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SOME CONVERGENCE RESULTS FOR JUNGCK-AM ITERATIVE PROCESS IN HYPERBOLIC SPACES

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ABSTRACT. In this paper, we introduce a new three steps iterative process called Jungck-AM iterative process and show that the proposed iterative process can be used to approximate fixed points of Jungck-contractive type mappings and Jungck-Suzuki type mappings. In addition, we establish some strong and Δ-convergence results for the approximation of fixed points of Jungck-Suzuki type mappings in the frame work of uniformly convex hyperbolic space. Furthermore, we show that the newly proposed iterative process has a better rate of convergence compare to the Jungck-Noor, Jungck-SP, Jungck-CR and some existing iterative processes in the literature. Finally, stability, data dependency results for Jungck-AM iterative process is established and we present an analytical proof and numerical examples to validate our claim.

Key words and phrases: Jungck-Suzuki nonexpansive mapping; Jungck-AM iterative process; Uniformly convex hyperbolic space; stability; data dependency; strong and Δ-convergence theorems.

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1. INTRODUCTION

Some of the physical problems in engineering, physics, economics and so on are usually formulated into a fixed point problem: Find $x \in X$ such that

$$Tx = x, \tag{1.1}$$

where T is a nonlinear mapping (self or nonself) of an arbitrary space, say X. For the past 50 years researchers have paid a very good attention to finding an analytical solution to (1.1), but this have been almost practically impossible. In view of this, iterative method has been adopted in finding an approximate solution to (1.1). A good number of iterative processes (explicit, implicit, Jungck-type and so on) have been introduced and studied by many authors, (see [1, 2, 11, 12, 13, 14, 17, 18, 23, 24, 25, 27, 30, 31, 32, 33, 34] and the reference there in). However, a good and reliable fixed point iterative process is required to posses at least the following attributes:

(1) it should converge to a fixed point of an operator;
(2) it should be T-stable;
(3) it should be fast compare to other existing iteration in literature;
(4) it should show data dependence result.

In [15], Jungck introduced and studied an iterative process which involves the use of two mappings. This iterative process is very useful in the approximation of common fixed point of these mappings. The likes of Olatinwo and Postolache [29], Sahin and Basair [37], Razani and Begherboum [35], Khan, Kumar and Hussain [17] and so on, have introduced and studied different types of Jungck-type iterative processes in the frame work of Banach and metric spaces.

Let X be a convex metric space, Y an arbitrary nonempty set and $S, T : Y \to X$ such that $T(Y) \subseteq S(Y)$. Singh, Bhatnagar and Mishra [38] defined the Jungck-Mann iterative process as follows:

$$Sx_{n+1} = W(Sx_n, Tx_n, \alpha_n) \quad n \in \mathbb{N}, \tag{1.2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$. They introduced and studied the stability of Jungck and Jungck-Mann iterative processes for the mappings S and T satisfying the Jungck-Osilike type and the Jungck-contraction conditions

$$d(Tx, Ty) \leq \delta d(Sx, Sy), \quad \delta \in [0,1)$$

and

$$d(Tx, Ty) \leq \delta d(Sx, Sy) + Ld(Sx, Tx), \quad \delta \in [0,1), L \geq 0,$$

respectively. Jungck and Hussain in [16] also used the iterative process (1.2) to approximate the common fixed point of the mappings S and T satisfying the Jungck-contraction condition.

Olatinwo introduced and studied the Jungck-Ishikawa [25, 28] and Jungck-Noor [27] iterative processes defined as follows:

$$\begin{aligned} Sx_{n+1} &= W(Sx_n, Ty_n, \alpha_n), \\ Sy_n &= W(Sx_n, Tx_n, \beta_n), \quad n \in \mathbb{N} \end{aligned}$$

and

$$\begin{aligned} Su_{n+1} &= W(Su_n, Tv_n, \alpha_n), \\ Sv_n &= W(Su_n, Tw_n, \beta_n), \\ Sw_n &= W(Su_n, Tu_n, \gamma_n), \quad n \in \mathbb{N}, \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}, \{\gamma_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. He established some qualitative features such as convergence and stability using a Jungck-Zamfirescu operator for a pair (S,T), satisfying the following conditions: for all $x, y \in Y$ at least one of the following is true:

(1) $d(Tx,Ty) \le ad(Sx,Sy)$,
(2) $d(Tx,Ty) \le b[d(Sx,Tx) + d(Sy,Ty)]$,
(3) $d(Tx,Ty) \le c[d(Sx,Ty) + d(Sy,Tx)]$,

where $a \in [0,1]$, $b \ge 0$ and $c \le \frac{1}{2}$.
Olatinwo [25] proved stability and strong convergence results for some iterative processes using a more general Jungck-type mapping called Jungck-contractive like mapping.

Definition 1.1. The pair of nonself mappings $S, T : Y \to X$ is said to be Jungck-contractive like if there exists $\delta \in [0,1)$ and a monotone increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$, and for all $x, y \in Y$, such that

$$d(Tx,Ty) \le \delta d(Sx,Sy) + \phi(d(Sx,Tx)). \tag{1.4}$$

In [6], Chugh and Kumar defined the Jungck-SP iterative process as follows;

$$\begin{aligned} Sp_{n+1} &= W(Sq_n, Tq_n, \alpha_n), \\ Sq_n &= W(Sr_n, Tr_n, \beta_n), \\ Sr_n &= W(Sp_n, Tp_n, \gamma_n), \quad n \in \mathbb{N}, \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. They proved strong convergence as well as stability results for a pair of nonself mappings.
In [10], Hussain, Kumar and Kutbi defined the Jungck-CR iterative process as follows;

$$\begin{aligned} Sa_{n+1} &= W(Sb_n, Tb_n, \alpha_n), \\ Sb_n &= W(Ta_n, Tc_n, \beta_n), \\ Sc_n &= W(Sa_n, Ta_n, \gamma_n), \quad n \in \mathbb{N}, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. They proved its strong convergence to a common fixed point of the pair (S,T) using the fact that S, T are weakly compatible and that $Y = X$.

Definition 1.2. [45] Let C be a nonempty subset of a convex metric space X and T be a self mapping on C. Then T is said to be Suzuki generalized nonexpansive mapping if for all $x, y \in C$

$$\frac{1}{2}d(x,Tx) \le d(x,y) \Rightarrow d(Tx,Ty) \le d(x,y). \tag{1.7}$$

Singh and Mishra [39] introduced and studied the Jungck-Suzuki type nonexpansive mappings more general than Suzuki generalized nonexpansive mapping introduced and studied in [45] for a pair of mappings (S,T) satisfying the following: for all $x, y \in Y$,

$$\frac{1}{2}d(Tx,Sx) \le d(Sx,Sy) \Rightarrow d(Tx,Ty) \le d(Sx,Sy). \tag{1.8}$$

Clearly, if $Sx = x$, we obtain the Suzuki generalized nonexpansive mapping. They established some results on coincidence and fixed point theorems of mappings satisfying condition (1.8). Furthermore, they gave examples of mappings that satisfies condition (1.8), but does not satisfy condition (1.7).
In what follows, we also present an example of a pair of operator S, T that satisfy condition (1.8), but S does not satisfy condition (1.7).

Example 1.1. *Let $X = Y = [0,3]$ and $d(x,y) = \max\{x,y\}$ for all $x, y \in X$. Clearly, (X,d) is a metric space. Let $S, T : X \to X$ be defined by*

$$Tx = \begin{cases} 0 & \text{if} \quad x \in [0,1) \\ \frac{1}{2x} & \text{if} \quad x \in [1,3] \end{cases}$$

and

$$Sx = \begin{cases} 3x & \text{if} \quad x \in [0,1) \\ 4-x & \text{if} \quad x \in [1,3]. \end{cases}$$

Then the pair S, T satisfy condition (1.8) but S does not satisfy condition (1.7).

