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Application of Fixed Point Results for Modified Generalized F-Contraction Mappings to Solve Boundary Value Problems

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Abstract

The aim of this paper is to introduce the notion of (α, β)-cyclic admissible mapping and modified generalized F-contraction mapping in the framework of metric-like spaces. Fixed point theorems for modified generalized F-contraction mapping in complete metric-like spaces are established. Furthermore, we present examples to support our main results, using this examples, we establish that our main results is a generalization of the fixed point result of Karapinar et al. [Fixed points of conditionally F-contractions in complete metric-like spaces, Fixed Point Theory and Appl. 2015] and a host of others in the literature. As an application, we apply our result to find the existence of solution of second order differential equations. Finally, we correct an anomaly detected in the work of Karapinar et al.[17]. Our results improve and extend corresponding results in the literature.

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1 Introduction

For the past three to four decades, the theory of fixed point has played an important role in nonlinear functional analysis and known to be very useful in establishing the existence and uniqueness theorems for nonlinear differential and integral equations. In fact, the theory of fixed point has been applied to solve many real life problems, for instance; equilibrium problems, variational inequalities, and optimization problems. Banach [7] in 1922 proved the well celebrated Banach contraction principle in the frame work of metric spaces. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. Due to its importance and fruitful applications, researchers in this area generalize the concept by considering classes of nonlinear mappings and spaces which are more general than contraction mappings and metric spaces, respectively (see [1, 16, 24] and the references therein). For example, Suzuki [25] introduced a class of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for this class of mappings.

Definition 1.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.2. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying condition (C) for all $x, y \in X$. Then T has a unique fixed point.*

In 2012, Wardowski [27] introduced the notion of F-contractions. This class of mappings is defined as follows:

Definition 1.3. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;*

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:
(F_1) *F is strictly increasing;*
(F_2) *for all sequences $\{\alpha_n\} \subseteq \mathbb{R}^+, \lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;*
(F_3) *there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.*

He also established the following result:

Theorem 1.4. *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$, the sequence $\{T^n x_0\}$ converges to x^*.*

Remark 1.5. *[27] If we suppose that $F(t) = \ln t$, an F-contraction mapping becomes the Banach contraction mapping.*

In [19], Piri et al. used the continuity condition instead of condition (F_3) and proved the following result:

Theorem 1.6. *Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. *Then T has a unique fixed point $z \in X$, and for every $x \in X$, the sequence $\{T^n x\}$ converges to z.*

Secelean in [22] proved the following lemma.

Lemma 1.7. *[22] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold:*

1. *if* $\lim_{n\to\infty} F(\alpha_n) = -\infty$ *then* $\lim_{n\to\infty} \alpha_n = 0$;
2. *if* $\inf F = -\infty$ *and* $\lim_{n\to\infty} \alpha_n = 0$ *then* $\lim_{n\to\infty} F(\alpha_n) = -\infty$.

Furthermore, the authors in [22] replaced the condition F_2 in the definition of F-contraction with the following condition.
(F_*) $\inf F = -\infty$
or, also by
(F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$.
We denote by $\mathcal{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy conditions
$(F_1^{'})$ F is strictly increasing,
$(F_2^{'})$ $\inf F = -\infty$,
or, also by,
$(F_3^{'})$ there exists a sequence $\{\alpha_n\}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$,
$(F_4^{'})$ F is continuous on $(0, \infty)$. Samet et al. [23] introduced the notion of α-admissible mapping and obtain some fixed point results for this class of mappings.

Definition 1.8. *[23] Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that a self mapping $T : X \to X$ is α-admissible if for all $x, y \in X$,*

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.9. *[16] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is a triangular α-admissible if*

1. *T is α-admissible and*
2. *$\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$.*

Theorem 1.10. *[23] Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions hold:*

1. *for all* $x, y \in X$, *we have* $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, *where* $\psi : [0, \infty) \to [0, \infty)$ *is a nondecreasing function such that* $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ *for all* $t > 0$;

2. *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$;

3. *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\alpha(x_n, x) \geq 1$.

Then T *has a fixed point.*

In 2016, Chandok et al. [12] introduced another class of mappings, called the TAC-contractive and established some fixed point results in the frame work of complete metric spaces.

Definition 1.11. *Let* $T : X \to X$ *be a mapping and let* $\alpha, \beta : X \to \mathbb{R}^+$ *be two functions. Then* T *is called a cyclic* (α, β)*-admissible mapping, if*

1. $\alpha(x) \geq 1$ *for some* $x \in X$ *implies that* $\beta(Tx) \geq 1$,

2. $\beta(x) \geq 1$ *for some* $x \in X$ *implies that* $\alpha(Tx) \geq 1$.

Definition 1.12. *Let* (X, d) *be a metric space and let* $\alpha, \beta : X \to [0, \infty)$ *be two mappings. We say that* T *is a TAC-contractive mapping, if for all* $x, y \in X$,

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))),$$

where ψ *is a continuous and nondecreasing function with* $\psi(t) = 0$ *if and only if* $t = 0$, ϕ *is continuous with* $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ *and* $f : [0, \infty)^2 \to \mathbb{R}$ *is continuous,* $f(a, t) \leq a$ *and* $f(a, t) = a \Rightarrow a = 0$ *or* $t = 0$ *for all* $s, t \in [0, \infty)$.

Theorem 1.13. *Let* (X, d) *be a complete metric space and let* $T : X \to X$ *be a cyclic* (α, β)*-admissible mapping. Suppose that* T *is a TAC contraction mapping. Assume that there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ *and either of the following conditions hold:*

1. T *is continuous,*

2. *if for any sequence* $\{x_n\}$ *in* X *with* $\beta(x_n) \geq 1$, *for all* $n \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x) \geq 1$.

