d(x, Tz) + d(Tz, y) \\ &\leq d(x, z) + d(z, y) \\ &= d(x, W(x, y, \beta)) + d(W(x, y, \beta), y) \\ &\leq (1-\beta)d(x, x) + \beta d(x, y) + (1-\beta)d(x, y) + \beta d(y, y) \\ &= d(x, y). \end{aligned}$$

Thus $d(x, Tz) = d(x, z)$ and $d(Tz, y) = d(z, y)$, because if $d(x, Tz) < d(x, z)$ or $d(Tz, y) < d(z, y)$, then we have a contradiction $d(x, y) < d(x, y)$. Therefore $Tz = W(x, y, \beta)$ and so $Tz = z$. Thus, $W(x, y, \beta) \in F(T)$. Hence $F(T)$ is convex. □

Lemma 3.2. *Let C be a nonempty subset of a hyperbolic space X. Let $T : C \to C$ be a Banach operator and $F(T) \neq \emptyset$, then T is quasi-nonexpansive.*

Proof . Let $y \in F(T)$ and $x \in C$. Then, it follows that

$$\begin{aligned} d(Tx, y) = d(Tx, T^2 y) \leq kd(x, Ty) \leq d(x, Ty) = d(x, y) \\ \Rightarrow d(Tx, y) \leq d(x, y). \end{aligned}$$

□

Lemma 3.3. *Let C be a nonempty subset of a hyperbolic space X. Let $T : C \to C$ be a Banach operator. Then*

$$d(x, Ty) \le 3d(Tx, x) + d(x, y).$$

Proof . Let $x, y \in C$, we have that

$$\begin{aligned} d(x, Ty) &\le d(x, Tx) + d(Tx, T^2x) + d(T^2x, Ty) \\ &\le d(x, Tx) + kd(x, Tx) + kd(Tx, y) \\ &\le d(x, Tx) + d(x, Tx) + d(Tx, y) \\ &\le 2d(x, Tx) + d(Tx, x) + d(x, y) \\ &= 3d(x, Tx) + d(x, y). \end{aligned}$$

Hence, the result holds. □

Lemma 3.4 (Demiclosedness principle). *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let C be a nonempty closed and convex subset of X and $T : C \to C$ be a Banach operator. If $\{x_n\}$ is a bounded sequence in C such that $\Delta-\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $x \in F(T)$.*

Proof . Since $\{x_n\}$ is a bounded sequence in X, then from Lemma 2.8, $\{x_n\}$ has a unique asymptotic centre in C. Also, since $\Delta - \lim_{n\to\infty} x_n = x$, we have that $A(\{x_n\}) = \{x\}$. Using Lemma 3.3, we have that

$$d(x_n, Tx) \le 3d(Tx_n, x_n) + d(x_n, x),$$

taking the $\limsup_{n\to\infty}$ of both sides and using the fact that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

$$\limsup_{n\to\infty} d(x_n, Tx) \le \limsup_{n\to\infty} d(x_n, x).$$

Then, by the uniqueness of asymptotic centre, we have $Tx = x$. Hence, the result holds. □

3.2. Strong and Δ-Convergence Theorems for Banach Operator.

We now introduce a modified iterative process of (1.5) in a hyperbolic space. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X and $T : C \to C$ be a Banach operator. The sequence $\{x_n\}$ is generated as follows:

$$\begin{cases} x_1 = x \in C, \\ z_n = W(x_n, T^2x_n, \beta_n), \\ y_n = W(z_n, T^2z_n, \gamma_n), \\ x_{n+1} = W(T^2y_n, 0, 0), \quad n \in \mathbb{N}, \end{cases} \tag{3.3}$$

where $\{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. We now state and prove the following lemmas which will be needed in the proof of our main theorems.