Proof. Clearly, $T(0) = S(0) = 0$, else, we have $Tx \neq Sx$ for all $x \in X$. We consider four cases to show that the pair of mappings S and T satisfied condition (1.8).

Case 1: When $x, y \in [0,1)$, we have

$$\begin{aligned} \frac{1}{2}d(Sx,Tx) = \frac{1}{2}d(3x,0) = \frac{3x}{2} &\leq \max\{3x,3y\} = d(Sx,Sy) \\ \Rightarrow d(Tx,Ty) = d(0,0) = 0 &\leq \max\{3x,3y\} = d(Sx,Sy). \end{aligned}$$

Case 2: When $x, y \in [1,3]$, we have

$$\begin{aligned} \frac{1}{2}d(Sx,Tx) = \frac{1}{2}d(4-x,\frac{1}{2x}) = \frac{4-x}{2} &\leq \max\{4-x,4-y\} = d(Sx,Sy) \\ d(Tx,Ty) = d(\frac{1}{2x},\frac{1}{2y}) = \max\{\frac{1}{2x},\frac{1}{2y}\} &\leq \max\{4-x,4-y\} = d(Sx,Sy). \end{aligned}$$

Case 3: When $x \in [0,1)$ and $y \in [1,3]$, we have

$$\begin{aligned} \frac{1}{2}d(Sx,Tx) = \frac{1}{2}d(3x,0) = \frac{3x}{2} &\leq \max\{3x,4-y\} = d(Sx,Sy) \\ d(Tx,Ty) = d(0,\frac{1}{2y}) = \frac{1}{2y} &\leq \max\{3x,4-y\} = d(Sx,Sy). \end{aligned}$$

Case 4: When $y \in [0,1)$ and $x \in [1,3]$, we have

$$\begin{aligned} \frac{1}{2}d(Sx,Tx) = \frac{1}{2}d(4-x,\frac{1}{2x}) = \frac{4-x}{2} &\leq \max\{4-x,3y\} = d(Sx,Sy) \\ d(Tx,Ty) = d(\frac{1}{2x},0) = \max\{\frac{1}{2x},0\} = \frac{1}{2x} &\leq \max\{4-x,3y\} = d(Sx,Sy). \end{aligned}$$

Thus, S and T satisfy condition (1.8). Clearly, 0 is the unique common fixed point of S and T. It is easy to see in the above example that T satisfy the generalized Suzuki nonexpansive mapping defined in [45].

To show that S does not satisfy the generalized Suzuki nonexpansive mapping defined in [45]. Let $x = 0$ and $y = 1$. Note that

$$\frac{1}{2}d(x,Sx) = \frac{1}{2}d(0,0) = 0 < 1 = \max\{0,1\} = d(x,y)$$

but

$$d(Sx,Sy) = d(0,4-1) = \max\{0,3\} = 3 > \max\{0,1\} = d(x,y).$$

This complete the proof. ∎

Motivated by the above facts and the research in this direction, our aim in this work is to:

(1) introduce a new Jungck-type iterative process and study its qualitative features, such as convergence, stability and data dependency for a Jungck-type contractive mappings;

(2) prove that our newly introduced iterative process has a better rate of convergence and more efficient as compared to some Jungck-type iterative processes in the literature;
(3) show some strong and Δ-convergence results for a Jungck-type Suzuki mappings using our newly proposed iterative process in the frame work of uniformly convex hyperbolic spaces;
(4) apply our iterative process and some existing iterative processes in literature to solve Legendre polynomial equation and quadratic equation.

2. Preliminaries

Throughout this paper, we carry out all our study in the frame work of hyperbolic space introduced by Kohlenbach [20].

Definition 2.1. A hyperbolic space (X, d, W) is a metric space (X, d) together with a convex mapping $W : X^2 \times [0, 1] \to X$ satisfying

(1) $d(u, W(x, y, \alpha)) \leq \alpha d(u, x) + (1 - \alpha)d(u, y)$;
(2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$;
(3) $W(x, y, \alpha) = W(y, x, 1 - \alpha)$;
(4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$;
for all $w, x, y, z \in X$ and $\alpha, \beta \in [0, 1]$.

Example 2.1. [44] *Let X be a real Banach space which is equipped with norm $||.||$. Define the function*
$d : X^2 \to [0, \infty)$ *by*

$$d(x, y) = ||x - y||$$

as a metric on X. Then, we have that (X, d, W) is a hyperbolic space with mapping $W : X^2 \times [0, 1] \to X$ defined by $W(x, y, \alpha) = (1 - \alpha)x + \alpha y$.

It is well-known that Banach spaces are examples of hyperbolic spaces and some other important examples are CAT(0) spaces, Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and $\mathbb{R}$-trees. The reader should please see [8, 9, 20, 36] for more discussion and examples of hyperbolic spaces.

Definition 2.2. [44] Let X be a hyperbolic space with a mapping $W : X^2 \times [0, 1] \to X$.

(i) A nonempty subset C of X is said to be convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$.
(ii) X is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided $d(x, z) \leq r, d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$.
(iii) A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for a given $r > 0$ and $\epsilon \in (0, 2]$ is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r (for a fixed ϵ).

Definition 2.3. Let C be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in C. For $x \in X$, let $r(\cdot, \{x_n\}) : X \to [0, \infty)$ be a continuous functional defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(C, \{x_n\})$ of $\{x_n\}$ with respect to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x, \{x_n\}) = \inf\{r(y, \{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C, \{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then we simply denote them by $r(\{x_n\})$ and $A(\{x_n\})$ respectively.
It is well-known that in uniformly convex Banach spaces and CAT(0) spaces, bounded sequences have unique asymptotic center with respect to closed convex subsets.

Definition 2.4. [19]. A sequence $\{x_n\}$ in X is said to $\triangle$-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{n_k}\}$ for every subsequence $\{x_{nk}\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim\limits_{n\to\infty} x_n = x$.

Remark 2.1. [21]. We note that $\triangle$-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.1. [22] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.*

Lemma 2.2. [5] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{n_k}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.*

Lemma 2.3. [18] *Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x^*) \leq c,\ \limsup_{n\to\infty} d(y_n, x^*) \leq c$ and $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x^*) = c$, for some $c > 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Definition 2.5. [16] Let X be a nonempty set and $S, T : X \to X$ be any two mappings.

(1) A point $x \in X$ is called:
  (a) coincidence point of S and T if $Sx = Tx$,
  (b) common fixed point of S and T if $x = Sx = Tx$.
(2) If $y = Sx = Tx$ for some $x \in X$, then y is called the point of coincidence of S and T.
(3) A pair (S, T) is said to be:
  (a) commuting if $TSx = STx$ for all $x \in X$,
  (b) weakly compatible if they commute at their coincidence points, that is $STx = TSx$, whenever $Sx = Tx$.

The set of coincidence points of S and T is denoted by $C(S, T)$ and the set of common fixed point of S and T is denoted by $F(S, T)$.

Definition 2.6. Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (A), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(t) > 0 \,\forall\, t \in (0, \infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T))$ denotes distance from x to $F(T)$.

Definition 2.7. Let C be a subset of a normed space X. The mappings $S, T : C \to C$ is said to satisfy condition (A^*), if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(t) > 0 \,\forall\, t \in (0, \infty)$ and that $\|Sx - Tx\| \geq f(d(Sx, F(S, T)))$ for all $x \in C$, where $d(Sx, F(S, T))$ denotes distance from Sx to $F(S, T)$.

