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Article *in* Asian-European Journal of Mathematics · March 2020

DOI: 10.1142/S1793557121500583

Asian-European Journal of Mathematics
(2021) 2150058 (24 pages)

DOI: 10.1142/S1793557121500583

A viscosity-type proximal point algorithm for monotone equilibrium problem and fixed point problem in an Hadamard space

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Communicated by L. A. Bokut
Received September 23, 2019
Revised February 28, 2020
Accepted
Published

In this paper, we introduce a viscosity-type proximal point algorithm comprising of a finite composition of resolvents of monotone bifunctions and a generalized asymptotically nonspreading mapping recently introduced by Phuengrattana [*Appl. Gen. Topol.* **18** (2017) 117–129]. We establish a strong convergence result of the proposed algorithm to a common solution of a finite family of equilibrium problems and fixed point problem for a generalized asymptotically nonspreading and nonexpansive mappings, which is also a unique solution of some variational inequality problems in an Hadamard space. We apply our result to solve convex feasibility problem and to approximate a common solution of a finite family of minimization problems in an Hadamard space.

Keywords: Equilibrium problems; monotone bifunctions; generalized nonspreading mappings; viscosity iterations; CAT(0) space.

AMS Subject Classification: 47H09, 47H10, 49J20, 49J40

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1. Introduction

The approximation of fixed points of nonlinear mappings is one of the most flourishing areas of research in mathematics that has enjoyed a prosperous development in the recent years (see, for example, [1, 10, 12, 16, 24, 27–31, 38, 44, 47, 48, 52] and the references therein). Due to its wide application in solving many mathematical problems; namely, inverse problems, variational inequality problems, minimization problems, problems emanating from game theory and fuzzy theory, among others, it has continued to attract the interest of numerous authors. These authors have introduced several nonlinear mappings whose fixed points are solutions to the aforementioned problems. For instance, Kohsaka and Takahashi [35] introduced the class of *nonspreading mappings* defined as follows: Let C be a nonempty closed and convex subset of a real smooth, strictly convex and reflexive Banach space E. A mapping $T : C \to C$ is called *nonspreading*, if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x), \quad \forall\, x, y \in C, \tag{1.1}$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ and J is the duality mapping on C. If $E = H$, where H is a real Hilbert space, then J is the identity mapping and $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Thus, for a nonempty, closed and convex subset C of H, $T : C \to C$ is called *nonspreading,* if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall\, x, y \in C. \tag{1.2}$$

Using the class of nonspreading mappings, Kohsaka and Takahashi [35] studied the resolvents of maximal monotone operators in Banach spaces. Later in 2013, Naraghirad [37] continued along this line and introduced the class of *asymptotically nonspreading* mappings in a real Banach space, which he defined as follows: Let C be a nonempty, closed and convex subset of a real Banach space E. A mapping $T : C \to C$ is called *asymptotically nonspreading*, if

$$\|T^n x - T^n y\|^2 \leq \|x - y\|^2 + 2\langle x - T^n x, J(y - T^n y)\rangle \quad \forall\, x, y \in C \text{ and } n \in \mathbb{N}, \tag{1.3}$$

where J is the duality mapping on C. One can easily verify that in a real Hilbert space, (1.3) is equivalent to

$$2\|T^n x - T^n y\|^2 \leq \|T^n x - y\|^2 + \|T^n y - x\|^2 \quad \forall\, x, y \in C \text{ and } n \in \mathbb{N}. \tag{1.4}$$

Clearly, if $n = 1$, then T is nonspreading. Naraghirad [37] proved some weak and strong convergence theorems for approximating fixed points of asymptotically nonspreading mappings in real Banach spaces.

Based on the work of Naraghirad [37], Phuengrattana [43] introduced a new class of nonlinear mappings in a convex metric space as follows: Let C be a nonempty subset of a convex metric space X. A mapping $T : C \to C$ is called *generalized* *asymptotically nonspreading*, if there exist two functions $f, g : C \to [0, \gamma]$, $\gamma < 1$ such that

$$d^2(T^n x, T^n y) \leq f(x) d^2(T^n x, y) + g(x) d^2(T^n y, x) \quad \forall x, \ y \in C, \ n \in \mathbb{N},$$

and

$$0 < f(x) + g(x) \leq 1 \quad \forall x \in C.$$

Phuengrattana [43] established some existence theorems and demiclosed principle for the class of generalized asymptomatically nonspreading mappings. Furthermore, he proved a Δ-convergence of the Mann-type iteration to a fixed point of this class of mappings in a complete CAT(0) space. As remarked by Phuengrattana [43], if $f(x) = \frac{1}{2} = g(x)$ for all $x \in C$, then T reduces to an asymptotically nonspreading mapping. Thus, the class of generalized asymptotically nonspreading mappings includes the class of asymptotically nonspreading mappings, as well as the class of nonspreading mappings. The following example was given by Phuengrattana [43] to show that this inclusion is actually proper.

Example 1.1 ([43]). Let $T : [0, \infty) \to [0, \infty)$ be defined by

$$Tx = \begin{cases} 0.9, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then, T is not an asymptotically nonspreading mapping. To see this, take $x = 1.2$ and $y = 0.7$. However, T is a generalized asymptotically nonspreading mapping.

Another area of mathematics that has received a lot of attention in recent time is optimization theory. One of the most important problems in optimization theory is the following Equilibrium Problem (EP):

$$\text{Find } x^* \in C \quad \text{such that } f(x^*, y) \geq 0, \ \forall y \in C, \tag{1.5}$$

where f is a bifunction from $C \times C$ into $\mathbb{R}$. The point x^* for which (1.5) is satisfied is called an equilibrium point of f. Throughout this paper, we shall denote the solution set of problem (1.5) by EP(f, C). Problem (1.5) includes many important mathematical problems as special cases such as variational inequality problems, minimization problems, complementarity problems, among others. EPs have been widely studied in Hilbert, Banach and topological vector spaces by many authors (see [6, 7, 15, 21, 26, 42, 49]), as well as in Hadamard manifolds (see [14, 39, 40]). Very recently, Khatibzadeh and Mohebbi [33] extended these studies to Hadamard spaces. More precisely, they studied the existence of an equilibrium point of the bifunction f under some appropriate conditions on f. Furthermore, Khatibzadeh and Mohebbi [33] proved the unique existence of the sequence generated by the Proximal Point Algorithm (PPA) (or equivalently, the unique existence of the resolvent) associated with the bifunction f. They also proved the convergence of the resolvent of f to an equilibrium point of f. More so, they obtained a Δ-convergence and a strong convergence of the PPA and the Halpern-type algorithm, respectively, to an equilibrium point of f.

