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SOME FIXED POINT RESULTS FOR TAC-SUZUKI CONTRACTIVE MAPPINGS

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Abstract. In this paper, we introduce the notion of modified TAC-Suzuki-Berinde type F-contraction and modified TAC-(ψ, ϕ)-Suzuki type rational mappings in the frame work of complete metric spaces, we also establish some fixed point results regarding this class of mappings and we present some examples to support our main results. The results obtained in this work extend and generalize the results of Dutta et al. [9], Rhoades [18], Doric, [8], Khan et al. [13], Wardowski [25], Piri et al. [17], Sing et al. [23] and many more results in this direction.

1. Introduction and preliminaries

Banach contraction principle [2] can be seen as the pivot of the theory of fixed point and its applications. The theory of fixed point plays an important role in nonlinear functional analysis and it is very useful for showing the existence and uniqueness theorems for nonlinear differential and integral equations. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. The Banach contraction principle have been extended and generalized by researchers in this area by considering classes of nonlinear mappings and spaces which are more general than the class of a contraction mappings and metric spaces (see [1, 7, 10, 14–16, 19, 22] and the references therein). For example, Geraghty [11] introduced a generalized contraction mapping called Geraghty-contraction and established the fixed point theorem for this class of contraction mappings in the frame work of metric spaces. We recall that for a metric space (X, d), a mapping $T : X \to X$ is said to be an α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall \quad x, y \in X. \tag{1.1}$$

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Definition 1.1 ([11])**.** Let (X,d) be a metric space. A mapping $T: X \to X$ is called a Geraghty-contraction mapping if

$$d(Tx,Ty) \leq \phi(d(x,y))d(x,y) \tag{1.2}$$

for all $x, y \in X$, where $\phi : \mathbb{R}^+ \to [0,1)$ satisfies the following condition:

$$\phi(t_n) \to 1 \quad \text{as} \quad n \to \infty \Rightarrow t_n \to 0 \quad \text{as} \quad n \to \infty.$$

The following is the result of Geraghty [11].

Theorem 1.2. *Let (X,d) be a complete metric space and $T: X \to X$ be a self map that satisfies condition* (1.2)*. Then T has a unique fixed point $x^* \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = x^*$.*

Jaggi [12] introduced a class of contraction mappings involving rational expressions and proved some fixed point results for this class of mappings. Khan et al. [13] introduced the concept of alternating distance function, which is defined as follows: A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an alternating distance function if the following conditions are satisfied:

(1) $\psi(0) = 0$,
(2) ψ is monotonically nondecreasing,
(3) ψ is continuous.

They established the following result.

Theorem 1.3. *Let (X,d) be a complete metric space, ψ an altering distance function, and $T : X \to X$ be a self mapping which satisfies the following condition*

$$\psi(d(Tx,Ty)) \leq \delta\psi(d(x,y)) \tag{1.3}$$

for all $x, y \in X$, where $\delta \in (0,1)$. Then T has a unique fixed point.

Remark 1.4. Clearly, if we take $\psi(x) = x$ for all $x \in X$ in (1.3), we obtain condition (1.1).

Using the concept of alternating distance function Rhoades [18], Dutta et al. [9] and Doric [8] established some fixed points results for weak contraction and generalized contraction mappings in the frame work of metric spaces. We recall that for a metric space (X,d), a mapping $T : X \to X$ is said to be weakly contractive if for all $x, y \in X$

$$d(Tx,Ty) \leq d(x,y) - \psi(d(x,y)),$$

$\psi : [0,\infty) \to [0,\infty)$ is continuous and nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$.

Theorem 1.5 ([18])**.** *Let (X,d) be a complete metric space and T a weakly contractive map. Then T has a unique fixed point.*

Theorem 1.6 ([9]). *Let (X, d) be a complete metric space. Suppose the mappings $T : X \to X$ satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{1.4}$$

for all $x, y \in X$, where ψ, ϕ are alternating distance functions. Then T has a fixed point.

Theorem 1.7 ([8]). *Let X be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{1.5}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, ψ an alternating distance function and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In 2008, Suzuki [24] introduced the concept of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for such class of mappings.

Definition 1.8. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.9. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying condition (C) for all $x, y \in X$. Then T has a unique fixed point.*

Furthermore, Berinde [4, 5] introduced and studied some class of contractive mappings, in particular he gave the definition for a mapping been almost contractive as follows:

Definition 1.10. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be a generalized almost contraction if there exist a constant $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for all $x, y \in X$.

Recently, Sing et al. [23] obtain the following result.

Theorem 1.11. *Let X be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)),$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, $\psi : [0, \infty) \to [0, \infty)$ is continuous, nodecreasing and $\psi(t) = 0$ if and only if $t = 0$ and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In [26] Yan et al. proved the following result in the fame work of partially ordered metric spaces.

Theorem 1.12. *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y))$$

for all $x \geq y$, where ψ is an alternating distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Lemma 1.13 ([26]). *If ψ is an alternating distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with condition $\psi(t) > \phi(t)$, then $\phi(0) = 0$.*

In 2012 Wardowski [25] introduced the notion of F-contractions. He defined F-contraction as follows:

Definition 1.14. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be an F-contraction if there exists $\tau > 0$ such that for all $x, y \in X$;

$$d(Tx, Ty) > 0 \Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)), \tag{1.6}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

- (F_1) F is strictly increasing;
- (F_2) for all sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$,
  $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
- (F_3) there exists $0 < k < 1$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

He also established the following result.

Theorem 1.15. *Let (X, d) be a complete metric space and $T : X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for each $x_0 \in X$ a sequence $\lim_{n\to\infty} T^n x_0 = x^*$.*

Remark 1.16. If we suppose that $F(t) = \ln t$, an F-contraction mapping becomes the Banach contraction.