Definition 2.8. [4] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n-a|}{|b_n-b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2.9 ([17])**.** Let $(S,T),(\overline{S},\overline{T}) : Y \to X$ be nonself-mapping pairs on an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and $\overline{T}(Y) \subseteq \overline{S}(Y)$. We say that the pair $(\overline{S},\overline{T})$ is an approximate mapping pair of (S,T) if for all $x \in Y$ and for fixed $\epsilon_1 > 0$ and $\epsilon_2 > 0$, we have

$$d(Tx,\overline{T}x) \le \epsilon_1, \ \ d(Sx,\overline{S}x) \le \epsilon_2.$$

Definition 2.10. [38] Let $S,T : Y \to X$ be nonself-mapping for an arbitrary set Y such that $T(Y) \subseteq S(Y)$ and x^* a point of coincidence of S and T. Let $\{Sx_n\} \subset X$, be sequence generated by an iterative procedure $Sx_{n+1} = f(T,x_n)$. Suppose $\{Sx_n\}$ converging to x^*, $\{Sy_n\} \subset X$ an arbitrary sequence and set $\epsilon_n = d(Sy_n, f(T,y_n)),\ \forall\ n \in \mathbb{N}$. Then, the iterative process is said to be $(S,T)-$ stable or stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} Sy_n = x^*$.

Definition 2.11. [42] Let $(S,T) : Y \to X$ be nonself-mapping pairs on an arbitrary set Y such that $T(Y) \subseteq S(Y)$ Two sequences say $\{Sx_n\} \subset X$ and $\{Sy_n\} \subset X$ are said to be equivalence if the $\lim_{n\to\infty} d(Sx_n, Sy_n) = 0$.

Definition 2.12. [42] Let S,T be two mappings such that $T(Y) \subseteq S(Y)$ and x^* a point of coincidence of S and T. Let $\{Sx_n\} \subset X$ be sequence generated by an iterative procedure $Sx_{n+1} = f(T,x_n)$. Suppose $\{Sx_n\}$ converging to x^*. If for any equivalent sequence $\{Sx_n\}$ and $\{Sy_n\}$

$$\lim_{n\to\infty} d(Sy_{n+1}, f(T,x_n)) = 0 \qquad \Rightarrow \lim_{n\to\infty} Sy_n = x^*,$$

then the iteration process is weak w^2- stable with respect of (S,T).

Lemma 2.4. [43] *Let $\{\eta_n\}$ and $\{\tau_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$\eta_{n+1} \le (1-\gamma_n)\eta_n + \tau_n,$$

where $\gamma_n \in (0,1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} \gamma_n = \infty$ and $\lim_{n\to\infty} \frac{\tau_n}{\gamma_n} = 0$, then $\lim_{n\to\infty} \eta_n = 0$.

Lemma 2.5. [41] *Let $\{\eta_n\}$ and $\{\tau_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$\eta_{n+1} \le (1-\gamma_n)\eta_n + \tau_n,$$

where $\gamma_n \in (0,1)$ for all $n \in \mathbb{N}, \sum_{n=0}^{\infty} \gamma_n = \infty$, then

$$0 \le \limsup_{n\to\infty} \eta_n \le \limsup_{n\to\infty} \tau_n.$$

3. Rate of Convergence, Stability and Data Dependency

In this section, we introduce and study our newly proposed iterative process. Let X be a uniformly convex hyperbolic space, Y an arbitrary set and $S,T : Y \to X$ be mappings satisfying condition (1.4) such that $T(Y) \subseteq S(Y)$. The sequence $\{Sx_n\}$ is define recursively as follows:

$$\begin{aligned} Sx_{n+1} &= W(Sy_n, Ty_n, \alpha_n), \\ Sy_n &= W(Tz_n, 0, 0), \\ Sz_n &= W(Sx_n, Tx_n, \beta_n), \quad n \in \mathbb{N}, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Theorem 3.1. *Let X be an hyperbolic space and $S,T : Y \to X$ be nonself mappings on an arbitrary set Y satisfying (1.4) such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X. Let $z \in C(S,T)$ such that $Sz = Tz = x^*$ (say) and suppose $\{Sx_n\}$ is the iterative process defined by (3.1) with $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then for $x_0 \in Y$, the iterative process $\{Sx_n\}$ converges strongly to x^*. In addition, x^* is the unique common fixed point of the pair (S,T) provided that $X = Y$ and S,T are weakly compatible.*

Proof. Using (3.1) and (1.4), we have

$$\begin{aligned} d(Sz_n, x^*) &= W(d(Sx_n, Tx_n, \beta_n), x^*) \\ &\leq (1-\beta_n)d(Sx_n, x^*) + \beta_n d(Tx_n, x^*) \\ &= (1-\beta_n)d(Sx_n, x^*) + \beta_n d(Tz, Tx_n) \\ &\leq (1-\beta_n)d(Sx_n, x^*) + \beta_n \delta d(Sz, Sx_n) + \beta_n \phi(d(Sz, Tz)) \\ &= (1-\beta_n)d(Sx_n, x^*) + \beta_n \delta d(Sx_n, x^*) \\ &= (1-(1-\delta)\beta_n)d(Sx_n, x^*) \\ &\leq d(Sx_n, x^*). \end{aligned} \tag{3.2}$$

Using (3.1), (3.2) and (1.4), we obtain

$$\begin{aligned} d(Sy_n, x^*) &= d(W(Tz_n, 0, 0), x^*) \\ &\leq d(Tz_n, x^*) \\ &\leq \delta d(Sz_n, x^*) \\ &\leq \delta d(Sx_n, x^*). \end{aligned} \tag{3.3}$$

Using (3.1), (3.3) and (1.4), we get

$$\begin{aligned} d(Sx_{n+1}, x^*) &= d(W(Sy_n, Ty_n, \alpha_n), x^*) \\ &\leq (1-\alpha_n)d(Sy_n, x^*) + \alpha_n d(Ty_n, x^*) \\ &= (1-\alpha_n)d(Sy_n, x^*) + \alpha_n d(Tz, Tx_n) \\ &\leq (1-\alpha_n)d(Sy_n, x^*) + \alpha_n \delta d(Sz, Sy_n) \\ &= (1-\alpha_n)d(Sy_n, x^*) + \alpha_n \delta d(Sy_n, x^*) \\ &= (1-(1-\delta)\alpha_n)d(Sy_n, x^*) \\ &\leq \delta(1-(1-\delta)\alpha_n)d(Sx_n, x^*). \end{aligned} \tag{3.4}$$

From (3.4), we have

$$\begin{aligned} d(Sx_{n+1}, x^*) &\leq \delta(1-(1-\delta)\alpha_n)d(Sx_n, x^*) \\ d(Sx_n, x^*) &\leq \delta(1-(1-\delta)\alpha_{n-1})d(Sx_{n-1}, x^*) \\ &\vdots \\ d(Sx_1, x^*) &\leq \delta(1-(1-\delta)\alpha_0)d(Sx_0, x^*). \end{aligned} \tag{3.5}$$

From (3.5), we have that

$$d(Sx_{n+1}, x^*) \leq d(Sx_0, x^*)\delta^{n+1} \prod_{m=0}^{n} (1-(1-\delta)\alpha_m). \tag{3.6}$$

Since $\{\alpha_n\}$ is in $(0,1)$ and δ in $[0,1)$, we have $(1-(1-\delta)\alpha_n) \in (0,1)$. We recall the inequality $1-x \leq e^{-x}$ for all $x \in [0,1]$, thus from (3.6), we have

$$d(Sx_{n+1}, x^*) \leq \frac{\delta^{n+1} d(Sx_0, x^*)}{e^{(1-\delta)\sum_{m=0}^{n} \alpha_m}}.$$

Taking the limit of both sides of the above inequalities, we have $\lim_{n\to\infty} d(Sx_n, x^*) = 0$.
In what follows, we now show that x^* is the unique common fixed point of S and T, when $Y = X$ and S, T are weakly compatible.
Suppose there exists another point of coincidence y^* of the pair (S,T). It follows that we can find say $z^* \in C(S,T)$ such that $Sz^* = Tz^* = y^*$. By definition, we obtain

$$d(x^*, y^*) = d(Tz, Tz^*) \leq \delta d(Sz, Sz^*) + \phi(d(Sz, Tz)) = \delta d(Sz, Sz^*) \leq d(x^*, y^*).$$