Motivated by the results of Phuengrattana [43], Khatibzadeh and Mohebbi [33], we introduce a viscosity-type PPA (since viscosity-type algorithms have higher rate of convergence than the Halpern-types, and Halpern-type convergence theorems imply viscosity convergence theorems, see for example [45]) and prove its strong convergence to a common solution of a finite family of equilibrium problems and fixed point problem for a generalized asymptotically nonspreading and nonexpansive mappings, which is also a unique solution of some variational inequality problems in an Hadamard space. Furthermore, we apply our results to solve convex feasibility problem and to approximate a common solution of a finite family of minimization problems in an Hadamard space.

2. Preliminary

2.1. *Geometry of CAT(0) spaces*

Definition 2.1. Let (X, d) be a metric space and $x, y \in X$. A geodesic path joining x to y is a mapping $c\colon [0, t] \subset \mathbb{R} \to X$ such that $c(0) = x$, $c(t) = y$ and $d(c(k), c(k')) = |k - k'|$ for all $k, k' \in [0, t]$. In this case, c is called an isometry and $d(x, y) = t$. The image of c is called a geodesic segment joining x to y. When this image is unique, it is denoted by $[x, y]$.

The metric space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and it is said to be a uniquely geodesic space if every two points of X are joined by exactly one geodesic segment. A subset C of a geodesic space X is said to be convex, if for all $x, y \in C$, the segment $[x, y]$ is in C. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic space (X, d) consists of three points x_1, x_2, x_3 in X (known as the vertices of Δ) and a geodesic segment between each pair of vertices (known as the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for all $i, j \in \{1, 2, 3\}$. A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in X, respectively, then δ is said to satisfy the CAT(0) inequality for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$ if

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.1}$$

Also, a geodesic space is a CAT(0) space if and only if it satisfies the following inequality, called the (CN) inequality of Bruhat and Titis [9] (see [8]): If x, y, z are points in X and y_0 is the midpoint of the segment $[y, z]$, then

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z). \tag{2.2}$$

Definition 2.2 ([5]). Let X be a CAT(0) space. Denote the pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle \cdot, \cdot \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)) \quad \forall\, a,b,c,d \in X$$

is called a quasilinearization mapping.

It is easy to check that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a,b,c,d,e \in X$. A geodesic space X is said to satisfy the Cauchy–Schwarz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d)$ $\forall$ $a,b,c,d \in X$. It has been established in [5] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwarz inequality. It is generally known that CAT(0) spaces are uniquely geodesic spaces (see for example [19]), and complete CAT(0) spaces are called Hadamard spaces. Examples of CAT(0) spaces include: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, $\mathbb{R}$-trees, Hilbert ball [20] and Hyperbolic spaces [44].

Definition 2.3 (see [25]). Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \left\{ \bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v \in X} \limsup_{n\to\infty} d(v, x_n) \right\}.$$

It is generally known that in a Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write Δ-$\lim_{n\to\infty} x_n = \bar{v}$ (see [18]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [36]. Kirk and Panyanak [34] later introduced and studied this concept in CAT(0) spaces, and proved that it is very similar to the weak convergence in Banach space setting.

2.2. *Existence and uniqueness of resolvent of monotone bifunctions*

Definition 2.4 ([33]). Let C be a nonempty closed and convex subset of an Hadamard space X and $\circ$ be an arbitrary but fixed point in X. The point $\circ$ is called a base-point of X. To study the EP (1.5) in X, we consider the following assumptions:

$P1$: $f(x,x) = 0$ for all $x \in C$.
$P2$: $f(\cdot, y) : C \to \mathbb{R}$ is upper semicontinuous for all $y \in C$.
$P3$: $f(x, \cdot) : C \to \mathbb{R}$ is convex and lower semicontinuous for all $x \in C$.

$P4$: f is monotone, that is, $f(x,y)+f(y,x)\leq 0$, for all $x,y\in C$.

$P4^*$: f is pseudo-monotone, that is, whenever $f(x,y)\geq 0$, we have that $f(y,x)\leq 0$.

$P4^{**}$: f is θ-undermonotone, that is, there exists $\theta\geq 0$ such that $f(x,y)+f(y,x)\leq \theta d^2(x,y)$ for all $x,y\in C$.

$P5$: For any sequence $\{x_n\}$ in C with $\lim_{n\to\infty} d(x_n,\circ)=+\infty$, there exists $v\in C$ and $n_0\in\mathbb{N}$ such that $f(x_n,v)\leq 0$, for all $n\geq n_0$.

The following theorem guarantees the existence of solution of EP (1.5).

Theorem 2.5 ([33]). *Let C be a nonempty closed and convex subset of an Hadamard space X and $f:C\times C\to\mathbb{R}$ be a bifunction such that f satisfies $P1$, $P2$, $P3$ and $P4^*$. Then, EP* (1.5) *has a solution if and only if $P5$ holds.*

Note that if $P4^*$ is replaced by $P4$ in Theorem 2.5, then the conclusion of Theorem 2.5 still holds.

Definition 2.6 ([33]). A function $f:C\times C\to\mathbb{R}$ is said to be cyclic monotone if for each $n\in\mathbb{N}$ and $x_1,x_2,\ldots,x_n\in X$, we have

$$f(x_1,x_2)+f(x_2,x_3)+\cdots+f(x_n,x_1)\leq 0.$$

In [33], the authors proposed the following PPA for finding an equilibrium point of f: Given an arbitrary $x_0\in X$, inductively for $\{x_{n-1}\}$ in C, $\{x_n\}$ satisfies the following inequality:

$$f(x_n,y)+\lambda_{n-1}\langle\overrightarrow{x_{n-1}x_n},\overrightarrow{x_ny}\rangle\geq 0,\quad \forall\, y\in C, \tag{2.3}$$

where $\{\lambda_n\}$ is a sequence of positive numbers. The existence and uniqueness of (2.3) has been established in real Hilbert space setting for a θ-undermonotone bifunction (see [22]). Also, Khatibzadeh and Mohebbi [33] proved the existence of the sequence generated by (2.3) in Hadamard space settings by considering the following auxiliary bifunction:

$$\bar{f}(x,y)=f(x,y)+\lambda\langle\overrightarrow{\bar{x}x},\overrightarrow{xy}\rangle, \tag{2.4}$$

where $\bar{x}\in X$, $\lambda>\theta\geq 0$ and f is a bifunction that satisfies $P1$, $P2$, $P3$ and $P4^{**}$. More precisely, they proved the following existence result.

Theorem 2.7. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f:C\times C\to\mathbb{R}$ be a cyclic monotone bifunction which satisfies $P1$, $P2$ and $P3$. Then, $\bar{f}$ has a solution.*

Furthermore, they established the uniqueness result by employing assumption $P4^{**}$ (see [33, p. 16]). Note that for the uniqueness result, we can replace assumption $P4^{**}$ with the monotonicity assumption of f.