Secelean [21] established the following lemma.

Lemma 1.17. *Let $F : \mathbb{R}^+ \to \mathbb{R}$ be an increasing mapping and $\{\alpha_n\}$ be a sequence of positive integers. Then the following assertion hold*:

1. *if $\lim_{n\to\infty} F(\alpha_n) = -\infty$, then $\lim_{n\to\infty} \alpha_n = 0$;*
2. *if $\inf F = -\infty$ and $\lim_{n\to\infty} \alpha_n = 0$, then $\lim_{n\to\infty} F(\alpha_n) = -\infty$.*

He replaced the condition F_2 with the following condition.

- (F_*) $\inf F = -\infty$ or,
- (F_{**}) there exists a sequence $\{\alpha_n\}$ of positive real numbers such that

$$\lim_{n\to\infty} F(\alpha_n) = -\infty.$$

In [17] Piri et al. introduced the continuity condition to replace (F_3). That is (F_{3^*}) F is continuous on $(0,\infty)$. He established the following result.

Theorem 1.18. *Let X be a complete metric space and $T : X \to X$ be a selfmap of X. Assume that there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$,*

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)),$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is continuous strictly increasing and $\inf F = -\infty$. *Then T has a unique fixed point $z \in X$ and every $x \in X$, the sequence $\{T^n x\}$ converges to z.*

We denoted by $\mathbb{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ which satisfy $(F_1), (F_*)$ and (F_{3^*}).

In 2016, Chandok et al. [6] introduced the concept of TAC-contractive mappings and established some fixed point results in the frame work of complete metric spaces.

Definition 1.19. Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. One can say T is a cyclic (α,β)-admissible mapping if

(1) $\alpha(x) \geq 1$ for some $x \in X$ implies that $\beta(Tx) \geq 1$,
(2) $\beta(x) \geq 1$ for some $x \in X$ implies that $\alpha(Tx) \geq 1$.

Definition 1.20. Let (X,d) be a metric space and let $\alpha, \beta : X \to [0,\infty)$ be two mappings. We say that T is a TAC-contractive mapping if for all $x, y \in X$ with

$$\alpha(x)\beta(y) \geq 1 \Rightarrow \psi(d(Tx,Ty)) \leq f(\psi(d(x,y)), \phi(d(x,y))),$$

where ψ is a continuous and nondecreasing function with $\psi(0) = 0$ if and only if $t = 0$, ϕ is continuous with $\lim_{n\to\infty} \phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$ and $f : [0,\infty)^2 \to \mathbb{R}$ is continuous, $f(a,t) \leq a$ and $f(a,t) = a \to a = 0$ or $t = 0$ for all $s, t \in [0,\infty)$.

Theorem 1.21. *Let (X,d) be a complete metric space and let $T : X \to X$ be a cyclic (α,β)-admissible mapping. Suppose that T is a TAC contraction mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1, \beta(x_0) \geq 1$ and either of the following conditions hold*:

(1) *T is continuous,*
(2) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$ for all $x, y \in Fix(T)$, then T has a unique fixed point.

Lemma 1.22 ([3])**.** *Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X such that* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. *If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(m_k, n_k) \geq \epsilon$. For each $k > 0$, corresponding to*

m_k, we can choose n_k to be the smallest positive integer such that $d(m_k, n_k) \geq \epsilon, d(m_k, n_{k-1}) < \epsilon$ and

(1) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$,
(2) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
(3) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_{k+1}}) = \epsilon$.

Remark 1.23. Using similar approach as Lemma 1.22, we get that

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{m_{k+1}}) = \epsilon$$

and

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon.$$

Motivated by the research works described above, our purpose in this paper is to generalize the results of Rhoades [18], Dutta et al. [9], Doric [8], Samet et al. [20], Sing et al. [23] and Yan et al. [26], by introducing a modified TAC-Suzuki-Berinde type F-contraction and modified TAC-(ψ, ϕ)-Suzuki type rational mappings in the frame work of complete metric spaces.

2. Main result

2.1. TAC-Suzuki-Berinde-F contraction type mappings

In this section, we introduce the notion of TAC-Suzuki Berinde-F contraction type mapping and established the existence and uniqueness results of the fixed point for this class of mappings.

Definition 2.1. Let (X, d) be a metric space, $\alpha, \beta, \varphi : X \to [0, \infty)$ be three functions and T be a self map on X. The mapping T is said to be a cyclic (α, β)-admissible mapping with respect to φ if

(1) $\alpha(x) \geq \varphi(x)$ for some $x \in X$ implies that $\beta(Tx) \geq \varphi(Tx)$,
(2) $\beta(x) \geq \varphi(x)$ for some $x \in X$ implies that $\alpha(Tx) \geq \varphi(Tx)$.

Remark 2.2. We note that if $\varphi(x) = 1$, then the definition reduces to Definition 1.19.

Definition 2.3. Let (X, d) be a metric space, $\alpha, \beta, \varphi : X \to [0, \infty)$ be three functions and T be a self map on X. The mapping T is said to be a TAC-Suzuki-Berinde-F contraction type mapping if

$$\begin{aligned} &\alpha(x)\beta(y) \geq \varphi(x)\varphi(y) \quad \text{and} \\ &\frac{1}{2}d(x, Tx) \leq d(x, y) \\ &\Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) + LN(x, y)\ \forall\ x, y \in X, \end{aligned} \tag{2.1}$$

where $\tau > 0, L \geq 0,\ N(x, y) = \min\{d(x, Tx), d(y, Ty), d(y, Tx), d(x, Ty)\}$ and $F \in \mathbb{F}$.