Clearly, we have that $d(x^*, y^*) = d(x^*, y^*)$, if not we get a contradiction $d(x^*, y^*) < d(x^*, y^*)$. Hence, we have that $x^* = y^*$. Since, S and T are weakly compatible and $x^* = Tz = Sz$, then $Tx^* = TTz = TSz = STz = Sx^*$. Thus, Tx^* is a point of coincidence of S and T, the point of coincidence is unique, we then have $x^* = Tx^*$. Hence, $Tx^* = Sx^* = x^*$ and therefore x^* is unique common fixed point of S and T. ∎

Theorem 3.2. *Let X be an hyperbolic space and $S, T : Y \to X$ be nonself mappings on an arbitrary set Y satisfying (1.4) such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X. Let $z \in C(S,T)$ such that $Sz = Tz = x^*$ (say) and suppose $\{Su_n\}, \{Sp_n\}$ and $\{Sa_n\}$ are the iterative processes defined by (1.3), (1.5) and (1.6) respectively with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n \beta_n = \infty$. Then for u_0, p_0 and a_0 in Y, the iterative processes $\{Su_n\}, \{Sp_n\}$ and $\{Sa_n\}$ converges strongly to x^*. In addition, x^* is the unique common fixed point of the pair (S,T) provided that $X = Y$ and S, T are weakly compatible.*

Proof. The proof follows the same line of argument as in Theorem 3.1. ∎

Theorem 3.3. *Let X be an hyperbolic space and $S, T : Y \to X$ be nonself mappings on an arbitrary set Y satisfying (1.4) such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X. Let $z \in C(S,T)$ such that $Sz = Tz = x^*$(say), and suppose $\{Sx_n\}$ is the iterative process defined by (3.1) with $\sum_{n=1}^{\infty} \alpha_n = \infty$ which converges strongly to x^*. Then, for $x_0 \in Y$, the iterative process $\{Sx_n\}$ is (S,T)- stable.*

Proof. Let $\{Sp_n\} \subset X$ be an arbitrary sequence and suppose that
$\epsilon_n = d(Sp_{n+1}, W(Sq_n, Tq_n, \alpha_n))$, where $Sq_n = W(Tr_n, 0, 0)$ and $Sr_n = W(Sp_n, Tp_n, \beta_n)$. Let $\lim_{n\to\infty} \epsilon_n = 0$, using condition (1.4) and triangle inequality , we have

$$\begin{aligned}
d(Sp_{n+1}, x^*) &\leq d(Sp_{n+1}, W(Sq_n, Tq_n, \alpha_n)) + d(W(Sq_n, Tq_n, \alpha_n), x^*) \\
&\leq \epsilon_n + (1-\alpha_n) d(Sq_n, x^*) + \alpha_n d(Tq_n, x^*) \\
&\leq \epsilon_n + (1-\alpha_n) d(Sq_n, x^*) + \alpha_n \delta d(Sq_n, x^*) \\
&= \epsilon_n + (1-(1-\delta)\alpha_n) d(Sq_n, x^*) \\
&= \epsilon_n + (1-(1-\delta)\alpha_n) d(W(Tr_n, 0, 0), x^*) \qquad (3.7) \\
&\leq \epsilon_n + (1-(1-\delta)\alpha_n) d(Tr_n, x^*) \\
&\leq \epsilon_n + (1-(1-\delta)\alpha_n) \delta d(Sr_n, x^*) \\
&= \epsilon_n + (1-(1-\delta)\alpha_n) \delta d(W(Sp_n, Tp_n, \beta_n), x^*) \\
&\leq \epsilon_n + (1-(1-\delta)\alpha_n) \delta [(1-(1-\delta)\beta_n) d(Sp_n, x^*)] \\
&\leq \epsilon_n + (1-(1-\delta)\alpha_n) d(Sp_n, x^*).
\end{aligned}$$

Since $\{\alpha_n\}$ is in $(0,1)$ and δ is in $[0,1)$, we have $(1-(1-\delta)\alpha_n) \in (0,1)$. Hence, using Lemma 2.4, we obtain $\lim_{n\to\infty} Sp_n = x^*$.
Conversely, let $\lim_{n\to\infty} Sp_n = x^*$. Then using triangle inequality and condition (1.4), we obtain

$$\begin{aligned}\epsilon_n &= d(Sp_{n+1}, W(Sq_n, Tq_n, \alpha_n))\\ &\le d(Sp_{n+1}, x^*) + d(W(Sq_n, Tq_n, \alpha_n), x^*)\\ &\le d(Sp_{n+1}, x^*) + \delta(1-(1-\delta)\alpha_n)d(Sp_n, x^*)\\ &\le d(Sp_{n+1}, x^*) + (1-(1-\delta)\alpha_n)d(Sp_n, x^*).\end{aligned}$$

Thus, $\lim_{n\to\infty}\epsilon_n = 0$. Then, the iterative scheme $\{Sx_n\}$ is (S,T)- stable. ∎

Theorem 3.4. *Let X be an hyperbolic space and $S, T : Y \to X$ be nonself mappings on an arbitrary set Y satisfying (1.4) such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X. Let $z \in C(S,T)$ such that $Sz = Tz = x^*$(say), and suppose $\{Sx_n\}$ is the iterative scheme defined by (3.1) with $\sum_{n=1}^{\infty}\alpha_n = \infty$ which converges strongly to x^*. Then, for $x_0 \in Y$, the iterative process $\{Sx_n\}$ is weak w^2-stable with respect to (S,T).*

Proof. Let $\{Sp_n\} \subset X$ be an equivalent sequence of $\{Sx_n\}$ and suppose that that $\epsilon_n = d(Spn+1, W(Sq_n, Tq_n, \alpha_n))$, where $Sq_n = W(Tr_n, 0, 0)$ and $Sr_n = W(Sp_n, Tp_n, \beta_n)$. Let $\lim_{n\to\infty}\epsilon_n = 0$, using condition (1.4) and triangle inequality, we have

$$\begin{aligned}
&d(Sp_{n+1}, x^*)\\
\le&d(Sp_{n+1}, Sx_{n+1}) + d(Sx_{n+1}, x^*)\\
\le&d(Sp_{n+1}, W(Sq_n, Tq_n, \alpha_n)) + d(W(Sq_n, Tq_n, \alpha_n), W(Sy_n, Ty_n, \alpha_n))\\
&+ d(Sx_{n+1}, x^*)\\
\le&\epsilon_n + (1-\alpha_n)d(Sq_n, Sy_n) + \alpha_n d(Ty_n, Tq_n) + d(Sx_{n+1}, x^*)\\
\le&\epsilon_n + (1-\alpha_n)d(Sy_n, Sq_n) + \alpha_n\delta d(Sy_n, Sq_n) + \alpha_n\phi(d(Sy_n, Ty_n))\\
&+ d(Sx_{n+1}, x^*)\\
=&\epsilon_n + (1-(1-\delta)\alpha_n)d(Sy_n, Sq_n) + \alpha_n\phi(d(Sy_n, Ty_n)) + d(Sx_{n+1}, x^*)\\
\le&\epsilon_n + (1-(1-\delta)\alpha_n)d(W(Tz_n, 0, 0), W(Tr_n, 0, 0)) + \alpha_n\phi(d(Sy_n, Ty_n))\\
&+ d(Sx_{n+1}, x^*)\\
\le&\epsilon_n + (1-(1-\delta)\alpha_n)\delta d(Sz_n, Tz_n) + (1-(1-\delta)\alpha_n)\phi(d(Sz_n, Tz_n))\\
&+ \alpha_n\phi(d(Sy_n, Ty_n)) + d(Sx_{n+1}, x^*)\\
=&\epsilon_n + (1-(1-\delta)\alpha_n)\delta d(W(Sx_n, Tx_n, \beta_n), W(Sp_n, Tp_n, \beta_n))\\
&+ (1-(1-\delta)\alpha_n)\delta\phi(d(Sz_n, Tz_n)) + \alpha_n\phi(d(Sy_n, Ty_n)) + d(Sx_{n+1}, x^*) \qquad (3.8)\\
\le&\epsilon_n + (1-(1-\delta)\alpha_n)\delta[(1-(1-\delta)\beta_n)d(Sx_n, Sp_n) + \beta_n\phi(d(Sx_n, Tx_n))]\\
&+ (1-(1-\delta)\alpha_n)\delta\phi(d(Sz_n, Tz_n)) + \alpha_n\phi(d(Sy_n, Ty_n)) + d(Sx_{n+1}, x^*)\\
=&(1-(1-\delta)\alpha_n)\delta(1-(1-\delta)\beta_n)d(Sx_n, Sp_n) + (1-(1-\delta)\alpha_n)\delta\beta_n\phi(d(Sx_n, Tx_n))\\
&+ (1-(1-\delta)\alpha_n)\delta\phi(d(Sz_n, Tz_n)) + \alpha_n\phi(d(Sy_n, Ty_n)) + d(Sx_{n+1}, x^*) + \epsilon_n.
\end{aligned}$$