In addition, if $\alpha(x) \geq 1$ *and* $\beta(y) \geq 1$ *for all* $x, y \in F(T)$ *(where* $F(T)$ *denotes the set of fixed points of* T*), then* T *has a unique fixed point.*

Question. Is it possible to generalize the concept of α-admissible mapping using the concept of cyclic (α, β)-admissible mapping?

One of the interesting generalization of metric space and partial metric space is the concept of metric-like space introduced by Amini-Harandi in [5]. He proved some fixed point theorems and gave σ-completeness in this frame work. Thereafter, several papers have been published on the fixed point theory of various classes of single-valued and multi-valued operators in the frame work of metric-like spaces (see[2, 3, 4, 6] and the references therein). In particular Karapinar et al. [17] introduced the notion of conditionally F-contraction and studied the fixed point theorem of such mappings in the frame work of metric-like spaces.

Definition 1.14. *[17] Let (X,σ) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (A) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with*

$$\sigma(Tx,Ty) > 0, \frac{1}{2}\sigma(x,Tx) < \sigma(x,y) \Rightarrow \tau + F(\sigma(Tx,Ty)) \leq F(M_T(x,y)),$$

where

$$M_T(x,y) = \max\left\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty), \frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\right\}.$$

Definition 1.15. *[17] Let (X,σ) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (B) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with*

$$\sigma(Tx,Ty) > 0, \frac{1}{2}\sigma(x,Tx) < \sigma(x,y) \Rightarrow$$

$$\tau + F(\sigma(Tx,Ty)) \leq F(\max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty)\}).$$

Theorem 1.16. *[17] Let (X,σ) be a complete metric-like space. If T is a conditionally F-contraction of type (A), then T has a fixed point $x^* \in X$.*

Theorem 1.17. *[17] Let (X,σ) be a complete metric-like space. If T is a conditionally F-contraction of type (B), then T has a fixed point $x^* \in X$.*

Claim. We claim that Definition 1.14 and Definition 1.15 are the same, then we conclude that Theorem 1.16 and Theorem 1.17 are just repetition.
Inspired by the work of Wardowski [27], Piri et al. [19], Samet et al. [23] and Chandok et al. [12], Karapinar et al. [17], we provide and affirmative answer to the question raised by introducing the concept of (α,β)-cyclic admissible mapping, we also introduce the concept of modified generalized F-contraction mappings in the frame work of metric-like spaces. In addition, we establish the existence and uniqueness theorems of fixed points for modified generalized F-contraction in this frame work, present some examples to support our main results, apply our fixed point result to establish the existence of solution of second order differential equation. Finally, we establish that the above claim is true using analytical approach with an example to validate our claim.

2 Preliminaries

In this section, we recall some results and definitions for our main results.

Definition 2.1. *[18] Let X be a nonempty set and $p : X \times X \to [0, \infty]$ be a function satisfying the following conditions, for all $x, y, z \in X$,*

1. $p(x, x) = p(y, y) = p(x, y)$ *if and only if* $x = y$,
2. $p(x, x) \leq p(x, y)$,
3. $p(x, y) = p(y, x)$,
4. $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then p is said to be a partial metric on X and the pair (X, p) is called a partial metric space.

An example of a partial metric space is the pair (X, p), where $X = [0, \infty)$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. One of the main properties of the metric-like spaces that generalizes metric spaces is the non-zero self distance properties.

Definition 2.2. *[5] Let X be a nonempty set. A function $\sigma : X \times X \to [0, \infty]$ is said to be a metric-like on X if for all $x, y, z \in X$, the following conditions hold:*

1. $\sigma(x, y) = 0 \Rightarrow x = y$,
2. $\sigma(x, y) = \sigma(y, x)$,
3. $\sigma(x, y) \leq \sigma(x, z) + \sigma(z, y)$.

Then the pair (X, σ) is called a metric-like space.

It is well-known that every metric space is a partial metric space and each partial metric space is a metric-like space, but the converse may not be true.

Example 2.3. *[5] Let $X = \{0, 1\}$ and $\sigma : X \times X \to [0, \infty]$ is defined by*

$$\sigma(x, y) = \begin{cases} 2 & \textit{if} \quad x = y = 0 \\ 1 & \textit{otherwise.} \end{cases}$$

Then (X, σ) is a metric-like space, but it is neither a metric space nor a partial metric space, since $\sigma(0, 0) > \sigma(0, 1)$.

Example 2.4. *Let $X = \{1, 2, 3\}$ and $\sigma : X \times X \to [0, \infty]$ is defined by*

$$\sigma(1,1) = 0, \quad \sigma(1,2) = \sigma(2,1) = \frac{11}{13}, \quad \sigma(1,3) = \sigma(3,1) = \frac{9}{13}$$
$$\sigma(2,2) = \frac{12}{13}, \quad \sigma(2,3) = \sigma(3,2) = \frac{10}{13}, \quad \sigma(3,3) = 1$$

Then (X, σ) is a metric-like space, but it is neither a metric space nor a partial metric space, since $\sigma(2, 2) > \sigma(1, 2)$.

Definition 2.5. *[5] Let (X, σ) be a metric-like space. Then:*

1. *A sequence $\{x_n\}$ in X converges to $x \in X$ if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$.*
2. *A sequence $\{x_n\}$ in X is called Cauchy in X if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite.*
3. *A metric-like space X is said to be complete if and only every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$ so that*
$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x).$$
4. *A mapping $T : X \to X$ is σ-continuous if for any sequence $\{x_n\}$ in X such that*
$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x),$$
we have
$$\lim_{n\to\infty} \sigma(Tx_n, Tx) = \sigma(Tx, Tx),$$

Lemma 2.6. *[15] Let (X, σ) be a metric-like space. Then:*

1. *if $\sigma(x, y) = 0$ then $\sigma(x, x) = \sigma(y, y) = 0$,*
2. *if $\{x_n\}$ is a sequence such that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$, then*
$$\lim_{n\to\infty} \sigma(x_n, x_n) = \lim_{n\to\infty} \sigma(x_{n+1}, x_{n+1}) = 0.$$

Lemma 2.7. *[5] If $\{x_n\}$ is a sequence in a metric-like space (X, σ) such that $x_n \to x$ and $x_n \to y$, then $x = y$.*

3 Main Result

In this section, we introduce the concept of (α, β)-cyclic admissible mapping, modified generalized F-contraction mapping in the frame work of metric-like spaces and prove the existence and uniqueness theorems of fixed points for modified generalized F-contraction mappings in the frame work of complete metric-like space.