This unique solution of the EP associated with $\bar{f}$, is denoted by $J_\lambda^f \bar{x}$ and it is called the resolvent of f of order $\lambda > 0$ at $\bar{x}$ (see [33]). In other words, the resolvent of the bifunction f is the set-valued mapping $J_\lambda^f : X \to 2^C$ defined by

$$J_\lambda^f(x) = \{z \in C : f(z, y) + \lambda\langle \overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ \forall\, y \in C\} \quad \text{for all } x \text{ in } X. \tag{2.5}$$

Thus, we have the following important remark which follows from Theorem 2.7 and the uniqueness result found in [33, p. 16].

Remark 2.8. If C is a nonempty closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ is a cyclic monotone bifunction which satisfies $P1$, $P2$ and $P3$, then for $\lambda > 0$, the resolvent J_λ^f of f exists and it is unique.

See [33, Problem 3.11] for more discussion on the existence and uniqueness of the PPA (2.3) or equivalently, the unique existence of the resolvent of monotone bifunctions in Hadamard spaces.

2.3. *Fundamental properties of resolvent of monotone bifunctions*

Definition 2.9. Let C be a nonempty subset of a metric space X. A mapping $T : C \to C$ is said to be *uniformly L-Lipschitzian* if there exists $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y) \quad \forall\, n \geq 1,\ \ x, y \in C.$$

If $L = 1$ and $n = 1$, then T is called *nonexpansive*. T is said to be *asymptotically regular*, if $\lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0\ \forall\, x \in C$. Furthermore, T is *firmly nonexpansive* if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle \quad \forall\, x, y \in X.$$

By Cauchy–Schwartz inequality, it is clear that firmly nonexpansive mappings are nonexpansive. Recall that a point $v \in C$ is called a fixed point of a nonlinear mapping $T : C \to C$, if $Tv = v$. We denote the set of fixed points of T by $F(T)$.

Lemma 2.10 ([33, Proposition 4.2]). *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction such that $J_\lambda^f x$ exists for $\lambda > 0$. If f is monotone, then the mapping $x \mapsto J_\lambda x$ is firmly nonexpansive.*

Remark 2.11. By Lemma 2.10, we see easily that J_λ^f is nonexpansive and EP(f, C) $= F(J_\lambda^f)$. Also note that, under the assumptions of Lemma 2.10, we have that J_λ^f singlevalued.

For the rest of this paper, we shall simply write J_λ for the resolvent of a monotone bifunction f.

Lemma 2.12. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction such that $J_\lambda x$ exists for $\lambda > 0$. Then,* *the following hold*:

(i) *If f is monotone and $F(J_\lambda) \neq \emptyset$, then*

$$d^2(J_\lambda x, x) \leq d^2(x, v) - d^2(J_\lambda x, v) \quad \forall\, x \in X, \ \ v \in F(J_\lambda).$$

(ii) *If $0 < \mu \leq \lambda$, then $d(J_\mu x, J_\lambda x) \leq \sqrt{1 - \frac{\mu}{\lambda}} d(x, J_\mu x)$, which implies that $d(x, J_\lambda x) \leq 2d(x, J_\mu x)$ $\forall\, x \in X$.*

Proof. (i) By Lemma 2.10, we have that $d^2(J_\lambda x, J_\lambda v) \leq \langle \overrightarrow{J_\lambda x J_\lambda v}, \overrightarrow{xv} \rangle$, which follows from the definition of quasilinearization that

$$d^2(x, J_\lambda x) \leq d^2(x, v) - d^2(v, J_\lambda x) \quad \forall\, x \in X, \ \ v \in F(J_\lambda).$$

(ii) Let $x \in X$ and $0 < \mu \leq \lambda$, then we have that

$$f(J_\lambda x, y) + \lambda \langle \overrightarrow{x J_\lambda x}, \overrightarrow{J_\lambda x y} \rangle \geq 0, \quad \forall\, y \in C \tag{2.6}$$

and

$$f(J_\mu x, y) + \mu \langle \overrightarrow{x J_\mu x}, \overrightarrow{J_\mu x y} \rangle \geq 0, \quad \forall\, y \in C. \tag{2.7}$$

By letting $y = J_\mu x$ in (2.6) and $y = J_\lambda x$ in (2.7), and summing up, we have

$$f(J_\lambda x, J_\mu x) + f(J_\mu x, J_\lambda x) + \lambda \langle \overrightarrow{x J_\lambda x}, \overrightarrow{J_\lambda x J_\mu x} \rangle + \mu \langle \overrightarrow{x J_\mu x}, \overrightarrow{J_\mu x J_\lambda x} \rangle \geq 0.$$

Also, by the monotonicity of f, we obtain that

$$\langle \overrightarrow{J_\lambda x x}, \overrightarrow{J_\mu x J_\lambda x} \rangle \geq \frac{\mu}{\lambda} \langle \overrightarrow{J_\mu x x}, \overrightarrow{J_\mu x J_\lambda x} \rangle.$$

By the definition of quasilinearization, we have that

$$\begin{aligned} &d^2(x, J_\mu x) - d^2(J_\lambda x, J_\mu x) - d^2(x, J_\lambda x) \\ &\quad \geq \frac{\mu}{\lambda}(d^2(J_\mu x, J_\lambda x) + d^2(x, J_\mu x) - d^2(x, J_\lambda x)). \end{aligned}$$

That is,

$$\left(\frac{\mu}{\lambda} + 1\right) d^2(J_\mu x, J_\lambda x) \leq \left(1 - \frac{\mu}{\lambda}\right) d^2(x, J_\mu x) + \left(\frac{\mu}{\lambda} - 1\right) d^2(x, J_\lambda x).$$

Since $\frac{\mu}{\lambda} \leq 1$, we obtain that

$$\left(\frac{\mu}{\lambda} + 1\right) d^2(J_\mu x, J_\lambda x) \leq \left(1 - \frac{\mu}{\lambda}\right) d^2(x, J_\mu x),$$

which implies

$$d(J_\mu x, J_\lambda x) \leq \sqrt{1 - \frac{\mu}{\lambda}} d(x, J_\mu x). \tag{2.8}$$

Furthermore, by triangle inequality and (2.8), we obtain that

$$d(x, J_\lambda x) \leq 2d(x, J_\mu x). \qquad \square$$

2.4. *Important lemmas*

We now recall some important lemmas which will be needed in the proof of our main results.

Lemma 2.13. *Let X be a $CAT(0)$ space, $x, y, z \in X$ and $t \in [0, 1]$. Then*

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$ (*see* [19]).
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$ (*see* [19]).
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x, z) + (1-t)^2 d^2(y, z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (*see* [17]).
(iv) $d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w, y) + (1-t)d(x, z)$ (*see* [8]).
(v) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x, y)$ (*see* [11]).