Since $\{Sx_n\}$ and $\{Sp_n\}$ are equivalent sequences, so $\lim_{n\to\infty} d(Sx_n, Sp_n) = 0$. Also, since $\{Sx_n\}$ converges to x^*, clearly, $\{Sx_{n+1}\}$ converges to x^*. In addition, observe that

$$\begin{aligned}d(Sx_n, Tx_n) &\le d(Sx_n, x^*) + d(Tx_n, x^*)\\ &\le (1+\delta)d(Sx_n, x^*) \to 0 \quad \text{as} \quad n\to\infty.\end{aligned}$$

Using similar argument, it is easy to obtain $\lim_{n\to\infty} d(Sz_n, Tz_n) = \lim_{n\to\infty} d(Sy_n, Ty_n) = 0$. Clearly, we have that $\lim_{n\to\infty} d(Sy_n, Ty_n) = \phi(\lim_{n\to\infty} d(Sy_n, Ty_n)) = 0$, which also holds for others.
Thus $\lim_{n\to\infty} d(Sp_{n+1}, x^*) = 0$, consequently $\lim_{n\to\infty} d(Sp_n, x^*) = 0$. Then, the iterative scheme $\{Sx_n\}$ is weak w^2-stable with respect to (S, T). ∎

Theorem 3.5. *Let X be an hyperbolic space and $S, T : Y \to X$ be nonself operators on an arbitrary set Y satisfying (1.4) such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X. Let $x^* \in F(S,T)$ that is $Sx^* = Tx^* = x^*$, and for $x_0 = u_0 = p_0 = a_0 \in Y$, the sequences $\{Sx_n\}, \{Su_n\}, \{Sa_n\}$ and $\{Sp_n\}$ defined by (3.1), (1.3), (1.5) and (1.6) such that $\alpha \leq \alpha_n \leq 1, \alpha\beta \leq \alpha_n\beta_n \leq 1$ with $\alpha, \beta > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \alpha_n = \sum_{n=1}^{\infty} \beta_n\alpha_n = \sum_{n=1}^{\infty} \alpha_n = \infty$. Then, the iterative process (3.1) converges faster to x^* than (1.3), (1.5) and (1.6).*

Proof. From (3.6) in Theorem 3.1, and using our assumption, we have that

$$\begin{aligned} d(Sx_{n+1}, x^*) &\leq d(Sx_0, x^*)\delta^{n+1} \prod_{m=0}^{n} (1 - (1-\delta)\alpha_m) \\ &= d(Sx_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha_m]^{n+1} \\ &\leq d(Sx_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha]^{n+1}. \end{aligned}$$

Using similar argument as in Theorem 3.1 and our assumption, we have the Jungck-Noor iteration (1.3) takes the form

$$d(Su_{n+1}, x^*) \leq d(Su_0, x^*)[1 - (1-\delta)\alpha]^{n+1}.$$

Using similar argument as in Theorem 3.1 and our assumption, we have the Jungck-SP iteration (1.5) takes the form

$$d(Sp_{n+1}, x^*) \leq d(Sp_0, x^*)[1 - (1-\delta)\alpha]^{n+1}.$$

Using similar argument as in Theorem 3.1 and our assumption, we have the Jungck-CR iteration (1.6) takes the form

$$d(Sa_{n+1}, x^*) \leq d(Sa_0, x^*)\delta^{n+1} 1 - (1-\delta)\alpha\beta]^{n+1}.$$

Now, let

$$\begin{aligned} a_n &= d(Sx_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha]^{n+1} \\ b_n &= d(Su_0, x^*)[1 - (1-\delta)\alpha]^{n+1} \\ c_n &= d(Sp_0, x^*)[1 - (1-\delta)\alpha]^{n+1} \\ d_n &= d(Sa_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha\beta]^{n+1} \end{aligned}$$

and

$$\begin{aligned} \Phi_n &= \frac{a_n}{b_n} = \frac{d(Sx_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha]^{n+1}}{d(Su_0, x^*)[1 - (1-\delta)\alpha]^{n+1}} \to \infty \quad \text{as} \quad n \to 0, \\ \Psi_n &= \frac{a_n}{c_n} = \frac{d(Sx_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha]^{n+1}}{d(Sp_0, x^*)[1 - (1-\delta)\alpha]^{n+1}} \to \infty \quad \text{as} \quad n \to 0, \\ \Gamma_n &= \frac{a_n}{b_n} = \frac{d(Sx_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha]^{n+1}}{d(Sa_0, x^*)\delta^{n+1}[1 - (1-\delta)\alpha\beta]^{n+1}} \to \infty \quad \text{as} \quad n \to 0. \end{aligned}$$

It is easy to see that

$$\frac{1 - (1-\delta)\alpha}{1 - (1-\delta)\alpha\beta} < 1.$$

Thus, the proof is complete. ∎

One of the interesting area of research in fixed point theory is data dependence. Some times, researchers find it challenging or maybe impossible to find the fixed point of some nonlinear mappings. When faced with such situations, instead of trying to find the fixed point of such mappings, we look for another nonlinear mapping which is an approximation of the nonlinear mapping we intend to find the fixed point, more so the fixed point of the approximating nonlinear mapping must be known. Having such an approximate nonlinear mapping, we can find the approximate location of the fixed point of the nonlinear mapping that is proofing difficult to get. For this reason the concept of data dependency is of great importance in both theoretical and application point of view. For further details about data dependency, the reader should (see Brinde [3], Espinola and Petrusel [7], Olatinwo [26] Soltuz [40], Soltuz and Grosan [41] and the references there in).