Definition 3.1. *Let X be a nonempty set, $T : X \to X$ be a mapping and $\alpha, \beta : X \times X \to \mathbb{R}^+$ be two functions. We say that T is an (α, β)-cyclic admissible mapping, if for all $x, y \in X$*

1. *$\alpha(x, y) \geq 1 \Rightarrow \beta(Tx, Ty) \geq 1$,*
2. *$\beta(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.*

Remark 3.2. *Clearly, if $\beta(x, y) = \alpha(x, y)$, we obtain Definition 1.8.*

Definition 3.3. *Let (X, σ) be a metric-like space, $\alpha, \beta : X \times X \to [0, \infty)$ be two functions and T be a self map on X. The mapping T is said to be a modified generalized F-contraction mapping, if there exists $F \in \mathcal{F}$ and $\tau > 0$ such that for all $x, y \in X$,*

$$\alpha(x, Tx)\beta(y, Ty) \geq 1 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\psi(M(x, y))) \tag{2}$$

with $\sigma(Tx, Ty) > 0$, where $M(x, y) = \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\}, \psi : [0, \infty) \to [0, \infty)$, such that

1. *ψ is monotonic increasing, i.e., $t_1 \leq t_2 \Rightarrow \psi(t_1) \leq \psi(t_2)$.*

2. *ψ is continuous and $\psi(t) < t$ for each $t > 0$, and that $\psi(0) = 0$.*

Example 3.4. *Let $X = [0, \infty)$ and $\sigma : X \times X \to [0, \infty)$ be defined as $\sigma(x, y) = \max\{x, y\}$ for all $x, y \in X$. It is clear that (X, σ) is a metric-like space. We defined $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x}{7} & if \quad x \in [0, 1] \\ 8x & if \quad x \in (1, \infty), \end{cases}$$

$\alpha, \beta : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if \quad x, y \in [0, 1] \\ 0 & if \quad x, y \in (1, \infty), \end{cases}$$

$$\beta(x, y) = \begin{cases} 2 & if \quad x, y \in [0, 1] \\ 0 & if \quad x, y \in (1, \infty), \end{cases}$$

$\psi : [0, \infty) \to [0, \infty)$, defined by $\psi(t) = \frac{t}{2}$ and $F(t) = -\frac{1}{t} + t$. Then T is a modified generalized F-contraction but not a conditionally F-contraction of type (A) and type (B) as defined by Karapnar et al.[16].

Proof. It is easy to see that for any $x, y \in [0, 1]$, we have that $\alpha(x, Tx) = 1$ and $\beta(y, Ty) = 2$ as such we have that $\alpha(x, Tx)\beta(y, Ty) > 1$. Since $\alpha(x, Tx)\beta(y, Ty) > 1$ if $x, y \in [0, 1]$. For other cases, it is easy to see that $\alpha(x, Tx)\beta(y, Ty) = 0$ as such, we need to show that $\tau + F(\sigma(Tx, Ty)) \leq F(\psi(M(x, y)))$ for any $x, y \in [0, 1]$. Let $x, y \in [0, 1]$ and without loss of generality we suppose that $x \leq y$. Note that

$$\begin{aligned} M(x, y) &= \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\} \\ &= \max\{\sigma(x, y), \sigma(x, \frac{x}{7}), \sigma(y, \frac{y}{7})\} \\ &= \max\{y, x, y\} = y. \end{aligned}$$

Observe that, for $\tau = 1$, we obtain

$$\begin{aligned}\tau + F(\sigma(Tx, Ty)) &= 1 + F(\sigma(\frac{x}{7}, \frac{y}{7})) \\ &= 1 + F(\max\{\frac{x}{7}, \frac{y}{7}\}) = 1 + F(\frac{y}{7}) \\ &= 1 + \frac{y}{7} - \frac{7}{y}\end{aligned}$$

and

$$F(\psi(M(x, y))) = F(\psi(y)) = F(\frac{y}{2}) = \frac{y}{2} - \frac{2}{y}.$$

Thus, for any $x, y \in (0, 1]$, we have that

$$\tau + F(\sigma(Tx, Ty)) \leq F(\psi(M(x, y))).$$

Hence, T is a modified generalized F contraction.
However to show T is not a conditionally F-contraction of type (A) and type (B) as defined by Karapnar et al.[16]. Suppose $x = 0$ and $y = 2$. We have that Note that

$$M_T(x, y) = \max\left\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{\sigma(x, Ty) + \sigma(y, Tx)}{4}\right\} = 16$$

and

$$\max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\} = 16.$$

Now observe that

$$\begin{aligned}\sigma(Tx, Ty) = \sigma(0, 16) = 16 > 0 \text{ and } \quad \frac{1}{2}\sigma(x, Tx) &= \frac{1}{2}\sigma(0, 0) = 0 < 2 \\ &= \max\{0, 2\} = \sigma(x, y)\end{aligned}$$

but

$$\begin{aligned}\tau + F(d(Tx, Ty)) = 1 + F(\sigma(0, 16)) &= 1 + 16 - \frac{1}{16} = 17 - \frac{1}{16} \\ &> 16 - \frac{1}{16} = F(M_T(x, y)).\end{aligned}$$