Definition 2.4. Let (X,d) be a metric space, $\alpha, \beta, \varphi : X \to [0,\infty)$ be three functions and T be a self map on X. The mapping T is said to be a TAC-Suzuki-F contraction type mapping if

$$\begin{aligned} &\alpha(x)\beta(y) \geq \varphi(x)\varphi(y) \text{ and} \\ &\frac{1}{2}d(x,Tx) \leq d(x,y) \\ &\Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y)) \ \forall \ x,y \in X, \end{aligned} \tag{2.2}$$

where $\tau > 0$ and $F \in \mathbb{F}$.

Theorem 2.5. *Let (X,d) be a complete metric space and $T : X \to X$ be a TAC-Suzuki-Berinde-F contraction type mapping. Suppose the following conditions hold*:

1. *T is a cyclic (α,β)-admissible mapping with respect to φ,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq \varphi(x_n)$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq \varphi(x)$.*

Then T has a fixed point.

Proof. From our hypothesis, there exists $x_0 \in X$ such that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a cyclic (α,β)-admissible mapping with respect to φ, and $\alpha(x_0) \geq \varphi(x_0)$, we have $\beta(x_1) = \beta(Tx_0) \geq \varphi(Tx_0) = \varphi(x_1)$ and this implies that $\alpha(x_2) = \alpha(Tx_1) \geq \varphi(Tx_1) = \varphi(x_2)$, continuing the process, we have

$$\alpha(x_{2k}) \geq \varphi(x_{2k}) \text{ and } \beta(x_{2k+1}) \geq \varphi(x_{2k+1}) \ \forall \ k \in \mathbb{N} \cup \{0\}. \tag{2.3}$$

Using similar argument, we have that

$$\beta(x_{2k}) \geq \varphi(x_{2k}) \text{ and } \alpha(x_{2k+1}) \geq \varphi(x_{2k+1}) \ \forall \ k \in \mathbb{N} \cup \{0\}. \tag{2.4}$$

It follows from (2.3) and (2.4) that $\alpha(x_n) \geq \varphi(x_n)$ and $\beta(x_n) \geq \varphi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n)\beta(x_{n+1}) \geq \varphi(x_n)\varphi(x_{n+1})$ and $\frac{1}{2}d(x_n,Tx_n) = \frac{1}{2}d(x_n,x_{n+1}) \leq d(x_n,x_{n+1})$, we have

$$\begin{aligned} \tau + F(d(x_{n+1},x_{n+2})) &= \tau + F(d(Tx_n,Tx_{n+1})) \\ &\leq F(d(x_n,x_{n+1})) + LN(x_n,x_{n+1}), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} N(x_n,x_{n+1}) &= \min\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), d(x_n,x_{n+1}), d(x_{n+1},x_{n+1}), d(x_n,x_{n+2})\} \\ &= 0. \end{aligned}$$

We then have that

$$\tau + F(d(x_{n+1},x_{n+2})) \leq F(d(x_n,x_{n+1}))$$

$$\Rightarrow F(d(x_{n+1}, x_{n+2})) \leq F(d(x_n, x_{n+1})) - \tau.$$

Using similar approach, we have that

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) \\ \Rightarrow F(d(x_n, x_{n+1})) &\leq F(d(x_{n-1}, x_n)) - \tau. \end{aligned}$$

Inductively, we have

$$F(d(x_n, x_{n+1})) \leq F(d(x_0, x_1)) - n\tau. \tag{2.6}$$

Since $F \in \mathbb{F}$, taking the limit as $n \to \infty$, we have that

$$\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty \Leftrightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.7}$$

In what follows, we now show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.22, there exist an $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(m_k, n_k) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(m_k, n_k) \geq \epsilon$, $d(m_k, n_{k-1}) < \epsilon$ and

(1) $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$,
(2) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$,
(3) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
(4) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_{k+1}}) = \epsilon$.

Since $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$, it is easy to see that $\alpha(x_{m_k}) \geq \varphi(x_{m_k})$ and $\beta(x_{n_k}) \geq \varphi(x_{n_k})$. It follows that $\alpha(x_{m_k})\beta(x_{n_k}) \geq \varphi(x_{m_k})\varphi(x_{n_k})$ and we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{1}{2} d(x_{m_k}, Tx_{m_k}) = \frac{1}{2}\epsilon \leq d(x_{m_k}, x_{n_k}).$$

Hence, for all $k \geq n_0$, we have

$$\begin{aligned} \tau + F(d(x_{m_{k+1}}, x_{n_{k+1}})) &= \tau + F(d(Tx_{m_k}, Tx_{n_k})) \\ &\leq F(d(x_{m_k}, x_{n_k})) + LN(x_{m_k}, x_{n_k}), \end{aligned} \tag{2.8}$$

where

$$N(x_{m_k}, x_{n_k}) = \min\{d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), d(x_{m_k}, x_{n_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}.$$

Using Lemma 1.22, (2.7), (F_{3^*}) and taking the limit as $k \to \infty$, we have $\tau + F(\epsilon) \leq F(\epsilon)$ which is a contradiction, therefore we have that $\{x_n\}$ is Cauchy. Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Suppose that T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx.$$

Thus, T has a fixed point.