Theorem 3.6. *Let X be an hyperbolic space and $(S,T),(\overline{S},\overline{T}):Y\to X$ be nonself-mappings on an arbitrary set Y with (S,T) satisfying condition (1.4) such that $d(Tx,\overline{T}x)\le\epsilon_1$ and $d(Sx,\overline{S}x)\le\epsilon_2$. Suppose $T(Y)\subseteq S(Y),\overline{T}(Y)\subseteq\overline{S}(Y)$, where $S(Y)$ and $\overline{S}(Y)$ are complete subspaces of X with $Sz=Tz=x^*$ and $\overline{S}\overline{z}=\overline{T}\overline{z}=\overline{x}^*$. Let $\{Sx_n\}$ be the iterative sequence generated by (3.1) and define an iterative process $\{\overline{S}\overline{x}_n\}$ as follows*

$$\begin{cases}\overline{S}\overline{z}_n=W(\overline{S}\overline{x}_n,\overline{T}\overline{x}_n,\beta_n),\\ \overline{S}\overline{y}_n=W(\overline{T}\overline{z}_n,0,0),\\ \overline{S}\overline{x}_{n+1}=W(\overline{S}\overline{y}_n,\overline{T}\overline{y}_n,\alpha_n)\ n\ge 1,\end{cases}\tag{3.9}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ and $\frac{1}{2}\le\alpha_n$ for all $n\in\mathbb{N}$ such that $\sum_{n=0}^{\infty}\alpha_n=\infty$. Suppose $\{Sx_n\}$ and $\{\overline{S}\overline{x}_n\}$ converges to x^* and $\overline{x}^*$ respectively. Then, we have

$$d(x^*,\overline{x}^*)\le\frac{5(\epsilon_2+\epsilon_1)}{1-\delta}.$$

Proof. Using (3.1) and (3.9), we have

$$\begin{aligned}d(S_{n+1},\overline{S}\overline{x}_{n+1})&=d(W(Sy_n,Ty_n,\alpha_n),W(\overline{S}\overline{y}_n,\overline{T}\overline{y}_n,\alpha_n))\\&\le(1-\alpha_n)d(Sy_n,\overline{S}\overline{y}_n)+\alpha_n d(Ty_n,\overline{T}\overline{y}_n)\\&\le(1-\alpha_n)d(Sy_n,\overline{S}\overline{y}_n)+\alpha_n d(Ty_n,T\overline{y}_n)+\alpha_n d(T\overline{y}_n,\overline{T}\overline{y}_n)\\&\le(1-\alpha_n)d(Sy_n,\overline{S}\overline{y}_n)+\alpha_n\delta d(Sy_n,S\overline{y}_n)+\alpha_n\phi(d(Sy_n,Ty_n))+\alpha_n\epsilon_n\\&\le(1-\alpha_n)d(Sy_n,\overline{S}\overline{y}_n)+\alpha_n\delta d(Sy_n,\overline{S}\overline{y}_n)+\alpha_n\delta d(S\overline{y}_n,\overline{S}\overline{y}_n)\\&\quad+\alpha_n\phi(d(Sy_n,Ty_n))+\alpha_n\epsilon_n(1-(1-\delta)\alpha_n)d(Sy_n,\overline{S}\overline{y}_n)+\alpha_n\phi(d(Sy_n,Ty_n))\\&\quad+\alpha_n\epsilon_n+\alpha_n\delta\epsilon_2.\end{aligned}\tag{3.10}$$

Using (3.1) and (3.9), we have

$$\begin{aligned}d(Sy_n,\overline{S}\overline{y}_n)&=d(Tz_n,\overline{T}\overline{z}_n)\\&\le d(Tz_n,T\overline{z}_n)+d(T\overline{z}_n,\overline{T}\overline{z}_n)\\&\le\delta d(Sz_n,S\overline{z}_n)+\phi(d(Sz_n,Tz_n))+\epsilon_1\\&\le\delta d(Sz_n,\overline{S}\overline{z}_n)+\delta d(\overline{S}\overline{z}_n,S\overline{z}_n)+\phi(d(Sz_n,Tz_n))+\epsilon_1\\&\le\delta d(Sz_n,\overline{S}\overline{z}_n)+\phi(d(Sz_n,Tz_n))+\delta\epsilon_2+\epsilon_1.\end{aligned}\tag{3.11}$$

Using (3.1) and (3.9), we obtain

$$\begin{aligned}
d(Sz_n,\overline{S}\overline{z}_n) &= d(W(Sx_n,Tx_n,\beta_n),W(\overline{S}\overline{x}_n,\overline{T}\overline{x}_n,\beta_n)) \\
&\leq (1-\beta_n)d(Sx_n,\overline{S}\overline{x}_n)+\beta_n d(Tx_n,\overline{T}\overline{x}_n) \\
&\leq (1-\beta_n)d(Sx_n,\overline{S}\overline{x}_n)+\beta_n d(Tx_n,T\overline{x}_n)+\beta_n d(T\overline{x}_n,\overline{T}\overline{x}_n) \\
&\leq (1-\beta_n)d(Sx_n,\overline{S}\overline{x}_n)+\beta_n\delta d(Sx_n,S\overline{x}_n)+\beta_n\phi(d(Sx_n,Tx_n))+\beta_n\epsilon_1 \qquad (3.12)\\
&\leq (1-\beta_n)d(Sx_n,\overline{S}\overline{x}_n)+\beta_n\delta d(Sx_n,\overline{S}\overline{x}_n)+\beta_n d(\overline{S}\overline{x}_n,S\overline{x}_n) \\
&\qquad +\beta_n\phi(d(Sx_n,Tx_n))+\beta_n\epsilon_1 \\
&\leq (1-(1-\delta)\beta_n)d(Sx_n,\overline{S}\overline{x}_n)+\phi(d(Sx_n,Tx_n))+\epsilon_1+\epsilon_2.
\end{aligned}$$

Substituting (3.12) into (3.11) and then (3.11) into (3.10), we obtain

$$\begin{aligned}
d(Sx_{n+1},\overline{S}\overline{x}_{n+1}) \leq{}& \delta(1-(1-\delta)\alpha_n)d(Sx_n,\overline{S}\overline{x}_n)+(1-(1-\delta)\alpha_n)\delta\phi(d(Sx_n,Tx_n)) \\
&+(1-(1-\delta)\alpha_n)\phi(d(Sz_n,Tz_n))+\alpha_n\phi(\|Sy_n-Ty_n\|) \\
&\qquad +(1-(1-\delta)\alpha_n)\delta^2\epsilon_2 \\
&+(1-(1-\delta)\alpha_n)\,\delta\epsilon_1+(1-(1-\delta)\alpha_n)\delta\epsilon_2 \\
&\qquad +(1-(1-\delta)\alpha_n)\epsilon_1+\alpha_n\delta\epsilon_2+\alpha_n\epsilon_1.
\end{aligned}$$

Since $\delta,\delta^2<1$, $\{\alpha_n\}$ in $(0,1)$ and our assumption that $\frac{1}{2}\leq\alpha_n$, we have

$$\begin{aligned}
1-\alpha_n &\leq \alpha_n \\
1-\alpha_n+\delta\alpha_n \leq 1-\alpha_n+\alpha_n &\leq \alpha_n+\alpha_n=2\alpha_n.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned}
&d(Sx_{n+1},\overline{S}\overline{x}_{n+1}) \\
\leq&(1-(1-\delta)\alpha_n)d(Sx_n,\overline{S}\overline{x}_n)+2\alpha_n\phi(d(Sx_n,Tx_n))+2\alpha_n\phi(d(Sz_n,Tz_n)) \\
&+\alpha_n\phi(d(Sy_n,Ty_n))+5\alpha_n(\epsilon_2+\epsilon_1) \\
=&(1-(1-\delta)\alpha_n)d(Sx_n,\overline{S}\overline{x}_n) \\
&+(1-\delta)\alpha_n\frac{2\phi(d(Sz_n,Tz_n))+2\phi(d(Sx_n,Tx_n))+\phi(d(Sy_n,Ty_n))+5(\epsilon_2+\epsilon_1)}{1-\delta}.
\end{aligned}$$

Let $\eta_n=d(Sx_n,\overline{S}\overline{x}_n)$, $\gamma_n=(1-\delta)\alpha_n$ and $\tau_n=\frac{2\phi(d(Sz_n,Tz_n))+2\phi(d(Sx_n,Tx_n))+\phi(d(Sy_n,Ty_n))+5(\epsilon_2+\epsilon_1)}{1-\delta}$. From Theorem 3.1, we have that $\lim_{n\to\infty}d(Sx_n,x^*)=0$. Also, observe that

$$\begin{aligned}
d(Sx_n,Tx_n) &\leq d(Sx_n,x^*)+d(Tx^*,Tx_n) \\
&\leq (1+\delta)d(Sx_n,x^*)\to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Using similar approach, we have that $\lim_{n\to\infty}d(Sy_n,Ty_n)=\lim_{n\to\infty}d(Sz_n,Tz_n)=0$. More so, we have that

$$\lim_{n\to\infty}\phi(d(Sy_n,Ty_n))=\phi(\lim_{n\to\infty}d(Sy_n,Ty_n))=0.$$

The same argument holds for others. Using Lemma 2.5, we have that

$$0\leq\limsup_{n\to\infty}d(Sx_n,\overline{S}\overline{x}_n)\leq\limsup_{n\to\infty}\frac{5(\epsilon_2+\epsilon_1)}{1-\delta}. \tag{3.13}$$