Also, we have

$$\begin{aligned}\tau + F(d(Tx, Ty)) = 1 + F(\sigma(0, 16)) &= 1 + 16 - \frac{1}{16} = 17 - \frac{1}{16} > 16 - \frac{1}{16} \\ &= F(\max\{\sigma(x, y), \sigma(x, Tx, \sigma(y, Ty)\}).\end{aligned}$$

□

Lemma 3.5. *Let X be a nonempty set and $T : X \to X$ be an (α, β)-cyclic admissible mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$. Define the sequence $x_{n+1} = Tx_n$, then $\alpha(x_m, x_{m+1}) \geq 1$ implies that $\beta(x_n, x_{n+1}) \geq 1$ and $\beta(x_m, x_{m+1}) \geq 1$ implies that $\alpha(x_n, x_{n+1}) \geq 1$, for all $n, m \in \mathbb{N} \cup \{0\}$ with $m < n$.*

Proof. Using the fact that T is an (α, β)-cyclic admissible mapping and our hypothesis, we have that there exists $x_0 \in X$ such that

$$\alpha(x_0, Tx_0) = \alpha(x_0, x_1) \geq 1 \Rightarrow \beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1$$

and

$$\beta(x_1, x_2) \geq 1 \Rightarrow \alpha(Tx_1, Tx_2) = \alpha(x_2, x_3) \geq 1.$$

Continuing this way, we obtain that

$$\alpha(x_{2n}, x_{2n+1}) \geq 1 \text{ and } \beta(x_{2n+1}, x_{2n+2}) \geq 1, \ \forall n \in \mathbb{N}.$$

Using similar approach, we obtain

$$\beta(x_{2n}, x_{2n+1}) \geq 1 \text{ and } \alpha(x_{2n+1}, x_{2n+2}) \geq 1, \forall n \in \mathbb{N}.$$

In similar sense, we obtain the same result for all $m \in \mathbb{N}$. That is

$$\alpha(x_{2m}, x_{2m+1}) \geq 1 \text{ and } \beta(x_{2m+1}, x_{2m+2}) \geq 1$$

and

$$\beta(x_{2m}, x_{2m+1}) \geq 1 \text{ and } \alpha(x_{2m+1}, x_{2m+2}) \geq 1, \forall m \in \mathbb{N}.$$

In addition, since

$$\alpha(x_m, x_{m+1}) \geq 1 \Rightarrow \beta(x_{m+1}, x_{m+2}) \geq 1 \Rightarrow \alpha(x_{m+2}, x_{m+3}) \geq 1 \cdots$$

with $m < n$, we deduce that

$$\alpha(x_m, x_{m+1}) \geq 1 \Rightarrow \beta(x_n, x_{n+1}) \geq 1.$$

Using similar approach, we have that

$$\beta(x_m, x_{m+1}) \geq 1 \Rightarrow \alpha(x_n, x_{n+1}) \geq 1.$$

□

Lemma 3.6. *Suppose that (X, σ) is a metric-like space and $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$ If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k \geq k$ such that $\sigma(x_{m_k}, x_{n_k}) \geq \epsilon, \sigma(x_{m_{k-1}}, x_{n_k}) < \epsilon$ and*

1. $\lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) = \epsilon$,
2. $\lim_{k\to\infty} \sigma(x_{m_{k+1}}, x_{n_k}) = \epsilon$,
3. $\lim_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
4. $\lim_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_{k+1}}) = \epsilon$,
5. $\lim_{k\to\infty} \sigma(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon$.

Proof. If $\{x_n\}$ is not a Cauchy sequence then there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k \geq k$ such that $\sigma(x_{m_k}, x_{n_k}) \geq \epsilon$. We choose m_k, the least positive integer satisfying $\sigma(x_{m_k}, x_{n_k}) \geq \epsilon$. We then obtain that $m_k > n_k > k$ with

$$\sigma(x_{m_k}, x_{n_k}) \geq \epsilon \quad \text{and} \quad \sigma(x_{m_{k-1}}, x_{n_k}) \leq \epsilon. \tag{3}$$

We now establish (1). Using triangular inequality and (3), we have that

$$\epsilon \leq \sigma(x_{m_k}, x_{n_k}) \leq \sigma(x_{m_k}, x_{m_{k-1}}) + \sigma(x_{m_{k-1}}, x_{n_k}).$$

Taking $\lim_{k\to\infty}$ and using our hypothesis that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$, we have that

$$\epsilon \leq \lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) \leq \epsilon$$

and by Sandwich theorem, we obtain

$$\lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) = \epsilon.$$

Hence (1) holds.
(2). Now observe that

$$\sigma(x_{m_k}, x_{n_k}) \leq \sigma(x_{m_k}, x_{m_{k+1}}) + \sigma(x_{m_{k+1}}, x_{n_k}),$$

using our hypothesis and taking the $\liminf_{k\to\infty}$, we have that

$$\epsilon \leq \liminf_{k\to\infty} \sigma(x_{m_{k+1}}, x_{n_k}). \tag{4}$$

Also, we have that

$$\sigma(x_{m_{k+1}}, x_{n_k}) \leq \sigma(x_{m_{k+1}}, x_{m_k}) + \sigma(x_{m_k}, x_{n_k}),$$

using our hypothesis and taking the $\limsup_{k\to\infty}$, we have that

$$\limsup_{k\to\infty} \sigma(x_{m_{k+1}}, x_{n_k}) \leq \epsilon. \tag{5}$$

From (4) and (5), we obtain

$$\lim_{k\to\infty} \sigma(x_{m_{k+1}}, x_{n_k}) = \epsilon.$$

Hence (2) holds.
(3). Since $\sigma(x_{m_{k-1}}, x_{n_k}) < \epsilon$, we have that

$$\limsup_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_k}) \leq \epsilon. \tag{6}$$