Lemma 2.14 ([19]). *Every bounded sequence in a Hadamard space always has a $\triangle$-convergent subsequence.*

Lemma 2.15 ([43]). *Let C be a nonempty, closed and convex subset of a Hadamard space X and $T : C \to C$ be a generalized asymptotically nonspreading mapping. Let $\{x_n\}$ be a bounded sequence in C such that $\{x_n\}$ Δ-converges to v and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Then, $Tv = v$.*

Lemma 2.16 ([32]). *Let X be a Hadamard space, $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty} \langle \overrightarrow{xx_n}, \overrightarrow{xy} \rangle \leq 0$ for all $y \in C$.*

Lemma 2.17 (Xu, [53]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 0)$, $\sum \gamma_n < \infty$. *Then, $a_n \to 0$ as $n \to \infty$.*

3. Main Results

Lemma 3.1. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a bifunction such that $J_{\lambda^{(i)}}x$ exists for each $i = 1, 2, \ldots, N$ and $\lambda^{(i)} > 0$. Let $\{y_n\}$ and $\{x_n\}$ be bounded sequences in C such that*

$$y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n,$$

where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence such that $0 < \lambda_n^{(i)} \leq \lambda^{(i)}$ for each $i = 1, 2, \ldots, N$ and $n \geq 1$. If $\lim_{n\to\infty} d(x_n, y_n) = 0$, f is monotone and $\bigcap_{i=1}^N F(J_{\lambda^{(i)}}) \neq \emptyset$, then $\lim_{n\to\infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0$, for each $i = 1, 2, \ldots, N$.

Proof. Let $v \in \bigcap_{i=1}^N F(J_{\lambda^{(i)}})$ and set $u_n^{(i+1)} = J_{\lambda_n^{(i)}} u_n^{(i)}$, for each $i = 1, 2, \ldots, N$, where $u_n^{(1)} = x_n$, for all $n \geq 1$. Then, $u_n^{(N+1)} = y_n$. Thus, by Lemma 2.12(i), we obtain

$$d^2(u_n^{(i)}, u_n^{(i+1)}) \leq d^2(v, u_n^{(i)}) - d^2(v, u_n^{(i+1)}),$$

which implies

$$\begin{aligned}\sum_{i=1}^N d^2(u_n^{(i)}, u_n^{(i+1)}) &\leq d^2(v, x_n) - d^2(v, u_n^{(N+1)}) \\ &\leq [d(v, y_n) + d(y_n, x_n)]^2 - d^2(v, y_n) \\ &= d^2(x_n, y_n) + 2d(x_n, y_n)d(v, y_n) \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} d(u_n^{(i)}, u_n^{(i+1)}) = 0, \quad i = 1, 2, \ldots, N. \tag{3.1}$$

From (3.1) and by triangle inequality, we obtain for each $i = 1, 2, \ldots, N$ that

$$\lim_{n\to\infty} d(x_n, u_n^{(i+1)}) = 0. \tag{3.2}$$

Since $0 < \lambda_n^{(i)} \leq \lambda^{(i)}$ for all $n \geq 1$, we obtain by Lemma 2.12(ii) and (3.1) that

$$d(u_n^{(i)}, J_{\lambda^{(i)}} u_n^{(i)}) \leq 2d(u_n^{(i)}, J_{\lambda_n^{(i)}} u_n^{(i)}) \to 0, \quad \text{as } n \to \infty,\ i = 1, 2, \ldots, N. \tag{3.3}$$

Again, since $J_{\lambda^{(i)}}$ is nonexpansive for each $i = 1, 2, \ldots, N$, we obtain from (3.1) and (3.2) that

$$\begin{aligned}d(J_{\lambda^{(i)}} x_n, J_{\lambda^{(i)}} u_n^{(i)}) &\leq d(J_{\lambda^{(i)}} x_n, J_{\lambda^{(i)}} u_n^{(i+1)}) + d(J_{\lambda^{(i)}} u_n^{(i+1)}, J_{\lambda^{(i)}} u_n^{(i)}) \\ &\leq d(x_n, u_n^{(i+1)}) + d(u_n^{(i+1)}, u_n^{(i)}) \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.4}$$

From (3.1) to (3.4), we obtain

$$\begin{aligned}d(J_{\lambda^{(i)}} x_n, x_n) \leq{}& d(J_{\lambda^{(i)}} x_n, J_{\lambda^{(i)}} u_n^{(i)}) + d(J_{\lambda^{(i)}} u_n^{(i)}, u_n^{(i)}) + d(u_n^{(i)}, u_n^{(i+1)}) \\ &+ d(u_n^{(i+1)}, x_n) \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

That is,

$$\lim_{n\to\infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0, \quad i = 1, 2, \ldots, N. \qquad \square$$

We now present our strong convergence theorems.

Theorem 3.2. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of cyclic monotone bifunctions satisfying $P1$, $P2$ and $P3$. Let $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is* *also asymptotically regular, and g be a contraction mapping on C with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := \bigcap_{i=1}^{N} EP(f_i, C) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n, \\ x_n = \alpha_n g(y_n) \oplus (1-\alpha_n) T^n y_n, \quad n \geq 1, \end{cases} \tag{3.5}$$

where $0 < \lambda_n^{(i)} \leq \lambda^{(i)}$ $\forall\ n \geq 1, i = 1, 2, \ldots, N$ and $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $L < (1-\alpha_n\gamma)/(1-\alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle \geq 0, \quad \forall\, u \in \Gamma. \tag{3.6}$$

Proof. Step 1. We show that (3.5) is well defined. By Remark 2.8, we have that $J_{\lambda_n} x$ exists for all $x \in C$. Let $S_n x_n := J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n$, then it follows from Remark 2.11 that S_n is nonexpansive for all $n \geq 1$. Now, define the mapping $T_n^g : C \to C$ as follows:

$$T_n^g x = \alpha_n g(S_n x) \oplus (1-\alpha_n) T^n S_n x.$$

Since T is uniformly L-Lipschitzian, we obtain from Lemma 2.13(iv) that

$$\begin{aligned} d(T_n^g x, T_n^g y) &\leq \alpha_n d(g(S_n x), g(S_n y)) + (1-\alpha_n) d(T^n S_n x, T^n S_n y) \\ &\leq \gamma \alpha_n d(S_n x, S_n y) + (1-\alpha_n) L d(S_n x, S_n y) \\ &\leq (\gamma \alpha_n + (1-\alpha_n) L) d(x,y), \end{aligned}$$

which implies by condition (ii) that T_n^g is a contraction for each $n \geq 1$. Therefore, by Banach contraction mapping principle, there exists a unique fixed point x_n of T_n^g for each $n \geq 1$. Hence, (3.5) is well defined.

Step 2: We show that $\{x_n\}$ is bounded. Let $v \in \Gamma$, then by Remark 2.11, we obtain that $v = J_{\lambda_n^{(i)}} v$ for each $i = 1, 2, \ldots, N$. Thus, $v = S_n v$. Again, since T is generalized asymptotically nonspreading, we obtain that

$$(1-g(v)) d^2(v, T^n y_n) \leq f(v) d^2(v, y_n),$$

which implies that

$$d(v, T^n y_n) \leq d(v, y_n), \tag{3.7}$$

since $0 < f(v) + g(v) \leq 1$.