Using our hypothesis that $\lim_{n\to\infty}\overline{S}\overline{x}_n=\overline{x}^*$, (3.13) and from Theorem 3.1, we conclude that

$$d(x^*,\overline{x}^*)\leq\frac{5(\epsilon_2+\epsilon_1)}{1-\delta}.$$

Hence, the proof is complete. ∎

4. STRONG AND Δ-CONVERGENCE THEOREMS

In this section, we establish some strong and Δ-convergence results for mappings satisfying condition (1.8) using the iterative process (3.1). In achieving this results, we suppose that $X = Y$ and S, T are weakly compatible.

Lemma 4.1. *Let C be a nonempty closed and convex subset of an Hyperbolic space X. Let $S, T : C \to C$ be mappings satisfying (1.8) and $F(S,T) \neq \emptyset$. Suppose that $\{Sx_n\}$ is defined by (3.1), where $\{\alpha_n\}, \{\beta_n\}$ are sequences in $(0,1)$, then the $\lim_{n\to\infty} d(Sx_n, x^*)$ exists for all $x^* \in F(S,T)$.*

Proof. Let $x^* \in F(S,T)$, that is $Sx^* = Tx^* = x^*$, we have

$$\frac{1}{2}d(Sx^*, Tx^*) = \frac{1}{2}d(x^*, x^*) \leq d(Sx^*, Sx_n),$$
$$\frac{1}{2}d(Sx^*, Tx^*) = \frac{1}{2}d(x^*, x^*) \leq d(Sx^*, Sy_n)$$

and

$$\frac{1}{2}d(Sx^*, Tx^*) = \frac{1}{2}d(x^*, x^*) \leq d(Sx^*, Sz_n).$$

Which now implies that

$$d(Tz_n, Tx^*) \leq d(Sz_n, Sx^*) = d(Sz_n, x^*),$$
$$d(Tx_n, Tx^*) \leq d(Sx_n, Sx^*) = d(Sx_n, x^*)$$

and

$$d(Ty_n, Tx^*) \leq d(Sy_n, Sx^*) = d(Sy_n, x^*).$$

Using (3.1), we have

$$\begin{aligned} d(Sz_n, x^*) &= d(W(Sx_n, Tx_n, \beta_n), x^*) \\ &\leq (1-\beta_n)d(Sx_n, x^*) + \beta_n d(Tx_n, x^*) \\ &\leq (1-\beta_n)d(Sx_n, x^*) + \beta_n d(Sx_n, x^*) \\ &= d(Sx_n, x^*). \end{aligned} \tag{4.1}$$

Using (3.1) and (4.1), we have

$$\begin{aligned} d(Sy_n, x^*) &= d(W(Tz_n, 0, 0), x^*) \\ &\leq d(Tz_n, x^*) \\ &\leq d(Sz_n, x^*) \\ &\leq d(Sx_n, x^*). \end{aligned} \tag{4.2}$$

Using (3.1) and (4.2), we have

$$\begin{aligned} d(Sx_{n+1}, x^*) &= d(W(Sy_n, Ty_n, \alpha_n), x^*) \\ &\leq (1-\alpha_n)d(Sy_n, x^*) + \alpha_n d(Ty_n, x^*) \\ &\leq (1-\alpha_n)d(Sy_n, x^*) + \alpha_n d(Sy_n, x^*) \\ &= d(Sy_n, x^*) \\ &\leq d(Sx_n, x^*). \end{aligned} \tag{4.3}$$

This shows that $\{d(Sx_n, x^*)\}$ is decreasing and bounded for all $x^* \in F(S,T)$. Thus, $\lim_{n\to\infty} d(Sx_n, x^*)$ exists. ∎

Lemma 4.2. *Let C be a nonempty closed and convex subset of an hyperbolic space X. Let $S, T : C \to C$ be mappings satisfying (1.8) and $F(S,T) \neq \emptyset$. Suppose that $\{Sx_n\}$ is defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, then $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$.*

Proof. Since $F(S,T) \neq 0$, let $x^* \in F(S,T)$. We have shown in Lemma 4.1 that $\{Sx_n\}$ is bounded and $\lim_{n\to\infty} d(Sx_n, x^*)$ exists. Suppose that $\lim_{n\to\infty} d(Sx_n, x^*) = c$. If we take c = 0, then we are done. Thus, we consider the case where $c > 0$.
From (4.1), we have $d(Sz_n, x^*) \leq d(Sx_n, x^*)$, it then follows that

$$\limsup_{n\to\infty} d(Sz_n, x^*) \leq c. \tag{4.4}$$

Also, we have $d(Tx_n, x^*) \leq d(Sx_n, x^*)$, it then follows that

$$\limsup_{n\to\infty} d(Tx_n, x^*) \leq c. \tag{4.5}$$

Using (4.2) and (4.3), we have

$$\begin{aligned} d(Sx_{n+1}, x^*) &\leq d(Sy_n, x^*) \\ &\leq d(Sz_n, x^*). \end{aligned}$$

Taking the $\liminf_{n\to\infty}$ of both sides, we get

$$c \leq \liminf_{n\to\infty} d(Sz_n, x^*). \tag{4.6}$$

From (4.4) and (4.6), we obtain that $\lim_{n\to\infty} d(Sz_n, x^*) = c$. That is,

$$\lim_{n\to\infty} d(W(Sx_n, Tx_n, \beta_n), x^*) = c.$$

Thus, by Lemma 2.3, we have

$$\lim_{n\to\infty} d(Sx_n, Tx_n) = 0.$$

∎

Theorem 4.3. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X, with monotone modulus of uniform convexity τ. Let $S, T : C \to C$ be mappings satisfying condition (1.8) and $F(S,T) \neq \emptyset$. Let $I - S$ and $I - T$ be demiclosed at zero and suppsoe that $\{Sx_n\}$ is defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Then $\{Sx_n\}$ Δ-converges to the common fixed point of S and T.*

Proof. Let $W_\Delta(\{Sx_n\}) := \cup A(\{Su_n\})$, where the union is taken over all subsequence $\{Su_n\}$ of $\{Sx_n\}$. We now show that $W_\Delta(Sx_n) \subset F(S,T)$ and that $W_\Delta(Sx_n)$ contains exactly one point.
Let $u \in W_\Delta(\{Sx_n\})$, then by Lemma 4.1, there exists a subsequence $\{Su_n\}$ of $\{Sx_n\}$ such that $A(\{Su_n\}) = \{u\}$. This implies from Lemma 2.1 that we can find a subsequence $\{Sv_n\}$ of $\{Su_n\}$ such that $\Delta - \lim_{n\to\infty} Sv_n = v$, for some $v \in C$. By Lemma 4.2, we have that $\lim_{n\to\infty} d(Sv_n, Tv_n) = 0$, which together with our hypothesis that $I - T$ demiclosed at zero (that is $v \in F(T)$) and $I - S$ demiclosed at zero (that is $v \in F(T)$), which follow that $v \in F(S,T)$. Therefore, $\{d(Su_n, v)\}$ converges and by Lemma 2.2, we have that $v = u \in F(S,T)$. Hence, $W_\Delta(Sx_n) \subset F(T)$.
Next, we show that $W_\Delta(Sx_n)$ contains only one point. Let $A(\{Sx_n\}) = \{x\}$ and $\{Su_n\}$ be arbitrary subsequence of $\{Sx_n\}$ such that $A(\{Su_n\}) = \{u\}$. Then by Lemma 4.1, we have that $\{d(Sx_n, u)\}$ converges, since $u \in F(S,T)$. Thus, by Lemma 2.2, we have that $u = x \in F(S,T)$. Hence, $W_\Delta(Sx_n) = \{x\}$. Therefore, $\{Sx_n\}$ $\triangle$-converges to a common fixed point of (S,T). ∎