More so, using the condition that $\beta(x_n) \geq \varphi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\beta(x) \geq \varphi(x)$. We now establish that T has a fixed point. Now, we claim that

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x)$$

or

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x).$$

Suppose on the contrary that there exists $m \in \mathbb{N} \cup \{0\}$, such that

$$\begin{aligned} \frac{1}{2}d(x_m, x_{m+1}) &\geq d(x_m, x), \\ \frac{1}{2}d(x_{m+1}, x_{m+2}) &\geq d(x_{m+1}, x). \end{aligned} \tag{2.9}$$

Now observe that

$$\begin{aligned} 2d(x_m, x) &\leq d(x_m, x_{m+1}) \\ &\leq d(x_m, x) + d(x, x_{m+1}) \\ \Rightarrow d(x_m, x) &\leq d(x, x_{m+1}). \end{aligned} \tag{2.10}$$

It follows from (2.9) and (2.10), we have

$$d(x_m, x) \leq d(x, x_{m+1}) \leq \frac{1}{2}d(x_{m+1}, x_{m+2}). \tag{2.11}$$

Since $\frac{1}{2}d(x_m, x_{m+1}) < d(x_m, x_{m+1})$ and $\alpha(x_m)\beta(x_{m+1}) \geq \varphi(x_m)\varphi(x_{m+1})$ we have that

$$\begin{aligned} \tau + F(d(x_{m+1}, x_{m+2})) &= \tau + F(d(Tx_m, Tx_{m+1})) \\ &\leq F(d(x_m, x_{m+1})) + LN(x_m, x_{m+1}), \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} N(x_m, x_{m+1}) &= \min\{d(x_m, x_{m+1}), d(x_{m+1}, x_{m+2}), d(x_m, x_{m+2}), d(x_{m+1}, x_{m+1})\} \\ &= 0. \end{aligned}$$

It follows that

$$\tau + F(d(x_{m+1}, x_{m+2})) \leq F(d(x_m, x_{m+1})). \tag{2.13}$$

Using the fact that F is strictly increasing, we have that

$$d(x_{m+1}, x_{m+2}) < d(x_m, x_{m+1}).$$

Using this fact, (2.9) and (2.10), we have

$$\begin{aligned} d(x_{m+1}, x_{m+2}) &< d(x_m, x_{m+1}) \\ &\leq d(x_m, x) + d(x, x_{m+1}) \\ &\leq \frac{1}{2}d(x_{m+1}, x_{m+2}) + \frac{1}{2}d(x_{m+1}, x_{m+2}) \\ &= d(x_{m+1}, x_{m+2}), \end{aligned} \tag{2.14}$$

which is a contradiction. Thus we must have that

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x)$$

or

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, x).$$

Thus, we have

$$\tau + F(d(x_{n+1}, Tx)) = \tau + (d(Tx_n, Tx)) \leq F(d(x_n, x)) + L\psi(N(x, y)),$$

where

$$N(x_n, x) = \min\{d(x_n, Tx_n), d(x, Tx), d(x, Tx_n), d(x_n, Tx)\}.$$

Using (F_*) and Lemma 1.17, we have

$$\lim_{n\to\infty} F(d(Tx_n, Tx)) = -\infty$$

so that

$$\lim_{n\to\infty} d(Tx_n, Tx) = 0.$$

Now, observe that

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n+1}, Tx) = \lim_{n\to\infty} d(Tx_n, Tx) = 0.$$

Clearly,

$$d(x, Tx) = 0 \Rightarrow x = Tx.$$ □

Theorem 2.6. *Suppose that the hypothesis of Theorem 2.5 holds and in addition suppose $\alpha(x) \geq \varphi(x)$ and $\beta(y) \geq \varphi(y)$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.*

Proof. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Using our hypothesis that $\alpha(x) \geq \varphi(x), \beta(y) \geq \varphi(y)$, we have $\alpha(x)\beta(y) \geq \varphi(x)\varphi(y)$ and $\frac{1}{2}d(x, Tx) = 0 \leq d(x, y)$, which implies that

$$F(d(x, y)) = F(d(Tx, Ty)) < \tau + F(d(Tx, Ty)) \leq F(d(x, y)) + L\psi(N(x, y))$$

$$\Rightarrow F(d(x, y)) < F(d(x, y)),$$

which is a contradiction, thus T has a unique fixed point. □

Theorem 2.7. *Let (X, d) be a complete metric space and $T : X \to X$ be a TAC-Suzuki-F contraction type mapping. Suppose the following conditions hold:*

(1) *T is a cyclic (α, β)-admissible mapping with respect to φ,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$,*
(3) *T is continuous,*
(4) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq \varphi(x_n)$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq \varphi(x)$.*

Then T has a fixed point.

Proof. The proof follows similar approach as in Theorem 2.5. □

Using Remark 2.2, we obtain the following result.

Corollary 2.8. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\begin{aligned} &\alpha(x)\beta(y) \geq 1 \quad \textit{and} \\ &\frac{1}{2}d(x, Tx) \leq d(x, y) \\ &\Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) + LN(x, y) \ \forall \ x, y \in X. \end{aligned} \tag{2.15}$$

Suppose the following conditions hold:

(1) *T is a cyclic (α, β)-admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
(3) *T is continuous,*
(4) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

Then T has a fixed point.

Corollary 2.9. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\begin{aligned} &\alpha(x)\beta(y) \geq 1 \quad \textit{and} \\ &\frac{1}{2}d(x, Tx) \leq d(x, y) \\ &\Rightarrow \tau + F(d(Tx, Ty)) \leq F(d(x, y)) \ \forall \ x, y \in X. \end{aligned} \tag{2.16}$$

Suppose the following conditions hold:

(1) *T is a cyclic (α, β)-admissible mapping,*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
(3) *T is continuous,*
(4) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

Then T has a fixed point.