Also, we have that

$$\sigma(x_{m_k}, x_{n_k}) \leq \sigma(x_{m_k}, x_{m_{k-1}}) + \sigma(x_{m_{k-1}}, x_{n_k}),$$

using our hypothesis and taking the $\liminf_{k\to\infty}$, we have that

$$\epsilon \leq \liminf_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_k}). \tag{7}$$

From (6) and (7), we obtain

$$\lim_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_k}) = \epsilon.$$

Hence (3) holds.
(4). Now observe that

$$\sigma(x_{m_k}, x_{n_k}) \leq \sigma(x_{m_k}, x_{m_{k-1}}) + \sigma(x_{m_{k-1}}, x_{n_{k+1}}) + \sigma(x_{n_{k+1}}, x_{n_k}),$$

using our hypothesis and taking the $\liminf_{k\to\infty}$, we have that

$$\epsilon \leq \liminf_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_{k+1}}). \tag{8}$$

Also, we have that

$$\sigma(x_{m_{k-1}}, x_{n_{k+1}}) \leq \sigma(x_{m_{k-1}}, x_{n_k}) + \sigma(x_{n_k}, x_{n_{k+1}}),$$

using our hypothesis and taking the $\limsup_{k\to\infty}$, we have that

$$\limsup_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_{k+1}}) \leq \epsilon. \tag{9}$$

From (8) and (9), we obtain

$$\lim_{k\to\infty} \sigma(x_{m_{k-1}}, x_{n_{k+1}}) = \epsilon.$$

Hence (4) holds.
(5). Using similar argument as (2), (3) and (4), it is easy to see that (5) holds. □

Lemma 3.7. *Let (X, σ) be a metric-like space. Suppose that $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} x_n = x$. Then all $x, y \in X$, we have that*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y).$$

Proof. For all $x, y \in X$, using triangular inequality we obtain

$$\sigma(x_n, y) - \sigma(x, y) \leq \sigma(x_n, x), \tag{10}$$

we also have that

$$\sigma(x, y) - \sigma(x_n, y) \leq \sigma(x, x_n) \tag{11}$$

From (10) and (11), we obtain

$$|\sigma(x_n, y) - \sigma(x, y)| \leq \sigma(x_n, x),$$

taking limit as $n \to \infty$ and using the Sandwich theorem, we obtain the desired result

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y).$$

□

Theorem 3.8. *Let (X, σ) be a complete metric-like space and $T : X \to X$ be a modified generalized F-contraction mapping. Suppose the following conditions hold:*

1. *T is a (α, β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *T is σ-continuous.*

Then T has a fixed point.

Proof. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a (α, β)-cyclic admissible mapping and $\alpha(x_0, x_1) \geq 1$, we have $\beta(Tx_0, Tx_1) = \beta(x_1, x_2) \geq 1$ and this implies that $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$, continuing the process, we have

$$\alpha(x_{2k}, x_{2k+1}) \geq 1 \quad \text{and} \quad \beta(x_{2k+1}, x_{2k+2}) \geq 1 \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{12}$$

Using similar argument, we have that

$$\beta(x_{2k}, x_{2k+1}) \geq 1 \quad \text{and} \quad \alpha(x_{2k+1}, x_{2k+2}) \geq 1 \quad \forall \quad k \in \mathbb{N} \cup \{0\}. \tag{13}$$

It follows from (12) and (13) that $\alpha(x_n, x_{n+1}) \geq 1$ and $\beta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n, x_{n+1})\beta(x_{n+1}, x_{n+2}) \geq 1$, we obtain from (2)

$$\begin{aligned}
\tau + F(\sigma(x_{n+1}, x_{n+2})) &= \tau + F(\sigma(Tx_n, Tx_{n+1})) \\
&\leq F(\psi(M(x_n, x_{n+1}))) \\
&\leq F(\psi \max\{\sigma(x_n, x_{n+1}, \sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2})\}))) \qquad (14) \\
&= F(\psi \max\{\sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2})\}))) \\
&< F(\max\{\sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2})\}).
\end{aligned}$$

If $\max\{\sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2})\} = \sigma(x_{n+1}, x_{n+2})$, we have (14) becomes

$$\tau + F(\sigma(x_{n+1}, x_{n+2})) < F(\sigma(x_{n+1}, x_{n+2})),$$

which is a contradiction since $\tau > 0$. Thus $\max\{\sigma(x_n, x_{n+1}), \sigma(x_{n+1}, x_{n+2})\}) = \sigma(x_n, x_{n+1})$, that is $\sigma(x_n, x_{n+1}) > \sigma(x_{n+1}, x_{n+2})$, which implies that (14) becomes

$$F(\sigma(x_{n+1}, x_{n+2})) < F(\sigma(x_n, x_{n+1})) - \tau.$$

Using similar approach, it is easy to see that

$$F(\sigma(x_n, x_{n+1})) < F(\sigma(x_{n-1}, x_n)) - \tau.$$

Thus by induction, we obtain

$$F(\sigma(x_n, x_{n+1})) < F(\sigma(x_0, x_1)) - n\tau, \quad \forall\ n \in \mathbb{N} \cup \{0\}. \tag{15}$$