Lemma 3.5. *Let C be a nonempty closed convex subset of a hyperbolic space X. Let $T : C \to C$ be a Banach operator and $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by (3.3) where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, then for each $x^* \in F(T)$, we have that* *(i)* $\{x_n\}$ *is bounded.*

(ii) $\lim_{n\to\infty} d(x_n, x^*)$ *exists.*

(iii) $\lim_{n\to\infty} d(x_n, F(T))$ *exists.*

Proof . In fact, from Lemma 3.2, T is quasi-nonexpansive mapping. Then for all $x^* \in F(T)$ and (3.3), we have

$$\begin{aligned} d(z_n, x^*) &= d(W(x_n, T^2x_n, \beta_n), x^*) \\ &\leq (1-\beta_n)d(x_n, x^*) + \beta_n d(T^2x_n, x^*) \\ &= (1-\beta_n)d(x_n, x^*) + \beta_n d(T^2x_n, Tx^*) \\ &\leq (1-\beta_n)d(x_n, x^*) + \beta_n k d(Tx_n, x^*) \\ &\leq (1-\beta_n)d(x_n, x^*) + \beta_n d(Tx_n, x^*) \\ &\leq (1-\beta_n)d(x_n, x^*) + \beta_n d(x_n, x^*) \\ &= d(x_n, x^*). \end{aligned} \tag{3.4}$$

Using (3.3) and (3.4), we obtain

$$\begin{aligned} d(y_n, x^*) &= d(W(z_n, T^2z_n, \gamma_n), x^*) \\ &\leq (1-\gamma_n)d(z_n, x^*) + \gamma_n d(T^2z_n, x^*) \\ &\leq (1-\gamma_n)d(z_n, x^*) + \gamma_n k d(Tz_n, x^*) \\ &\leq (1-\gamma_n)d(z_n, x^*) + \gamma_n d(Tz_n, x^*) \\ &\leq (1-\gamma_n)d(z_n, x^*) + \gamma_n d(z_n, x^*) \\ &= d(z_n, x^*) \\ &\leq d(x_n, x^*). \end{aligned} \tag{3.5}$$

From (3.3) and (3.5), we obtain

$$\begin{aligned} d(x_{n+1}, x^*) &= d(W(T^2y_n, 0, 0), x^*) \leq d(T^2y_n, x^*) \leq kd(Ty_n, x^*) \\ &\leq d(Ty_n, x^*) \leq d(y_n, x^*) \leq d(x_n, x^*), \end{aligned} \tag{3.6}$$

which implies that $d(x_{n+1}, x^*) \leq d(x_n, x^*)$ for all $x^* \in F(T)$. Hence, $\{x_n\}$ is Fejer monotone with respect to $F(T)$ and so by Proposition 2.12, $\{x_n\}$ is bounded, $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$ and $\lim_{n\to\infty} d(x_n, F(T))$ exists. □

Lemma 3.6. *Let X be a complete uniformly convex hyperbolic space with monotone convexity η and C be a nonempty closed and convex subset of X. Let $T : C \to C$ be a Banach operator and $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by* (3.3) *where $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in* $(0, 1)$, *then* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof . From Lemma 3.5, we have that $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F(T)$. Now suppose that $\lim_{n\to\infty} d(x_n, x^*) = c$. If we take c = 0, then we are done. Now let's us consider the case in which $c > 0$. It has been established in Lemma 3.2 that, T is quasi-nonexpansive mapping.