Example 2.10. Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$. We defined $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{12} & \text{if} \quad x \in [0, 1], \\ 4x & \text{if} \quad x \in (1, \infty), \end{cases}$$

$\beta, \alpha, \varphi : X \to [0, \infty)$ by

$$\beta(x) = \alpha(x) = \begin{cases} 2 & \text{if} \quad x \in [0, 1], \\ 0 & \text{if} \quad x \in (1, \infty), \end{cases}$$

$$\varphi(x) = \begin{cases} 1.5 & \text{if} \quad x \in [0,1], \\ 3 & \text{if} \quad x \in (1,\infty), \end{cases}$$

and $F(t) = \frac{-1}{t} + t$. Thus T satisfy condition in Theorem 2.6 and that T is a TAC-Suzuki Berinde-F contraction.

Proof. Clearly, for any $x \in [0,1]$, we have that $\alpha(x) > \varphi(x), \beta(x) > \varphi(x)$ and $Tx = \frac{x}{12}$, we also have that $\alpha(Tx) > \varphi(Tx)$ and $\beta(Tx) > \varphi(Tx)$. Clearly T is a cyclic (α,β)-admissible mapping with respect to φ. For any $x_0 \in [0,1]$, we have that $\alpha(x_0) > \varphi(x_0)$ and $\beta(x_0) > \varphi(x_0)$. Let $\{x_n\}$ be a sequence in X with $\beta(x_n) \geq \varphi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, using the definition of β, we must have that $\{x_n\} \subset [0,1]$ and thus $x \in [0,1]$. Hence $\beta(x) > \varphi(x)$. Since $\beta(x)\alpha(y) > \varphi(x)\varphi(y)$ if $x, y \in [0,1]$, we need to show that T is a TAC-Suzuki Berinde-F contraction for any $x, y \in [0,1]$ with $\frac{1}{2}d(x,Tx) \leq d(x,y)$. Let $x, y \in [0,1]$ and without loss of generality, we suppose that $x \leq y$. We then have that $\frac{1}{2}d(x,Tx) = \frac{1}{2}|x - \frac{x}{12}| = \frac{11x}{24}$. Thus for $\frac{1}{2}d(x,Tx) \leq d(x,y)$, we must have that $\frac{35x}{24} \leq y$. For $\tau = 1$ and $L > 2$, it is easy to see that

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)) + LN(x,y)$$

thus T satisfy all the hypothesis of Theorem 2.6, and $x = 0$ is the unique fixed point of T. □

Remark 2.11. We note that Theorem 1.18 and Definition 1.8 is not applicable to the above example. To see this, observe that for $x = 0$ and $y = 4$, we have that

$$\frac{1}{2}d(x,Tx) = 0 \leq 4 = d(x,y)$$

but

$$\begin{aligned} \tau + F(d(Tx,Ty)) &= 1 + F(16) = 1 + 16 - \frac{1}{16} \\ &> 4 - \frac{1}{4} = F(4) = F(d(x,y)). \end{aligned}$$

Also,

$$d(Tx,Ty) = 16 > 4 = d(x,y).$$

Furthermore, the above example is not applicable to Theorem 1.15, to see this observe that for $x = 0$ and $y = 4$, we have that

$$d(Tx,Ty) = 8 > 0$$

but

$$\begin{aligned} \tau + F(d(Tx,Ty)) &= 1 + F(16) = 1 + 16 - \frac{1}{16} \\ &> 4 - \frac{1}{4} = F(4) = F(d(x,y)). \end{aligned}$$

2.2. TAC-(ψ, ϕ)-Suzuki type rational contraction mappings

In this section, we introduce the notion of TAC-(ψ, ϕ)-Suzuki type rational contraction mapping and established the existence and uniqueness results of the fixed point for this class of mappings.

Definition 2.12. Let (X, d) be a metric space, $\alpha, \beta, \varphi : X \to [0, \infty)$ be three functions and T be a self map on X. The mapping T is said to be a TAC-(ψ, ϕ)-Suzuki type rational contraction mapping, if

$$\begin{aligned} &\alpha(x)\beta(y) \geq \varphi(x)\varphi(y) \text{ and} \\ &\frac{1}{2}d(x, Tx) \leq d(x, y) \\ &\Rightarrow \psi(d(Tx, Ty)) \leq \phi(M(x, y)) + L\psi(N(x, y)) \end{aligned} \tag{2.17}$$

for all $x, y \in X$, where $L \geq 0$, ψ is an alternating distance function, $\phi : [0, \infty) \to [0, \infty)$ is a continuous function,

$$\begin{aligned} M(x, y) = &\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2},\right. \\ &\left.\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}, \frac{d(y, Tx)[1 + d(x, Tx)]}{1 + d(x, y)}\right\} \text{ and} \\ N(x, y) = &\min\{d(x, Ty), d(x, Tx), d(y, Tx)\}. \end{aligned}$$

Theorem 2.13. *Let (X, d) be a complete metric space and $T : X \to X$ be a TAC-(ψ, ϕ)-Suzuki type rational contraction mapping. Suppose the following conditions hold:*

1. *$\psi(t) > \phi(t)$ for all $t > 0$,*
2. *T is a cyclic (α, β)-admissible mapping with respect to φ,*
3. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$,*
4. *T is continuous,*
5. *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq \varphi(x_n)$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq \varphi(x)$.*

Then T has a fixed point.