Since $F \in \mathcal{F}$, taking limit as $n \to \infty$ in (15), we have

$$\lim_{n\to\infty} F(\sigma(x_n, x_{n+1})) = -\infty. \tag{16}$$

It follows from $(F_3^{'})$ and Lemma 1.7 that

$$\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0. \tag{17}$$

In what follows, we now show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 3.6, there exists an $\epsilon > 0$ and sequences of positive integers $\{x_{n_k}\}$ and $\{x_{m_k}\}$ with $n_k > m_k \geq k$ such that $\sigma(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $\sigma(x_{m_k}, x_{n_k}) \geq \epsilon, \sigma(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and $(1) - (5)$ of Lemma 3.6 hold. Since, $\alpha(x_0, Tx_0) \geq 1$, $\beta(x_0, Tx_0) \geq 1$ and T is (α, β)-cyclic admissible mapping, from Lemma 3.5, we obtain that $\alpha(x_{m_k}, x_{m_{k+1}})\beta(x_{n_k}, x_{n_{k+1}}) \geq 1$. Hence for all $k \geq n_0$, using Lemma 3.6, $(F_4^{'})$, (17) and the properties of ψ we have

$$\begin{aligned}
\tau + F(\sigma(x_{m_{k+1}}, x_{n_{k+1}})) &\leq \tau + F(\sigma(Tx_{m_k}, Tx_{n_k})) \\
&\leq F(\psi(M(x_{m_k}, x_{n_k}))) \qquad (18) \\
&\leq F(\psi(\max \sigma(x_{m_k}, x_{n_k}), \sigma(x_{m_k}, x_{m_{k+1}}), \sigma(x_{n_k}, x_{n_{k+1}}))) \\
&= F(\psi(\max\{\epsilon, 0, 0\})) \\
&< F(\epsilon). \qquad (19)
\end{aligned}$$

That is

$$\tau + F(\epsilon) \leq F(\epsilon)$$

which is a contradiction. As such, we have that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0.$$

We therefore have that $\{x_n\}$ is Cauchy. Since (X,σ) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. From Lemma 3.7, it is easy to see that

$$\lim_{n\to\infty} \sigma(x_n, x_m) = \sigma(x, x_m) \quad \text{and} \quad \lim_{m\to\infty} \sigma(x, x_m) = \sigma(x, x).$$

As such, we obtain that

$$\begin{aligned} \sigma(x,x) &= \lim_{n\to\infty} \sigma(x_n, x) \\ &= \lim_{m\to\infty} [\lim_{n\to\infty} \sigma(x_n, x_m)] \\ &= \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \end{aligned} \tag{20}$$

Using Lemma 3.7 and the σ-continuity of T, we have that

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tx) = \sigma(x, Tx) \tag{21}$$

and

$$\lim_{n\to\infty} \sigma(x_{n+1}, Tx) = \lim_{n\to\infty} \sigma(Tx_n, Tx) = \sigma(Tx, Tx). \tag{22}$$

Comparing (21) and (22), and applying Lemma 2.7, we obtain

$$\sigma(Tx, Tx) = \sigma(x, Tx).$$

Using the fact that T is σ-continuous, we obtain from (20) that

$$\lim_{n\to\infty} \sigma(Tx_n, Tx) = \lim_{n\to\infty} \sigma(x_{n+1}, Tx) = \sigma(x, Tx) = 0.$$

Thus, we have that $x = Tx$. □

Theorem 3.9. *Let (X,d) be a complete metric-like space and $T : X \to X$ be a modified generalized F-contraction mapping. Suppose the following conditions hold:*

1. *T is a (α,β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Proof. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}\cup\{0\}$. In Theorem 3.8, we have establish that $\{x_n\}$ is Cauchy and since (X,σ) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$. Now suppose hypothesis (3) holds. We now establish that T has a fixed point. Since $\alpha(x_n, x_{n+1}) \geq 1$ and

$\beta(x, Tx) \geq 1$, we have that $\alpha(x_n, x_{n+1})\beta(x, Tx) \geq 1$. Using Lemma 3.6, $(F_4^{'})$, (17) and the properties of ψ, we obtain from (2)

$$\begin{aligned}\lim_{n\to\infty}[\tau + F(\sigma(x_{n+1}, Tx))] &= \tau + \lim_{n\to\infty} F(\sigma(Tx_n, Tx))\\ &\leq F(\psi(\lim_{n\to\infty} M(x_n, x)))\\ &\leq F(\psi(\sigma(x, Tx)))\\ &< F(\sigma(x, Tx)).\end{aligned}$$

Thus, we have that

$$\tau + F(\sigma(x, Tx)) < F(\sigma(x, Tx))$$

which is a contradiction, then, we have $x = Tx$. □

Theorem 3.10. *Suppose that the hypothesis of Theorem 3.9 holds and in addition suppose $\alpha(x, Tx) \geq 1$ and $\beta(y, Ty) \geq 1$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.*

Proof. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Since, $\alpha(x, Tx) \geq 1$ and $\beta(y, Ty) \geq 1$, we have $\alpha(x, Tx)\beta(y, Ty) \geq 1$, we obtain that

$$\begin{aligned}F(\sigma(x, y)) &= F(\sigma(Tx, Ty)) < \tau + F(\sigma(Tx, Ty))\\ &\leq F(\psi(M(x, y))) \leq F(\psi(\sigma(x, y)))\\ &< F(\sigma(x, y)),\end{aligned}$$

which implies that

$$F(\sigma(x, y)) < F(\sigma(x, y)).$$

Clearly, we get a contradiction, thus, T has a unique fixed point. □

Corollary 3.11. *Let (X, d) be a complete metric-like space and $T : X \to X$ be a mapping satisfying the following inequality*

$$\alpha(x, Tx)\beta(y, Ty) \geq 1 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\psi(\sigma(x, y))),$$

for all $x, y \in X$. Suppose the following conditions hold:

1. *T is a (α, β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *T is d continuous,*
4. *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Corollary 3.12. *Let (X, d) be a complete metric-like space and $T : X \to X$ be a mapping satisfying the following inequality*

$$\sigma(Tx, Ty) > 0 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\psi(\sigma(x, y)))$$

for all $x, y \in X$. Then T has a fixed point.