Observe that,

$$d(T^2x_n, x^*) \leq kd(Tx_n, x^*) \leq d(Tx_n, x^*) \leq d(x_n, x^*),$$

taking $\limsup_{n\to\infty}$ of the above inequality, we have

$$\limsup_{n\to\infty} d(T^2x_n, x^*) \leq c. \tag{3.7}$$

Also, from (3.4), we have

$$d(z_n, x^*) \leq d(x_n, x^*)$$

and by taking the $\limsup_{n\to\infty}$, we have

$$\limsup_{n\to\infty} d(z_n, x^*) \leq c. \tag{3.8}$$

More so, from (3.6) we obtain

$$d(x_{n+1}, x^*) \leq d(z_n, x^*),$$

taking $\liminf_{n\to\infty}$, we have

$$c \leq \liminf_{n\to\infty} d(z_n, x^*). \tag{3.9}$$

It follows from (3.8) and (3.9) that

$$\lim_{n\to\infty} d(z_n, x^*) = c. \tag{3.10}$$

So,

$$\begin{aligned} c &= \lim_{n\to\infty} d(z_n, x^*) = \lim_{n\to\infty} d(W(x_n, T^2x_n, \beta_n), x^*) \\ &\leq \lim_{n\to\infty} [(1-\beta_n)d(x_n, x^*) + \beta_n d(T^2x_n, x^*) \\ &= \lim_{n\to\infty} [(1-\beta_n) \limsup_{n\to\infty} d(x_n, x^*) + \beta_n \limsup_{n\to\infty} d(T^2x_n, x^*)], \end{aligned}$$

it follows that

$$c \leq \lim_{n\to\infty} [(1-\beta_n)c - \beta_n c] = c.$$

Therefore we have

$$\lim_{n\to\infty} d(W(x_n, T^2x_n, \beta_n), x^*) = c.$$

Then by Lemma 2.10, we have

$$\lim_{n\to\infty} d(x_n, T^2x_n) = 0. \tag{3.11}$$

Since T is a Banach operator, we obtain

$$d(x_n, Tx_n) \leq d(x_n, T^2x_n) + d(T^2x_n, Tx_n) \leq d(x_n, T^2x_n) + kd(Tx_n, x_n).$$

Thus, from (3.11), we have

$$d(x_n, Tx_n) \leq \frac{1}{1-k} d(x_n, T^2x_n) \to 0.$$

Hence,

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

□

Theorem 3.7. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $T : C \to C$ be a Banach operator with $F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by* (3.3)*, then the sequence $\{x_n\}$ Δ-converges to a point in $F(T)$.*

Proof . We need to show that $W_\Delta(\{x_n\}) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\}) \subset F(T)$ and that $W_\Delta\{x_n\}$ consists of just one point.

Let $u \in W_\Delta(\{x_n\})$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$, since $\{u_n\}$ is bounded by Lemma 3.5, it follows from Lemma 2.8 that there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$. By Lemma 3.6, we have that $d(v_n, Tv_n) = 0$, and by Lemma 3.4, we have $v \in F(T)$. Since $\{d(u_n, v)\}$ converges, from Lemma 2.9, we have $u = v \in F(T)$. Hence, $W_\Delta(\{x_n\}) \subset F(T)$.

Next, we establish that $W_\Delta(\{x_n\})$ contains exactly one point. Let $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. We have established that $u = v \in F(T)$. Now, since $d(x_n, x^*)$ is convergent for all $x^* \in F(T)$ by Lemma 3.5, it follows that $\{d(x_n, u)\}$ is convergent and so by Lemma 2.9, we have $x = v \in F(T)$. Hence, $W_\Delta(\{x_n\}) = \{x\}$. Therefore, $\{x_n\}$ Δ-convergence to a point in $F(T)$. □

Theorem 3.8. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X with a monotone modulus of uniform convexity η. Let $T : C \to C$ be a Banach operator with $F(T) \neq \emptyset$. If $\{x_n\}$ is the sequence defined by* (3.3)*, then the sequence $\{x_n\}$ converges strongly to some fixed point of T if and only if $\liminf_{n\to} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf_{x\in F(T)} d(x_n, x)$.*

Proof . Suppose that $\{x_n\}$ converges to a fixed point, say x^* of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$ and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 3.5, we have that

$$\lim_{n\to\infty} d(x_n, F(T))$$

exists and so, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Suppose that $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{p_k\}$ a sequence in $F(T)$ such that for all $k \in \mathbb{N}$,

$$d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

From (3.6), it follows that

$$d(x_{n+1}, p_k) \leq d(x_{n_k}, p_k) < \frac{1}{2^k}.$$

So, we have

$$d(p_{k+1}, p_k) \leq d(p_{k+1}, x_{n+1}) + d(x_{n+1}, p_k) < \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}.$$