Proof. From our hypothesis, there exists $x_0 \in X$ such that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$. We define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is a cyclic (α, β)-admissible mapping with respect to φ, and $\alpha(x_0) \geq \varphi(x_0)$, we have $\beta(x_1) = \beta(Tx_0) \geq \varphi(Tx_0) = \varphi(x_1)$ and this implies that $\alpha(x_2) = \alpha(Tx_1) \geq \varphi(Tx_1) = \varphi(x_2)$, continuing the process, we have

$$\alpha(x_{2k}) \geq \varphi(x_{2k}) \text{ and } \beta(x_{2k+1}) \geq \varphi(x_{2k+1}) \ \forall\ k \in \mathbb{N} \cup \{0\}. \tag{2.18}$$

Using similar argument, we have that

$$\beta(x_{2k}) \geq \varphi(x_{2k}) \text{ and } \alpha(x_{2k+1}) \geq \varphi(x_{2k+1}) \ \forall\ k \in \mathbb{N} \cup \{0\}. \tag{2.19}$$

It follows from (2.18) and (2.19) that $\alpha(x_n) \geq \varphi(x_n)$ and $\beta(x_n) \geq \varphi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n)\beta(x_{n+1}) \geq \varphi(x_n)\varphi(x_{n+1})$ and $\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1})$, we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \phi(M(x_n, x_{n+1})) + L\psi(N(x_n, x_{n+1})), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) = \ &\max\Bigg\{ d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &\frac{d(x_n, Tx_{n+1}) + d(x_{n+1,}, Tx_n)}{2}, \frac{d(x_n, Tx_n)d(x_{n+1}, Tx_{n+1})}{1 + d(x_n, x_{n+1})}, \\ &\frac{d(x_{n+1}, Tx_n)[1 + d(x_n, Tx_n)]}{1 + d(x_n, x_{n+1})}\Bigg\} \\ = \ &\max\Bigg\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \\ &\frac{d(x_n, x_{n+2}) + d(x_{n+1,}, x_{n+1})}{2}, \frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, \\ &\frac{d(x_{n+1}, x_{n+1})[1 + d(x_n, x_{n+1})]}{1 + d(x_n, x_{n+1})}\Bigg\} \\ = \ &\max\Bigg\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2})}{2}, \\ &\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1 + d(x_n, x_{n+1})}, 0\Bigg\}. \end{aligned}$$

Since $\frac{d(x_n, x_{n+1})}{1+d(x_n, x_{n+1})} < 1$, we obtain $\frac{d(x_n, x_{n+1})d(x_{n+1}, x_{n+2})}{1+d(x_n, x_{n+1})} < d(x_{n+1}, x_{n+2})$. We therefore have that

$$M(x_n, x_{n+1}) = \max\Big\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \Big\},$$

$$N(x_n, x_{n+1}) = \min\Big\{ d(x_n, x_{n+2}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+1} \Big\} = 0.$$

If we suppose that

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2}),$$

we then have that (2.20) becomes

$$\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_{n+1}, x_{n+2})),$$

which contradicts hypothesis (1), thus we must have that

$$M(x_n, x_{n+1}) = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1}),$$

which implies that $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, so that (2.20) becomes

$$\psi(d(x_{n+1}, x_{n+2})) \leq \phi(d(x_n, x_{n+1})).$$

Similarly, we have that

$$\psi(d(x_n, x_{n+1})) \leq \phi(d(x_{n-1}, x_n)), \tag{2.21}$$

using the properties of ψ and ϕ, we have that

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \quad n \in \mathbb{N} \cup \{0\}. \tag{2.22}$$

Therefore, $\{d(x_n, x_{n+1})\}$ is a nonincreasing sequence and bounded below. Thus, there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c. \tag{2.23}$$

Suppose that $c > 0$ and taking the limit of both sides of (2.21), we have that $\psi(c) \leq \phi(c)$, which is a contradiction to condition (1), thus we must have that $c = 0$. So, we have that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.24}$$

We now show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence, then by Lemma 1.22, there exist $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $d(x_{m_k}, x_{n_k}) \geq \epsilon, d(x_{m_k}, x_{n_{k-1}}) < \epsilon$ and

(1) $\lim_{k\to\infty} d(x_{n_k}, x_{m_{k+1}}) = \epsilon$,
(2) $\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \epsilon$,
(3) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_k}) = \epsilon$,
(4) $\lim_{k\to\infty} d(x_{m_{k-1}}, x_{n_{k+1}}) = \epsilon$.

Since $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$, we have that $\alpha(x_{m_k})\beta(x_{n_k}) \geq \varphi(x_{m_k})\varphi(x_{n_k})$ and we can choose $n_0 \in \mathbb{N} \cup \{0\}$ such that

$$\frac{1}{2} d(x_{m_k}, Tx_{m_k}) = \frac{1}{2}\epsilon \leq d(x_{m_k}, x_{n_k}).$$

Hence, for all $k \geq n_0$, we have where

$$\begin{aligned} M(x_{m_k}, x_{n_k}) = \ & \max\Bigg\{ d(x_{m_k}, x_{n_k}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{n_{k+1}}), \\ & \frac{d(x_{n_k}, x_{n_{k+1}}) + d(x_{n_k}, x_{m_{k+1}})}{2}, \frac{d(x_{m_k}, x_{m_{k+1}}) d(x_{n_k}, x_{n_{k+1}})}{1 + d(x_{m_k}, x_{n_k})}, \\ & \frac{d(x_{m_k}, x_{n_{k+1}})[1 + d(x_{n_k}, x_{n_{k+1}})]}{1 + d(x_{m_k}, x_{n_k})} \Bigg\}, \\ N(x_{m_k}, x_{n_k}) = \ & \min\{d(x_{m_k}, x_{n_{k+1}}), d(x_{m_k}, x_{m_{k+1}}), d(x_{n_k}, x_{m_{k+1}})\}. \end{aligned}$$

Using Lemma 1.22, (2.24) and taking the limit as $k \to \infty$, we obtain $\psi(\epsilon) \leq \phi(\epsilon)$ which contradicts condition (1) and thus applying Lemma 1.13, we get that $\epsilon = 0$. This contradicts the assumption that $\epsilon > 0$. We therefore have that $\{x_n\}$ is Cauchy.