Thus, by (3.5) and Lemma 2.13(i), we obtain

$$\begin{aligned} d(x_n, v) &\leq \alpha_n d(g(y_n), v) + (1-\alpha_n) d(T^n y_n, v) \\ &\leq \alpha_n \gamma d(y_n, v) + \alpha_n d(g(v), v) + (1-\alpha_n) d(y_n, v) \\ &\leq (1 - \alpha_n(1-\gamma)) d(x_n, v) + \alpha_n d(g(v), v), \end{aligned} \tag{3.8}$$

which implies that

$$d(x_n, v) \leq \frac{d(g(v), v)}{1-\gamma}.$$

Thus, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ $\{T^n y_n\}$ and $\{g(y_n)\}$ are all bounded.

Step 3. We show that $\lim_{n\to\infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0 = \lim_{n\to\infty} d(y_n, Ty_n)$, $i = 1, 2, \ldots, N$.

From (3.5), we obtain

$$\begin{aligned} d(x_n, T^n y_n) &= d(\alpha_n g(y_n) \oplus (1-\alpha_n) T^n y_n, T^n y_n) \\ &\leq \alpha_n d(g(y_n), T^n y_n) \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.9}$$

Again, from Lemma 2.13(ii) and (3.7), we obtain

$$\begin{aligned} d^2(x_n, v) &= d^2(\alpha_n g(y_n) \oplus (1-\alpha_n) T^n y_n, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(T^n y_n, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(y_n, v). \end{aligned} \tag{3.10}$$

Let $u_n^{(i+1)}$ be as defined in the proof of Lemma 3.1, then by Lemma 2.12(i), we obtain for each $i = 1, 2, \ldots, N$ that

$$d^2(u_n^{(i+1)}, v) \leq d^2(u_n^{(i)}, v) - d^2(u_n^{(i)}, u_n^{(i+1)}). \tag{3.11}$$

For $i = N$, we obtain from (3.10) and (3.11) that

$$\begin{aligned} d^2(x_n, v) &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N+1)}, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N)}, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &= \alpha_n (d^2(g(y_n), v) - d^2(x_n, v)) + d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}), \end{aligned}$$

which implies by condition (i) that

$$\lim_{n\to\infty} d^2(u_n^{(N)}, u_n^{(N+1)}) = 0. \tag{3.12}$$

Similarly, from (3.10) and (3.11), we obtain for $i = N-1$ that

$$\begin{aligned} d^2(x_{n+1}, v) &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N)}, v) \\ &\leq \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^p(u_n^{(N-1)}, v) - (1-\alpha_n) d^2(u_n^{(N-1)}, u_n^{(N)}) \\ &\leq d^2(g(y_n), v) + (1-\alpha_n) d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N-1)}, u_n^{(N)}), \end{aligned}$$

which implies by the condition of (i) that

$$\lim_{n\to\infty} d^2(u_n^{(N-1)}, u_n^{(N)}) = 0. \tag{3.13}$$

Continuing in this manner, we can show that

$$\lim_{n\to\infty} d(u_n^{(i)}, u_n^{(i+1)}) = 0, \quad i = 1, 2, \ldots, N-2, \tag{3.14}$$

which together with (3.12) and (3.13) yields

$$\lim_{n\to\infty} d(u_n^{(i)}, u_n^{(i+1)}) = 0, \quad i = 1, 2, \ldots, N. \tag{3.15}$$

From (3.15), and applying triangle inequality, we obtain for each $i = 1, 2, \ldots, N$, that

$$\lim_{n\to\infty} d(x_n, u_n^{(i+1)}) = 0. \tag{3.16}$$

In particular, for $i = N$, we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.17}$$

Thus, we obtain from Lemma 3.1 that

$$\lim_{n\to\infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0, \quad i = 1, 2, \ldots, N. \tag{3.18}$$

Furthermore, we obtain from (3.9) and (3.17) that

$$\lim_{n\to\infty} d(y_n, T^n y_n) = 0. \tag{3.19}$$

By the asymptotic regularity of T, we obtain

$$\begin{aligned} d(y_n, Ty_n) &\leq d(y_n, T^n y_n) + d(T^n y_n, T^{n+1} y_n) + d(T^{n+1} y_n, T y_n) \\ &\leq (1+L) d(y_n, T^n y_n) + d(T^{n+1} y_n, T^n y_n) \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.20}$$

By the boundedness of $\{x_n\}$, we obtain from Lemma 2.14, that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which $\triangle$-converges to w. It then follows from the boundedness of $\{y_n\}$ and (3.17) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ which $\triangle$-converges to w. Thus, from (3.18), (3.20), and Lemma 2.15, we obtain that $w \in \Gamma$.

Step 4. We now show that $\{x_n\}$ converges strongly to w. Since $\{y_{n_k}\}$ $\triangle$-converges to $w \in \Gamma$, we obtain by Lemma 2.16 that

$$\lim_{k\to\infty} \langle \overrightarrow{g(w)w}, \overrightarrow{y_{n_k}w} \rangle \leq 0. \tag{3.21}$$

Also, by Lemma 2.13(iii) and (3.5), we have

$$\begin{aligned} d^2(x_n, w) &= d^2(\alpha_n g(y_n) \oplus (1-\alpha_n)T^n y_n, w) \\ &\leq \alpha_n^2 d^2(g(y_n), w) + (1-\alpha_n) d^2(T^n y_n, w) \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{g(y_n)w}, \overrightarrow{T^n y_n w} \rangle \\ &\leq \alpha_n^2 d^2(g(y_n), w) + (1-\alpha_n) d^2(y_n, w) \\ &\quad + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{g(y_n)w}, \overrightarrow{T^n y_n y_n} \rangle + \langle \overrightarrow{g(y_n)g(w)}, \overrightarrow{y_n w} \rangle + \langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle] \\ &\leq \alpha_n^2 d^2(g(y_n), w) + (1-\alpha_n) d^2(y_n, w) \\ &\quad + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{g(y_n)w}, \overrightarrow{T^n y_n y_n} \rangle + \gamma d^2(y_n, w) + \langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle] \\ &\leq [(1-\alpha_n) + 2\gamma\alpha_n(1-\alpha_n)] d^2(x_n, w) + \alpha_n[\alpha_n d^2(g(y_n), w) \\ &\quad + 2(1-\alpha_n) d(T^n y_n, y_n)] d(g(y_n), w) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle. \end{aligned} \tag{3.22}$$

Therefore,

$$\begin{aligned} d^2(x_n, w) \leq{}& \frac{[\alpha_n d^2(g(y_n), w) + 2(1-\alpha_n) d(T^n y_n, y_n)] d(g(y_n), w)}{[1 - 2\gamma(1-\alpha_n)]} \\ &+ \frac{2(1-\alpha_n)\langle \overrightarrow{g(w)w}, \overrightarrow{y_n w} \rangle}{[1 - 2\gamma(1-\alpha_n)]}, \end{aligned} \tag{3.23}$$

which implies from condition (i), (3.19) and (3.21) that

$$\lim_{k\to\infty} d^2(x_{n_k}, w) = 0.$$

Therefore, $\lim_{k\to\infty} x_{n_k} = w$.