This shows that $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, by Lemma 2.9, $\{p_k\}$ is a convergent sequence in $F(T)$ and say it converges to $q \in F(T)$. Therefore,

$$d(x_{n_k}, q) \leq d(x_{n_k}, p_k) + d(p_k, q) \to 0 \text{ as } k \to \infty,$$

we have $\lim_{k\to\infty} d(x_{n_k}, q) = 0$ and so $\{x_{n_k}\}$ converges strongly to $q \in F(T)$. Since $\lim_{n\to\infty} d(x_n, q)$ exists, it follows that $\{x_n\}$ converges strongly to q. □

Theorem 3.9. *Suppose the assumptions in Theorem 3.7 holds and that there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that*

$$f(d(x, F(T)) \leq d(x, Tx)$$

for all $x \in C$. Then the sequence $\{x_n\}$ defined by (3.3) converges strongly to $x^* \in F(T)$.

Proof . From Lemma 3.5, we have $\lim_{n\to\infty} d(x_n, F(T))$ exist and by Lemma 3.6, we have

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Using the fact that $f(d(x, F(T)) \leq d(x, Tx)$ for all $x \in C$, we have that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0,\infty)$, it then follows that

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Hence, $\{x_n\}$ converges strongly to $x^* \in F(T)$ by Theorem 3.8. □

4. Numerical Example

In this section, we present a numerical example to show how the change in initial values affect the number of iteration for algorithm (3.3). Consider $X = \mathbb{R}$ with its usual metric and Let $C = [0, 4]$. Let $T = \dfrac{\sqrt{x}}{\sqrt{x}+1}$, chosen $\beta_n = \dfrac{1}{n+1}$, and $\gamma_n = \dfrac{5}{3(n+2)}$, then our algorithm (3.3) becomes

$$\begin{cases} x_1 = x \in C, \\ z_n = \dfrac{n}{n+1}x_n + \dfrac{1}{n+1}T^2 x_n, \\ y_n = \dfrac{3n+1}{3(n+2)}z_n + \dfrac{5}{3(n+2)}T^2 z_n, \\ x_{n+1} = T^2 y_n. \end{cases}$$

We make different choices of x_1 with stopping criterion $\frac{||x_{n+1}-x_n||}{||x_2-x_1||} < 10^{-4}$. Using Mathlab version 2016(b), we plot the graph of $||x_{n+1} - x_n||$ against the number of iteration in order to see how the change in initial values affect the number of iterations.

Case 1: Choose $x_1 = 0.5$,
Case 2: Choose $x_1 = 2$,
Case 3: Choose $x_1 = 2.5$.
See Figure 1, Figure 2 and Figure 3 for the graphs.

Remark 4.1. By the choice of our stopping criterion, we observe that different choices of x_1 have no significant effect in term of cpu time for the convergence of our algorithm.

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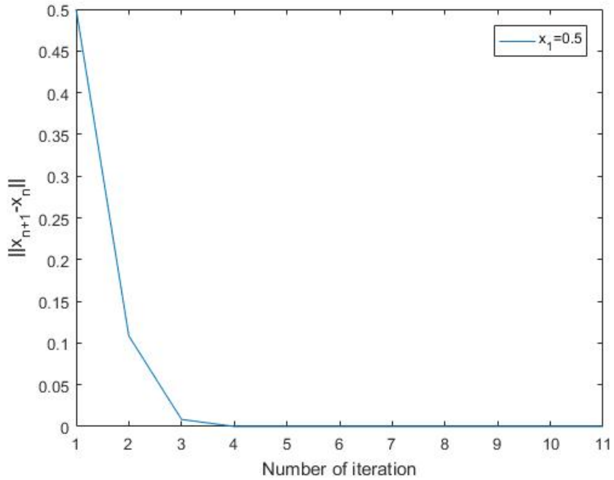

Figure 1: Case 1, $x_1 = 0.5$ (cpu time: 0.020sec).