Since (X, d) is complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Suppose that T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx.$$

Thus, T has a fixed point.

More so, using the condition that $\beta(x_n) \geq \varphi(x_n)$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\beta(x) \geq \varphi(x)$. We establish that T has a fixed point. Now suppose that

$$d(x_n, x) \leq \frac{1}{2} d(x_n, x_{n+1})$$

and

$$d(x_{n+1}, x) \leq \frac{1}{2} d(x_{n+1}, x_{n+2}).$$

Then using the fact that $d(x_{n+1}, x_{n+2}) \leq d(x_n, x_{n+1})$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, x) + d(x, x_{n+1}) \\ &< \frac{1}{2} d(x_n, x_{n+1}) + \frac{1}{2} d(x_{n+1}, x_{n+2}) \\ &= d(x_n, x_{n+1}). \end{aligned}$$

The above inequality is a contradiction, thus, we must have that

$$d(x_n, x) \geq \frac{1}{2} d(x_n, x_{n+1}) \quad \text{or} \quad d(x_{n+1}, x) \geq \frac{1}{2} d(x_{n+1}, x_{n+2}).$$

Hence, we have

$$\psi(d(x_{n+1}, Tx)) = \psi(d(Tx_n, Tx)) \leq \phi(M(x_n, x)) + L\psi(N(x, y)),$$

where

$$\begin{aligned} M(x_n, x) = & \max\left\{ d(x_n, x), d(x_n, Tx_n), d(x, Tx), \frac{d(x_n, Tx) + d(x, Tx_n)}{2}, \right. \\ & \left. \frac{d(x_n, Tx_n) d(x, Tx)}{1 + d(x_n, x)}, \frac{d(x, Tx_n)[1 + d(x_n, Tx_n)]}{1 + d(x_n, x)} \right\}, \\ N(x_n, x) = & \min\{d(x_n, Tx), d(x_n, Tx_n), d(x, Tx_n)\}. \end{aligned}$$

Taking limit as $n \to \infty$ and using the properties of ψ and ϕ, we have that

$$\psi(d(x, Tx)) \leq \phi(d(x, Tx)),$$

which contradicts condition (1) and by Lemma 1.13, we have that

$$d(x, Tx) = 0 \Rightarrow x = Tx.$$

Hence, T has a fixed point. □

Theorem 2.14. *Suppose that the hypothesis of Theorem 2.13 holds and in addition suppose $\alpha(x) \geq \varphi(x)$ and $\beta(y) \geq \varphi(y)$ for all $x, y \in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.*

Proof. Let $x, y \in F(T)$, that is $Tx = x$ and $Ty = y$ such that $x \neq y$. Using our hypothesis that $\alpha(x) \geq \varphi(x), \beta(y) \geq \varphi(y)$, we have $\alpha(x)\beta(y) \geq \varphi(x)\varphi(y)$ and $\frac{1}{2}d(x, Tx) = 0 \leq d(x, y)$, which implies that

$$\psi(d(x,y)) = \psi(d(x,y)) \leq \phi(M(x,y)) + L\psi(N(x,y)),$$

where

$$\begin{aligned} M(x,y) =\ & \max\left\{d(x,y), d(x,Tx), d(y,,Ty), \frac{d(x,Ty)+d(y,Tx)}{2},\right. \\ & \left.\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Tx)]}{1+d(x,y)}\right\} = d(x,y), \\ N(x,y) =\ & \min\{d(x,Ty), d(x,Tx), d(y,Tx)\} = 0. \end{aligned}$$

$$\psi(d(x,y)) \leq \phi(d(x,y)),$$

which contradicts condition (1) and by Lemma 1.13, we have that

$$d(x,y) = 0 \Rightarrow x = y.$$

Thus, T has a unique fixed point. □

Definition 2.15. Let (X, d) be a metric space, $\alpha, \beta : X \to [0, \infty)$ be two functions and T be a self map on X. The mapping T is said to be a TAC-1-(ψ, ϕ)-Suzuki type rational contraction mapping, if

$$\begin{aligned} & \alpha(x)\beta(y) \geq \varphi(x)\varphi(y) \quad \text{and} \\ & \frac{1}{2}d(x,Tx) \leq d(x,y) \\ \Rightarrow\ & \psi(d(Tx,Ty)) \leq \phi(M(x,y)) \end{aligned} \tag{2.25}$$

for all $x, y \in X$, where ψ is an alternating distance function, $\phi : [0, \infty) \to [0, \infty)$ is a continuous function,

$$\begin{aligned} M(x,y) =\ & \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2},\right. \\ & \left.\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Tx)]}{1+d(x,y)}\right\}. \end{aligned}$$

Theorem 2.16. *Let (X, d) be a complete metric space and $T : X \to X$ be a TAC-1-(ψ, ϕ)-Suzuki type rational contraction mapping. Suppose the following conditions hold:*

1. *$\psi(t) > \phi(t)$ for all $t > 0$,*
2. *T is a cyclic (α, β)-admissible mapping with respect to φ,*
3. *there exists $x_0 \in X$ such that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$,*
4. *T is continuous,*
5. *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq \varphi(x_n)$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq \varphi(x)$.*

Then T has a fixed point.

Proof. The proof follows similar approach as in Theorem 2.13. □

Using Remark 2.2, we obtain the following results.