Theorem 4.4. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X, with monotone modulus of uniform convexity τ. Let $S, T : C \to C$ be mappings satisfying condition (1.8) and $F(S,T) \neq \emptyset$. Suppose that $\{Sx_n\}$ is defined by (3.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. Then $\{Sx_n\}$ converges strongly to a point of $F(S,T)$ if and only if $\liminf_{n\to\infty} d(Sx_n, F(S,T)) = 0$.*

Proof. Suppose that $\{Sx_n\}$ converges to a fixed point, say x^* of (S,T). Then $\lim_{n\to\infty} d(Sx_n, x^*) = 0$, and since $0 \leq d(Sx_n, F(T)) \leq d(Sx_n, x^*)$, it follows that $\lim_{n\to\infty} d(Sx_n, F(S,T)) = 0$. Therefore, $\liminf_{n\to\infty} d(Sx_n, F(S,T)) = 0$.
Conversely, suppose that $\liminf_{n\to\infty} d(Sx_n, F(S,T)) = 0$. From Lemma 4.1, we have that $\lim_{n\to\infty} d(Sx_n, F(S,T))$ exists and so, it follows that $\lim_{n\to\infty} d(Sx_n, F(S,T)) = 0$. Suppose that $\{Sx_{n_k}\}$ is any arbitrary subsequence of $\{Sx_n\}$ and $\{p_k\}$ a sequence in $F(S,T)$ such that for all $n \geq 1$,

$$d(Sx_{n_k}, p_k) < \frac{1}{2^k}.$$

From (4.3), we obtain that

$$d(Sx_{n+1}, p_k) \leq d(Sx_{n_k}, p_k) < \frac{1}{2^k}.$$

Thus,

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, Sx_{n+1}) + d(Sx_{n+1}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in $F(S,T)$. Also, by our hypothesis that $F(S,T)$ is closed. Thus, $\{p_k\}$ is a convergent sequence in $F(S,T)$ and say it converges to $q \in F(S,T)$. Therefore, since

$$d(Sx_{n_k}, q) \leq d(Sx_{n_k}, p_k) + d(p_k, q) \to 0 \text{ as } n \to \infty,$$

we have $\lim_{n\to\infty} d(Sx_{n_k}, q) = 0$ and so $\{Sx_{n_k}\}$ converges strongly to $q \in F(S,T)$. Since, $\lim_{n\to\infty} d(Sx_n, q)$ exists, it follows that $\{Sx_n\}$ converges strongly to q. ∎

Theorem 4.5. *Let C be a nonempty closed and convex subset of a complete hyperbolic space X, with monotone modulus of uniform convexity τ. Let S, T be mappings satisfying condition (1.8), $\{Sx_n\}$ defined by (3.1) and $F(S,T) \neq \emptyset$. Let T, S satisfy condition (A^*), then $\{Sx_n\}$ converges strongly to a common fixed point of S and T.*

Proof. From Lemma 4.1, we have $\lim_{n\to\infty} d(Sx_n, F(S,T))$ exist and by Lemma 4.2, we have $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$. Using the fact that $f(d(Sx, F(S,T)) \leq d(Sx, Tx)$ for all $x \in C$, we have that $\lim_{n\to\infty} f(d(Sx_n, F(S,T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0,\infty)$, it then follows that $\lim_{n\to\infty} d(Sx_n, F(T)) = 0$. Hence, by Theorem 4.4 $\{Sx_n\}$ converges strongly to $x^* \in F(S,T)$. ∎

5. NUMERICAL EXAMPLE

In this section, we apply the newly introduced Jungck-AM iterative process to find the solution of a quadratic equation and a Legendre equation. We also show that the new iterative process converges faster to the solution of a given quadratic equation and the Legendre equation as compared to Jungck-Noor and Jungck-SP and Jungck-CR.

Example 5.1. *To find the roots of a Legendre equation* $\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x = 0$, *we write it in the form* $Sx = Tx$, *where the mappings* $S, T : [0,1] \to [0,70]$ *are define as* $Sx = 70^3x$ *and* $Tx = 63x^5 + 15x$. *It is easy to see that for* $x^* = 0.5384693$, *we have* $T(0.5384693) = S(0.5384693) = 10.9290$. *We take* $\gamma_n = \alpha_n = \beta_n = \frac{1}{\sqrt{5n+1}}$ *and* $x_0 = u_0 = p_0 = a_0 = 0.4$. *The comparison table for the iterative process is shown below.*

Step	*Jungck-AM* (x_{n+1})	*Jungck-CR* (a_{n+1})	*Jungck-SP* (p_{n+1})	*Jungck-Noor* (u_{n+1})
0	*0.4*	*0.4*	*0.4*	*0.4*
1	*0.4801090*	*0.4737563*	*0.4595051*	*0.4327705*
2	*0.5085783*	*0.5029036*	*0.4837408*	*0.4489964*
3	*0.5219236*	*0.5175876*	*0.4973600*	*0.4596309*
4	*0.5289282*	*0.5257728*	*0.5061551*	*0.4674490*
5	*0.5328290*	*0.5305856*	*0.5123018*	*0.4735725*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
31	*0.5384693*	*0.5384692*	*0.5370272*	*0.5178034*
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
38	*0.5384693*	*0.5384693*	*0.5376059*	*0.5213762*
39	*0.5384693*	*0.5384693*	*0.5376637*	*0.5218054*
40	*0.5384693*	*0.5384693*	*0.5377170*	*0.5222178*

Comparison shows that Jungck-AM iterative process converges faster.

Example 5.2. *To find the roots of a quadratic equation* $x^2 - 10x + 9 = 0$, *we write it in the form* $Sx = Tx$, *where the mappings* $S, T : [1,5] \to [1,70]$ *are define as* $Sx = 10x$ *and* $Tx = x^2 + 9$. *Clearly,* $x^* = 1$, *we have* $T(1) = S(1) = 10$. *We take* $\gamma_n = \alpha_n = \beta_n = \frac{1}{\sqrt{5n+4}}$ *and* $x_0 = u_0 = p_0 = a_0 = 2$. *The comparison table for the iterative process is shown below.*

Step	*Jungck-AM* (x_{n+1})	*Jungck-CR* (a_{n+1})	*Jungck-SP* (p_{n+1})	*Jungck-Noor* (u_{n+1})
0	*2*	*2*	*2*	*2*
1	*1.157048*	*1.200959*	*1.437948*	*1.733949*
2	*1.020748*	*1.032742*	*1.220616*	*1.577680*
3	*1.002792*	*1.005204*	*1.121943*	*1.471947*
4	*1.000391*	*1.000844*	*1.071969*	*1.394932*
5	*1.000057*	*1.000140*	*1.044604*	*1.336186*
6	*1.000008*	*1.000024*	*1.028715*	*1.289920*
7	*1.000001*	*1.000004*	*1.019059*	*1.252610*
8	*1.000000*	*1.000001*	*1.012973*	*1.221964*
9	*1.000000*	*1.000000*	*1.009019*	*1.196419*
10	*1.000000*	*1.000000*	*1.006385*	*1.174866*

Comparison shows that Jungck-AM iterative process converges faster.

6. Conclusion

We have shown that our newly proposed Jungck-type iterative process is more efficient and converges faster than recently introduced Jungck-type iterative processes in literature. In addition, it is clear from Section 5 that our newly proposed Jungck-type iterative process have a very good potential for further applications.

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