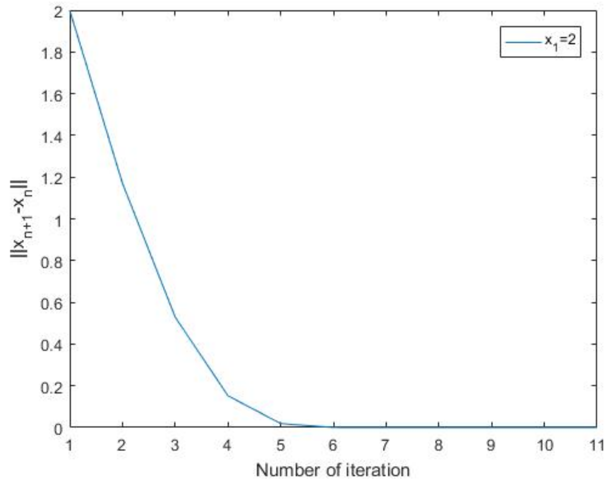

Figure 2: Case 2, $x_1 = 2$ (cpu time: 0.019sec).

References

[1] M. Abbas, Z. Kadelburg and D.R. Sahu, *Fixed point theorems for Lipschitzian type mappings in CAT(0) spaces,* Math. Comp. Model. 55 (2012) 1418–1427.

[2] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems,* MATEMATIQKI VESNIK 66 (2014) 223–234.

[3] O. Alagoz, B. Gunduz and S. Akbulut, *Convergence theorems for a family of multivalued nonexpansive mappings in hyperbolic spaces,* Open Math. 14 (2016) 1065–1073.

[4] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings,* J. Convex Anal. 8 (2007) 61–79.

[5] Y.I. Alber, C.E. Chidume and H. Zegeye, *Approximating fixed points of total asymptotically nonexpansive mappings,* Fixed Points Theory Appl. 2006 (2006), Article id 10673.

[6] S. Akbulut and B. Gunduz, *Strong convergence of a fatser iteration process in hyperbolic spaces,* Commun. Korean Math. Soc. 30 (2015) 209–219.

[7] H.H. Bauschke and P.L. Combettes, *Convex analysis and monotone operator theory in Hilbert spaces,* ser. CMS Books in Mathematics. Berlin: Springer, 2011.

[8] S.S. Chang, G. Wang, L. Wang, Y.K. Tang and Z.L. Ma, $\triangle$*-convergence theorems for multi-valued nonexpansive mappings in hyperbolic spaces,* Appl. Math. Comp. 249 (2014) 535–540.

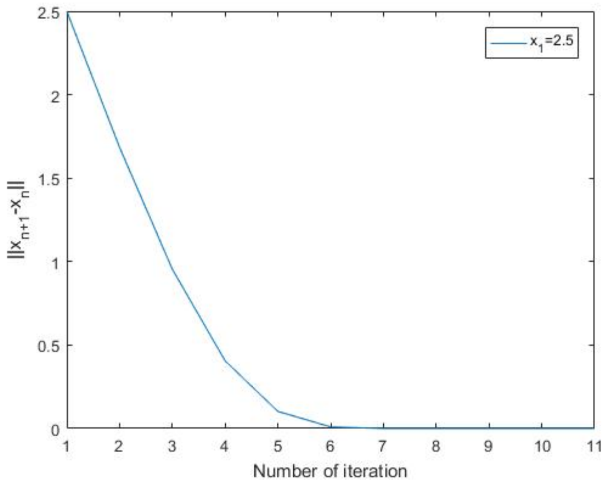

Figure 3: Case 3, $x_1 = 2.5$ (cpu time: 0.019sec).

[9] H. Fukhar-ur-din and M.A. Khan, *COnvergence analysis of a general iteration schema of nonlnear mappings in hyperbolic spaces,* Fixed Point Theory Appl. 2013 (2013): 238

[10] B. Gunduz and S. Akbulut, *Strong convergence theorems in hyperbolic spaces*, Miskolc Math. Note. 14 (2013) 915–925.

[11] B. Gunduz and S. Akbulut, *A one step implicit iterative process for a finite family of I-nonexpansive mappings in Kohlenbach hyperbolic spaces,* Math. Sci. 10 (2016) 55–61.