Step 5. Lastly, we show that w is a solution of (3.6). From Lemma 2.13(ii) and (3.5), we obtain for all $u \in \Gamma$ that

$$\begin{aligned} d^2(x_m, u) &\leq \alpha_m d^2(g(y_m), u) + (1-\alpha_m) d^2(T^m y_m, u) \\ &\quad - \alpha_m(1-\alpha_m) d^2(g(y_m), T^m y_m) \\ &\leq \alpha_m d^2(g(y_m), u) + (1-\alpha_m) d(x_m, u) \\ &\quad - \alpha_m(1-\alpha_m) d^2(g(y_m), T^m y_m), \end{aligned}$$

which implies

$$d^2(x_m, u) \le d^2(g(y_m), u) - (1-\alpha_m)d^2(g(y_m), T^m y_m).$$

Thus, taking limit as $m \to \infty$, we obtain

$$d^2(w, u) \le d^2(g(w), u) - d^2(g(w), w).$$

Hence,

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle = \frac{1}{2}(d^2(g(w), u) - d^2(w, u) - d^2(g(w), w)) \ge 0, \quad \forall\, u \in \Gamma.$$

Therefore, we have that w solves the variational inequality (3.6).

Now, assume that $\{x_{n_k}\}$ $\triangle$-converges to u. Then, by the same argument, we obtain that $u \in \Gamma$ solves the variational inequality (3.6). That is,

$$\langle \overrightarrow{ug(u)}, \overrightarrow{uw} \rangle \le 0. \quad \text{Also, } \langle \overrightarrow{wg(w)}, \overrightarrow{wu} \rangle \le 0.$$

Now, adding both, we get

$$\begin{aligned} 0 &\ge \langle \overrightarrow{wg(w)}, \overrightarrow{wu} \rangle - \langle \overrightarrow{ug(u)}, \overrightarrow{wu} \rangle \\ &= \langle \overrightarrow{wg(u)}, \overrightarrow{wu} \rangle + \langle \overrightarrow{g(u)g(w)}, \overrightarrow{wu} \rangle - \langle \overrightarrow{uw}, \overrightarrow{wu} \rangle - \langle \overrightarrow{wg(u)}, \overrightarrow{wu} \rangle \\ &= \langle \overrightarrow{wu}, \overrightarrow{wu} \rangle - \langle \overrightarrow{g(u)g(w)}, \overrightarrow{uw} \rangle \\ &\ge \langle \overrightarrow{wu}, \overrightarrow{wu} \rangle - d(g(u)g(w))d(u, w) \\ &\ge d^2(w, u) - \gamma d^2(u, w) = (1-\gamma)d^2(w, u), \end{aligned}$$

which implies that $d(w, u) = 0$. Hence, $w = u$. Therefore, $\{x_n\}$ converges strongly to w, which is a solution of the variational inequality (3.6). □

Corollary 3.3. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $T : C \to C$ be a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is also asymptotically regular. Let g be a contraction mapping on C with coefficient $\gamma \in (0, 1)$. Suppose that $F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$x_n = \alpha_n g(x_n) \oplus (1-\alpha_n)T^n x_n, \quad n \ge 1, \tag{3.24}$$

where $\{\alpha_n\}$ is in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $L < (1-\alpha_n\gamma)/(1-\alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in F(T)$ which solves the variational inequality

$$\langle \overrightarrow{wg(w)}, \overrightarrow{uw} \rangle \ge 0, \quad \forall\, u \in F(T). \tag{3.25}$$

Corollary 3.4. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of cyclic monotone bifunctions satisfying $P1$, $P2$ and $P3$. Let g be a contraction mapping on C with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := \bigcap_{i=1}^{N} EP(f_i, C) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n, \\ x_n = \alpha_n g(y_n) \oplus (1-\alpha_n) y_n, \quad n \geq 1, \end{cases} \tag{3.26}$$

where $0 < \lambda_n^{(i)} \leq \lambda^{(i)}$ $\forall n \geq 1, i = 1, 2, \ldots, N$, and $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $L < (1 - \alpha_n\gamma)/(1-\alpha_n)$.

Then, $\{x_n\}$ converges strongly to $w \in \Gamma$ which solves the variational inequality (3.6).

Theorem 3.5. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f_i : C \times C \to \mathbb{R}$, $i = 1, 2, \ldots, N$ be a finite family of cyclic monotone bifunctions satisfying $P1$, $P2$ and $P3$. Let $T : C \to C$ be a nonexpansive mapping and g be a contraction mapping on C with coefficient $\gamma \in (0,1)$. Suppose that $\Gamma := \bigcap_{i=1}^{N} EP(f_i, C) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}} x_n, \\ x_{n+1} = \alpha_n g(y_n) \oplus (1-\alpha_n) T y_n, \quad n \geq 1, \end{cases} \tag{3.27}$$

where $0 < \lambda_n^{(i)} \leq \lambda^{(i)}$ $\forall n \geq 1, i = 1, 2, \ldots, N$, and $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,
(iii) $\sum_{n=1}^{\infty} \left(\sqrt{1 - \frac{(\lambda_{n-1}^{(i)})}{(\lambda_n^{(i)})}}\right) < \infty$, $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $w \in \Gamma$.

Proof. First, we show that $\{x_n\}$ is bounded. Let $v \in \Gamma$, then by (3.27) and Lemma 2.13(i), we obtain

$$\begin{aligned} d(x_{n+1}, v) &\leq \alpha_n d(g(y_n), v) + (1-\alpha_n) d(Ty_n, v) \\ &\leq \alpha_n \gamma d(y_n, v) + \alpha_n d(g(v), v) + (1-\alpha_n) d(y_n, v) \\ &\leq (1 - \alpha_n(1-\gamma)) d(x_n, v) + \alpha_n d(g(v), v) \end{aligned}$$

$$\begin{aligned}&\le \max\left\{d(x_n, v), \frac{d(g(v), v)}{1-\gamma}\right\}\\ &\ \vdots\\ &\le \max\left\{d(x_1, v), \frac{d(g(v), v)}{1-\gamma}\right\}.\end{aligned} \tag{3.28}$$

Thus, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{Ty_n\}$ and $\{g(y_n)\}$ are all bounded.