Corollary 3.13. *Let (X, d) be a complete metric-like space and $T : X \to X$ be a mapping satisfying the following inequality*

$$\alpha(x, Tx)\beta(y, Ty) \geq 1 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\sigma(x, y)),$$

for all $x, y \in X$. Suppose the following conditions hold:

1. *T is a (α, β)-cyclic admissible mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$,*
3. *T is d continuous,*
4. *if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$ and $\alpha(x, Tx) \geq 1$.*

Then T has a fixed point.

Corollary 3.14. *Let (X, d) be a complete metric-like space and $T : X \to X$ be a mapping satisfying the following inequality*

$$\sigma(Tx, Ty) > 0 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\sigma(x, y))$$

for all $x, y \in X$. Then T has a fixed point.

Using Remark 3.2, we have next result. Let (X, d) be a complete metric-like space and $T : X \to X$ be a mapping satisfying the following inequality

$$\alpha(x, y) \geq 1 \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\psi(\sigma(x, y))),$$

for all $x, y \in X$. Suppose the following conditions hold:

1. T is a α-admissible mapping,
2. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
3. T is d continuous,
4. if for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\alpha(x, Tx) \geq 1$.

Then T has a fixed point.

4 Slip-up in a recent paper

In this section, we provide an affirmative answer to our claim by showing analytically and with examples that conditionally F-contraction type (A) is the same as conditionally F-contraction type (B).

Definition 4.1. *[17] Let (X, σ) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (A) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with*

$$\sigma(Tx, Ty) > 0, \frac{1}{2}\sigma(x, Tx) < \sigma(x, y) \Rightarrow \tau + F(\sigma(Tx, Ty)) \leq F(M_T(x, y)),$$

where

$$M_T(x, y) = \max\left\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{\sigma(x, Ty) + \sigma(y, Tx)}{4}\right\}.$$

Definition 4.2. *[17] Let (X, σ) be a metric-like space. A mapping $T : X \to X$ is said to be a conditionally F-contraction of type (B) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that, for all $x, y \in X$ with*

$$\sigma(Tx, Ty) > 0, \frac{1}{2}\sigma(x, Tx) < \sigma(x, y) \Rightarrow \tau + F(\sigma(Tx, Ty))$$
$$\leq F(\max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\}).$$

Theorem 4.3. *Let (X, σ) be a metric-like space. Let T be a conditionally F-contraction of type (A) mapping as defined in (4.1) and T_1 be a conditionally F-contraction of type (B) mapping as defined in (4.2). Then $T = T_1$.*

Proof. To establish that $T = T_1$, we just need to show that

$$\max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\}$$
$$= \max\left\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty), \frac{\sigma(x, Ty) + \sigma(y, Tx)}{4}\right\}.$$

To achieve this, we just need to show that

$$\frac{\sigma(x, Ty) + \sigma(y, Tx)}{4}$$
$$\leq \max\{\sigma(x, y), \sigma(x, Tx), \sigma(y, Ty)\}.$$

To do this, we have to show that

1. $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \leq \sigma(x, y)$,
2. $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \leq \sigma(x, Tx)$,
3. $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} \leq \sigma(y, Ty)$.

(1). Suppose that $\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}=\sigma(x,y)$, that is

$$\sigma(x,Tx)\le\sigma(x,y) \quad \text{and} \quad \sigma(y,Ty)\le\sigma(x,y).$$

Using triangular inequality, observe that

$$\begin{aligned}\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} &\le \frac{2\sigma(x,Ty)+\sigma(y,Ty)+\sigma(x,Tx)}{4}\\ &\le \frac{2\sigma(x,Ty)+\sigma(x,y)+\sigma(x,y)}{4}=\sigma(x,y).\end{aligned}$$

(2). Suppose that $\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}=\sigma(x,Tx)$, that is

$$\sigma(x,y)\le\sigma(x,Tx) \quad \text{and} \quad \sigma(y,Ty)\le\sigma(x,Tx).$$

Using triangular inequality, observe that

$$\begin{aligned}\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} &\le \frac{2\sigma(x,Ty)+\sigma(y,Ty)+\sigma(x,Tx)}{4}\\ &\le \frac{2\sigma(x,Tx)+\sigma(x,Tx)+\sigma(x,Tx)}{4}=\sigma(x,Tx).\end{aligned}$$

(3). Using similar approach, it is easy to see that **(3)** holds. □

Example 4.4. *Let $X=\{0,1,2\}$ and $\sigma : X\times X\to[0,\infty]$ be a mapping defined by*

$$\sigma(0,0)=\sigma(1,1)=0, \quad \sigma(0,1)=\sigma(0,2)=\sigma(1,0)=\sigma(2,0)=1,$$
$$\sigma(1,2)=\sigma(2,1)=\sigma(2,2)=2.$$

It is easy to see that σ is a metric-like space, but not a metric space. Let $T : X\to X$ be defined by

$$T0=T1=0 \quad \textit{and} \quad T2=1.$$

We will show that

$$\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}\le\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}$$

Proof. **Case 1.** If $x=y$, it is easy to see that $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}$
$\le\max\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty)\}$
Case 2. If $x=0$ and $y\ne0$, we consider the following sub-cases
Case 2a. If $x=0$ and $y=1$, we obtain that

$$\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4}=\frac{1}{4}$$
$$\sigma(x,y)=1, \sigma(x,Tx)=0 \quad \text{and} \quad \sigma(y,Ty)=1.$$

Clearly, we have that $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} = \frac{1}{4} < 1 = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty)\}$.
Case 2b. If $x = 0$ and $y = 2$, we obtain that

$$\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} = \frac{1}{2}$$
$$\sigma(x,y) = 1, \sigma(x,Tx) = 0 \quad \text{and} \quad \sigma(y,Ty) = 2.$$

Clearly, we have that $\frac{\sigma(x,Ty)+\sigma(y,Tx)}{4} = \frac{1}{2} < 2 = \max\{\sigma(x,y), \sigma(x,Tx), \sigma(y,Ty)\}$.
Since $\sigma(x,y) = \sigma(y,x)$ the results also holds for $y = 0$ and $x \neq 0$. □

Remark 4.5. *It is clear that Definition 4.1 and Definition 4.2 are the same. Then, we conclude that Theorem 2.2 and Theorem 2.5 of [17] are just repetition.*

5 Application

In this section, we apply our fixed point theory result to the boundary value problem for the second order differential equations.