[12] S.J.H. Ghoncheh and A. Razani, *Fixed point theorems for some generalized nonexpansive mappings in Ptolemy spaces,* Fixed Point Theory Appl. 2014 (2014): 76.

[13] K. Goebel and W.A. Kirk, *Iteration processes for nonexpansive mappings,* In Topological Methods in Nonlinear Functional Analysis, S. P. Singh, S. Thomeier, and B.Watson, Eds., vol. 21 of Contemporary Mathematics, 115–123, American Mathematical Society, Providence, RI, USA, 1983.

[14] K. Goebel and S. Reich, *Uniform convexity, Hyperbolic Geometry and Nonexpansive mappings,* Marcel Dekket, New York, 1984.

[15] M. Imdad and S. Dashputre, *Fixed point approximation of Picard normal S-iteration process for generalized nonexpansive mappings in hyperbolic spaces,* Math. Sci. 10 (2016) 131–138.

[16] S. Ishikawa, *Fixed points by new iteration method,* Proc. Am. Math. Soc. 149 (1974) 147–150.

[17] C. Jianren and L. Zhongkai, *Common fixed-points for Banach operator pairs in best approximation,* J. Math. Anal. Appl. 336 (2007) 1466–1475.

[18] N. Karaca and I. Yildirim, *Approximating fixed points of non-expansive mappings by faster iteration process,* J. Adv. Math. Stud. 8 (2015) 257–264.

[19] S.H. Khan and M. Abass, *Common fixed point of two multivalued nonexpansive maps in Kohlenbach hyperbolic spaces*, Fixed Point Theory Appl. 2014 (2014): 181.

[20] A. R. Khan, H. Fukhar-ud-din and M.A.A. Khan, *An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces,* Fixed Point Theory Appl. 2012 (2012), Article ID 54.

[21] W. A. Kirk and B. Panyanak, *A concept of convergence in geodesic spaces,* Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods, vol. 68 (2008) 3689–3696, .

[22] U. Kohlenbach, *Some logical metathorems with applications in functional analysis,* Trans. Amer. Math. Soc. 357 (2005) 89–128.

[23] T. Kuczumow, *An almost convergence and its applications,* Ann. Univ. Mariae Curie-Skodowska, Sect. A 32 (1978) 79–88.

[24] M.A. Krasnosel'skii, *Two remarks on the method of successive approximations,* Usp. Mat. Nauk. 10 (1955) 123–127

[25] L. Leustean, *Nonexpansive iteration in uniformly convex W-hyperbolic spaces,* In A. Leizarowitz, B.S. Mordukhovich, I. Shafrir, A. Zaslavski, Nonlinear Analysis and Optimization I. Nonlinear analysis Contemporary Mathematics. Providence. RI Ramat Gan American Mathematical Soc. Bar Ilan University 513 (2010) 193–210.

[26] W.R. Mann, *Mean value methods in iteration,* Proc. Am. Math. Soc. 4 (1953) 506–510.

[27] M.A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl. 251 (2000) 217–229.

[28] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces,* Nonlinear Anal. 15 (1990) 537–558.
[29] B. Rhoades, E.L.S. Kanhaya and J.H.M. Whitfield, *Fixed points for generalized nonexpansive mappings,* Commentationes Mathematicae Universitatis Carolinae 23 (1982) 443–451.
[30] D.R. Sahu, *Application of the S-iteration process to constrained minimization problem and split feasibility problems,* Fixed Point Theory 12 (2011) 187–204.
[31] H. Salahifard, S. M. Vaezpour and S. Dhompongsa, *Fixed point theorems for some generalized nonexpansive mappings in CAT (0) spaces,* J. Nonlinear Anal. Optim. 4 (2013) 241–248.
[32] C. Suanoom and C. Klin-eam, *Remark on fundamentally non-expansive mappings in hyperbolic spaces,* J. Nonlinear Sci. Appl. 9 (2016) 1952–1956.
[33] W.A. Takahashi, *A convexity in metric space and nonexpansive mappings,* I. Kodai Math. Sem. Rep. 22 (1970) 142–149.