Next, we show that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Let $u_n^{(i+1)}$ be as defined in the proof of Lemma 3.1. We may assume without loss of generality that $\lambda_{n-1}^{(i)} \le \lambda_n^{(i)}$, $i = 1, 2, \ldots, N$, $n \ge 1$. Thus, by Lemma 2.12(ii), we obtain

$$\begin{aligned} d(u_n^{(i+1)}, u_{n-1}^{(i+1)}) &\le d(J_{\lambda_n^{(i)}} u_n^{(i)}, J_{\lambda_n^{(i)}} u_{n-1}^{(i)}) + d(J_{\lambda_n^{(i)}} u_{n-1}^{(i)}, J_{\lambda_{n-1}^{(i)}} u_{n-1}^{(i)})\\ &\le d(u_n^{(i)}, u_{n-1}^{(i)}) + \left(\sqrt{1-\frac{(\lambda_{n-1}^{(i)})}{(\lambda_n^{(i)})}}\right) d(u_{n-1}^{(i)}, J_{\lambda_{n-1}^{(i)}} u_{n-1}^{(i)})\\ &\le d(J_{\lambda_n^{(i-1)}} u_n^{(i-1)}, J_{\lambda_n^{(i-1)}} u_{n-1}^{(i-1)}) + d(J_{\lambda_n^{(i-1)}} u_{n-1}^{(i-1)}, J_{\lambda_{n-1}^{(i-1)}} u_{n-1}^{(i-1)})\\ &\quad + \left(\sqrt{1-\frac{(\lambda_{n-1}^{(i)})}{(\lambda_n^{(i)})}}\right) d(u_{n-1}^{(i)}, J_{\lambda_{n-1}^{(i)}} u_{n-1}^{(i)})\\ &\le d(u_n^{(i-1)}, u_{n-1}^{(i-1)}) + \left(\sqrt{1-\frac{(\lambda_{n-1}^{(i-1)})}{(\lambda_n^{(i-1)})}}\right) d(u_{n-1}^{(i-1)}, J_{\lambda_{n-1}^{(i-1)}} u_{n-1}^{(i-1)})\\ &\quad + \left(\sqrt{1-\frac{(\lambda_{n-1}^{(i)})}{(\lambda_n^{(i)})}}\right) d(u_{n-1}^{(i)}, J_{\lambda_{n-1}^{(i)}} u_{n-1}^{(i)})\\ &\le d(u_n^{(i-(N-1))}, u_{n-1}^{(i-(N-1))}) + \sum_{j=0}^{N-1}\left(\sqrt{1-\frac{(\lambda_{n-1}^{(i-j)})}{(\lambda_n^{(i-j)})}}\right)\\ &\quad \times d(u_{n-1}^{(i-j)}, J_{\lambda_{n-1}^{(i-j)}} u_{n-1}^{(i-j)}). \end{aligned} \tag{3.29}$$

Again, we obtain from (3.27) and Lemma 2.13 that

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n g(y_n) \oplus (1-\alpha_n)Ty_n, \alpha_{n-1} g(y_{n-1}) \oplus (1-\alpha_{n-1})Ty_{n-1})\\ &\le d(\alpha_n g(y_n) \oplus (1-\alpha_n)Ty_n, \alpha_n g(y_{n-1}) \oplus (1-\alpha_n)Ty_{n-1})\\ &\quad + d(\alpha_n g(y_{n-1}) \oplus (1-\alpha_n)Ty_{n-1}, \alpha_{n-1} g(y_{n-1}) \oplus (1-\alpha_{n-1})Ty_{n-1}) \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n d(g(y_n),g(y_{n-1})) + (1-\alpha_n) d(Ty_n, Ty_{n-1}) \\
&\quad + |\alpha_n - \alpha_{n-1}| d(g(y_{n-1}), Ty_{n-1}) \\
&\le (1-\alpha_n(1-\gamma)) d(y_n, y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(g(y_{n-1}, Ty_{n-1}).
\end{aligned} \tag{3.30}$$

For $i = N$, we obtain from (3.29) and (3.30) that

$$\begin{aligned}
d(x_{n+1}, x_n) &\le (1-\alpha_n(1-\gamma)) \left[d(x_n, x_{n-1}) + \sum_{j=0}^{N-1} \left(\sqrt{1 - \frac{(\lambda_{n-1}^{(N-j)})}{(\lambda_n^{(N-j)})}} \right) \right. \\
&\qquad \left. \times d(u_{n-1}^{(N-j)}, J_{\lambda_{n-1}^{(N-j)}} u_{n-1}^{(N-j)}) \right] \\
&\qquad + |\alpha_n - \alpha_{n-1}| d(g(y_{n-1}), Ty_{n-1}) \\
&\le (1-\alpha_n(1-\gamma)) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(g(y_{n-1}), Ty_{n-1}) \\
&\qquad + \sum_{j=0}^{N-1} \left(\sqrt{1 - \frac{(\lambda_{n-1}^{(N-j)})}{(\lambda_n^{(N-j)})}} \right) d(u_{n-1}^{(N-j)}, J_{\lambda_{n-1}^{(N-j)}} u_{n-1}^{(N-j)}) \\
&\le (1-\alpha_n(1-\gamma)) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(g(y_{n-1}), Ty_{n-1}) \\
&\qquad + \sum_{j=0}^{N-1} \left(\sqrt{1 - \frac{(\lambda_{n-1}^{(N-j)})}{(\lambda_n^{(N-j)})}} \right) M,
\end{aligned} \tag{3.31}$$

where $M := \sup_{n\ge 1}\{\sum_{j=0}^{N-1} d(u_{n-1}^{(N-j)}, J_{\lambda_n^{(N-j)}} u_{n-1}^{(N-j)})\}$. Thus, using conditions (i)–(iii) of Theorem 3.5 and Lemma 2.17 in (3.31), we obtain that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.32}$$

We now show that $\lim_{n\to\infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0,\ i = 1, 2, \ldots, N$, and $\lim_{n\to\infty} d(y_n, Ty_n) = 0$.

We obtain from (3.27) that

$$\begin{aligned}
d(x_{n+1}, Ty_n) &= d(\alpha_n g(y_n) \oplus (1-\alpha_n) Ty_n, Ty_n) \\
&\le \alpha_n d(g(y_n), Ty_n) \to 0, \quad \text{as } n \to \infty.
\end{aligned} \tag{3.33}$$

Also, from Lemma 2.13(ii), we obtain

$$\begin{aligned}
d^2(x_{n+1}, v) &= d^2(\alpha_n g(y_n) \oplus (1-\alpha_n) Ty_n, v) \\
&\le \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(Ty_n, v) \\
&\le \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(y_n, v).
\end{aligned} \tag{3.34}$$

By Lemma 2.12(i), we have

$$d^2(u_n^{(i+1)}, v) \le d^2(u_n^{(i)}, v) - d^2(u_n^{(i)}, u_n^{(i+1)}). \tag{3.35}$$