5.1 Application to second order differential equations

In this section, we give an application on the existence of solution for the following second order differential equation of the form

$$\begin{aligned} x''(t) &= -f(t, x(t)), t \in I \\ x(0) &= x(1) = 0, \end{aligned} \tag{23}$$

where $I = [0,1], f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Consider the space $C(I)$ of continuous function defined on I. It is well-known that $C(I)$ with the metric-like

$$\sigma(x,y) = \|x-y\|_\infty + \|x\|_\infty + \|y\|_\infty \ \forall \ x,y \in C(I),$$

where

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)| \ \ \forall \ u \in C(I).$$

Since $d_\sigma(x,y) = 2\sigma(x,y) - \sigma(x,x) - \sigma(y,y) = 2\|x-y\|_\infty$, σ is also a partial metric on $C(I)$. Hence, $(C(I), \sigma)$ is a complete since the metric space $(C(I), \|\cdot\|_\infty)$ is complete. It is also well-known that the problem (23) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s)) ds, \tag{24}$$

for $t \in I$, where G is the Green function defined by

$$G(t,s) = \begin{cases} (1-t)s & \text{if} \quad 0 \le s \le t \le 1, \\ (1-s)t & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

If $x \in C^2(I)$, then $x \in C(I)$ is also solution of (23) if and only if it is a solution (24).

Theorem 5.1. *Let $X = C(I)$ and $T : X \to X$ be the operator given by*

$$Tx(t) = \int_0^1 G(t,s)f(s,x(s))ds$$

for all $x \in X$ and $t \in I = [0,1]$. Furthermore, suppose the following conditions hold:

1. *there exists functions $\varphi, \phi : I \to [0,\infty)$, $\alpha, \beta : X \times X \to [0,\infty)$ such that $\alpha(x,y) \geq 1 \Rightarrow \beta(Tx,Ty) \geq 1$ and $\beta(x,y) \geq 1 \Rightarrow \alpha(Tx,Ty) \geq 1$ for all $x, y \in X$, we have*

$$|f(s,u) - f(s,v)| \leq 8\varphi(s)|u - v|, \quad |f(s,u)| \leq \phi(s)|u|$$

   for some $s \in I$ and $u, v \in \mathbb{R}$.

2. $\sup_{s\in I} \tau(s) = K_1 < \frac{1}{10}$ *and* $\sup_{s\in I} \phi(s) = K_2 < \frac{1}{10}$
3. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\beta(x_0, Tx_0) \geq 1$;*
4. *for any sequence $\{x_n\}$ in X such that $x_n \to x$ as $n \to \infty$, then $\beta(x, Tx) \geq 1$.*

Then the second order differential equation (23) *has a solution.*

Proof. We define

$$\alpha(x,y) = \begin{cases} 2 & \text{if} \quad x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(x,y) = \begin{cases} 1 & \text{if} \quad x \preceq y \\ 0 & \text{otherwise} \end{cases}$$

and $\psi(t) = \frac{t}{3}$. We define $x, y \in X$, $x \preceq y$ if and only if $x(t) \leq y(t)$ for all $t \in [0,1]$. It is clear that if $x \preceq y$, we have that $\alpha(x,Tx)\beta(y,Ty) > 1$. Thus, we have that

$$\begin{aligned} |Ty(t) - Tx(t)| &\leq \int_0^1 G(t,s)|f(s,y(s)) - f(s,x(s))|ds \\ &\leq \int_0^1 G(t,s)\varphi(s)|y(s) - x(s)|ds \qquad (25) \\ &= 8K_1\|y - x\|_\infty \sup_{t\in[0,1]} \int_0^1 G(t,s)ds \\ &= 8K_1\|x - y\|_\infty \sup_{t\in[0,1]} \int_0^1 G(t,s)ds. \end{aligned}$$

It is well-known that for each $t \in I$, we have $\int_0^1 G(t,s)ds = \frac{t(1-t)}{2}$ as such, we obtain that

$$sup_{t\in[0,1]} \int_0^1 G(t,s)ds = \frac{1}{8}.$$

We have (25) becomes.

$$\|Tx - Ty\|_\infty \leq K_1\|x - y\|_\infty \tag{26}$$

It is also easy to see that

$$\|Tx\|_\infty \leq K_2\|x\|_\infty \tag{27}$$

and

$$\|Ty\|_\infty \leq K_2\|y\|_\infty. \tag{28}$$

Let $e^{-\tau} = 3(K_1 + 2K_2) < 1$. Using (26), (27) and (28), we get

$$\begin{aligned} \sigma(Tx, Ty) &= \|Tx - Ty\|_\infty + \|Tx\|_\infty + \|Ty\|_\infty \\ &\leq K_1\|x - y\|_\infty + K_2\|x\|_\infty + K_2\|y\|_\infty \\ &\leq (K_1 + 2K_2)(\|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty) \\ &= \frac{e^{-\tau}}{3}\sigma(x, y) \\ &= e^{-\tau}\psi(\sigma(x, y)). \end{aligned}$$

Taking the function $F(t) = \ln(t)$, we obtain

$$\tau + F(\sigma(Tx, Ty)) \leq F(\psi(\sigma(x, y))).$$

Clearly, all the conditions in Corollary 3.11 are satisfied, and so T has a fixed point. Thus the second order differential equation (23) has a solution. □

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