For $i = N$, we obtain from (3.34) and (3.35) that

$$\begin{aligned} d^2(x_{n+1}, v) &\le \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N+1)}, v) \\ &\le \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(u_n^{(N)}, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &\le \alpha_n d^2(g(y_n), v) + (1-\alpha_n) d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &= \alpha_n (d^2(g(y_n), v) - d^2(x_n, v)) + d^2(x_n, v) - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \\ &\le \alpha_n (d^2(g(y_n), v) - d^2(x_n, v)) + d^2(x_n, x_{n+1}) \\ &\quad + 2d(x_n, x_{n+1}) d(x_{n+1}, v) + d^2(x_{n+1}, v) \\ &\quad - (1-\alpha_n) d^2(u_n^{(N)}, u_n^{(N+1)}) \end{aligned}$$

which implies from (3.32) and condition (i) of Theorem 3.5 that

$$\lim_{n\to\infty} d^2(u_n^{(N)}, u_n^{(N+1)}) = 0. \tag{3.36}$$

Similarly, from (3.34), (3.35), (3.32) and condition (i) of Theorem 3.5, we can show that

$$\lim_{n\to\infty} d(u_n^{(i)}, u_n^{(i+1)}) = 0, \quad i = 1, 2, \ldots, N. \tag{3.37}$$

From (3.37), and applying triangle inequality, we obtain for each $i = 1, 2, \ldots, N$, that

$$\lim_{n\to\infty} d(x_n, u_n^{(i+1)}) = 0. \tag{3.38}$$

Thus, for $i = N$, we have

$$\lim_{n\to\infty} d(x_n, y_n) = 0. \tag{3.39}$$

Therefore, applying Lemma 3.1, we obtain

$$\lim_{n\to\infty} d(J_{\lambda^{(i)}} x_n, x_n) = 0, \quad i = 1, 2, \ldots, N. \tag{3.40}$$

Furthermore, we obtain from (3.33), (3.32) and (3.39) that

$$\lim_{n\to\infty} d(y_n, Ty_n) = 0. \tag{3.41}$$

Finally, we show that $\{x_n\}$ converges strongly to some point, say $w \in \Gamma$.

By the same argument as in the proof of Theorem 3.2, we obtain that $w \in \Gamma$ and

$$\lim_{k\to\infty} \langle \overrightarrow{g(w)w}, \overrightarrow{y_{n_k} w} \rangle \le 0. \tag{3.42}$$

Using (3.32) and following similar argument as in the proof of Theorem 3.2 (Step 4), we can show that

$$\lim_{k\to\infty} d^2(x_{n_k}, w) = 0.$$

Hence, $\lim_{k\to\infty} x_{n_k} = w$. Therefore, $\{x_n\}$ converges strongly to $w \in \Gamma$. □

By setting, $N = 1$, $T \equiv I$ and $g(x) = u$ for arbitrary but fixed $u \in C$ and for all $x \in C$, we obtain the following corollary.

Corollary 3.6. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $f : C \times C \to \mathbb{R}$ be a cyclic monotone bifunctions satisfying $P1$, $P2$ and $P3$. Suppose that $EP(f, C) \neq \emptyset$ and for arbitrary $x_1, u \in C$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n)J_{\lambda_n} x_n, \quad n \geq 1, \tag{3.43}$$

where $0 < \lambda_n \leq \lambda \; \forall n \geq 1$ and $\{\alpha_n\}$ is in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,
(iii) $\sum_{n=1}^{\infty} (\sqrt{1 - \frac{(\lambda_{n-1})}{(\lambda_n)}}) < \infty$.

Then, $\{x_n\}$ converges strongly to $w \in EP(f, C)$.

4. Applications

In this section, we give applications of our results to some well-known optimization problems. Throughout this section, X is an Hadamard space and C is a nonempty closed and convex subset X.

4.1. *Minimization problem*

Definition 4.1. A function $h : C \to (-\infty, \infty]$ is called

(i) *convex*, if

$$h(\lambda x \oplus (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y) \quad \forall x,\ y \in C,\ \lambda \in (0, 1),$$

(ii) *proper*, if $D(h) \neq \emptyset$,
(iii) *lower semicontinuous at a point $x \in D(h)$*, if

$$h(x) \leq \liminf_{n\to\infty} h(x_n), \quad \text{for each sequence } \{x_n\} \text{ in } D(f) \text{ such that } \lim_{n\to\infty} x_n = x.$$

Moreover, h is said to be lower semicontinuous on $D(h)$, if it is lower semicontinuous at any point in $D(h)$.

Let $h : C \to \mathbb{R}$ be a proper convex and lower semicontinuous function. The minimization problem is to find a point $x \in C$ such that

$$h(x) = \min_{u \in C} h(u). \tag{4.1}$$

We denote the set of solutions of problem (4.1) by $\arg\min_{u\in C} h(u)$. Minimization problems have been studied in Hadamard spaces by numerous authors (see for example [2–4, 23, 41, 46]), as well as in p-uniformly convex metric spaces (see [13, 25, 50, 51] and the references therein).

Now, consider the bifunction $f_h : C \times C \to \mathbb{R}$ defined by

$$f_h(x, y) = h(y) - h(x), \quad \forall\, x, y \in C.$$

It is known from [33] that $\mathrm{EP}(f_h, C) = \arg\min_{u\in C} h,\ J_\lambda^{f_h} = \mathrm{prox}_\lambda^h,\ \lambda > 0$ and $D(\mathrm{prox}_\lambda^h) = X$. Now, consider the following finite family of minimization problem and fixed point problem:

$$\text{Find } x \in F(T) \quad \text{such that } h_i(x) \le h_i(y), \quad \forall\, y \in C,\ i = 1, 2 \ldots, N, \tag{4.2}$$

where T is either a uniformly L-Lipschitzian generalized asymptotically nonspreading mapping which is also asymptotically regular or a nonexpansive mapping. Thus, by setting $J_{\lambda_n} = \mathrm{prox}_{\lambda_n}$ in either Algorithm (3.5) or Algorithm (3.27), we can apply Theorem 3.2 or Theorem 3.5 (respectively), to approximate solutions of problem (4.2).

4.2. *Convex feasibility problem*

Let $C_i, i = 1, 2, \ldots, N$ be a finite family of nonempty, closed and convex subsets of C such that $\bigcap_{i=1}^N C_i \neq \emptyset$. A convex feasibility problem is defined as

$$\text{Find } x \in F(T) \quad \text{such that } x \in \bigcap_{i=1}^{N} C_i. \tag{4.3}$$

We know that the indicator function $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases}$$

is a proper, convex and lower semicontinuous function. Thus, by letting $\delta_C = h$ and following similar argument as above, we obtain that $J_\lambda^{f_{\delta_C}} = \mathrm{prox}_\lambda^{\delta_C} = P_C$. Therefore, by setting $J_{\lambda_n}^{f_i} = P_{C_i},\ i = 1, 2, \ldots, N$ in either Algorithm 3.5 or Algorithm 3.27, we can apply either Theorem 3.2 or Theorem 3.5 to approximate solutions of problem (4.3).

Acknowledgments

The authors sincerely thank the anonymous reviewer for his careful reading, constructive comments and useful suggestions. The second and third authors acknowledge with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. The fourth author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant No. 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the COE-MASS and NRF.

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