Corollary 2.17. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\begin{aligned} &\alpha(x)\beta(y) \geq 1 \quad \text{and} \\ &\frac{1}{2}d(x,Tx) \leq d(x,y) \\ \Rightarrow\ &\psi(d(Tx,Ty)) \leq \phi(M(x,y)) + L\psi(N(x,y)) \end{aligned} \tag{2.26}$$

for all $x, y \in X$, where $L \geq 0$, ψ is an alternating distance function, $\phi : [0,\infty) \to [0,\infty)$ is a continuous function,

$$\begin{aligned} M(x,y) = \max\Big\{&d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Tx)]}{1+d(x,y)}\Big\} \quad \text{and} \\ N(x,y) = \min\{&d(x,Ty), d(x,Tx), d(y,Tx)\}. \end{aligned}$$

Suppose the following conditions hold:

(1) *$\psi(t) > \phi(t)$ for all $t > 0$,*
(2) *T is a cyclic (α,β)-admissible mapping,*
(3) *there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$,*
(4) *T is continuous,*
(5) *if for any sequence $\{x_n\}$ in X with $\beta(x_n) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x) \geq 1$.*

Then T has a fixed point.

Corollary 2.18. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\begin{aligned} &\alpha(x)\beta(y) \geq 1 \quad \text{and} \\ &\frac{1}{2}d(x,Tx) \leq d(x,y) \\ \Rightarrow\ &\psi(d(Tx,Ty)) \leq \phi(M(x,y)) \end{aligned} \tag{2.27}$$

for all $x, y \in X$, where ψ is an alternating distance function, $\phi : [0,\infty) \to [0,\infty)$ is a continuous function,

$$\begin{aligned} M(x,y) = \max\Big\{&d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}, \\ &\frac{d(x,Tx)d(y,Ty)}{1+d(x,y)}, \frac{d(y,Tx)[1+d(x,Tx)]}{1+d(x,y)}\Big\}. \end{aligned}$$

Suppose the following conditions hold:

(1) *$\psi(t) > \phi(t)$ for all $t > 0$,*
(2) *T is a cyclic (α,β)-admissible mapping,*

(3) *there exists* $x_0 \in X$ *such that* $\alpha(x_0) \geq 1$ *and* $\beta(x_0) \geq 1$,
(4) T *is continuous,*
(5) *if for any sequence* $\{x_n\}$ *in* X *with* $\beta(x_n) \geq 1$ *for all* $n \geq 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x) \geq 1$.

Then T *has a fixed point.*

Example 2.19. Let $X = [0, \infty)$ with the usual metric $d(x, y) = |x - y|$. We defined $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{6} & \text{if} \quad x \in [0,1], \\ 2x & \text{if} \quad x \in (1, \infty), \end{cases}$$

$\alpha, \beta, \varphi : X \to [0, \infty)$ by

$$\alpha(x) = \beta(x) = \begin{cases} 2 & \text{if} \quad x \in [0,1], \\ 0 & \text{if} \quad x \in (1, \infty), \end{cases}$$

$$\varphi(x) = \begin{cases} 1 & \text{if} \quad x \in [0,1], \\ 0 & \text{if} \quad x \in (1, \infty), \end{cases}$$

and $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t^2$ and $\phi(t) = \log(t + 1)$. Thus T satisfy condition in Theorem 2.14 and T is a TAC-(ψ, ψ)-Suzuki type mapping.

Proof. Clearly, for any $x \in [0, 1]$, we have that $\alpha(x) \geq \varphi(x), \beta(x) \geq \varphi(x)$ and $Tx = \frac{x}{12}$, we also have that $\alpha(Tx) \geq \varphi(Tx)$ and $\beta(Tx) \geq \varphi(Tx)$. Clearly T is a cyclic (α, β)-admissible with respect to φ. For any $x_0 \in [0, 1]$, we have that $\alpha(x_0) \geq \varphi(x_0)$ and $\beta(x_0) \geq \varphi(x_0)$. Let $\{x_n\}$ be a sequence in X with $\beta(x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, using the definition of β, we must have that $\{x_n\} \subset [0, 1]$ and thus $x \in [0, 1]$. Hence $\beta(x) \geq \varphi(x)$. Furthermore, it is clear that $\psi(t) \geq \phi(t)$ for all $t > 0$. Since $\beta(x)\alpha(y) \geq \varphi(x)\varphi(y)$ if $x, y \in [0, 1]$, we need to show that T is a TAC-(ψ, ψ)-Suzuki type mapping for any $x, y \in [0, 1]$ with $\frac{1}{2}d(x, Tx) \leq d(x, y)$. Let $x, y \in [0, 1]$ and without loss of generality, we suppose that $x \leq y$. We then have that $\frac{1}{2}d(x, Tx) = \frac{1}{2}|x - \frac{x}{6}| = \frac{5x}{12}$. Thus for $\frac{1}{2}d(x, Tx) \leq d(x, y)$, we must have that $\frac{17x}{12} \leq y$. Now, observe that for $L > 3$, we have

$$\begin{aligned} \psi(d(Tx, Ty)) &= \psi(|\frac{y}{6} - \frac{x}{6}|) = \psi(\frac{1}{6}|y - x|) = \frac{1}{36}|y - x|^2 \\ &\leq \log[|y - x| + 1] + |y - x|^2 \\ &= \phi(|y - x|) + \psi(|y - x|) \\ &\leq \phi(M(x, y)) + L\psi(N(x, y)), \end{aligned}$$

thus T satisfy all the hypothesis of Theorem 2.14 and $x = 0$ is the unique fixed point of T. □

Remark 2.20. We note that Theorem 1.11 and Definition 1.8 is not applicable to the above example. To see this, observe that for $x = 0$ and $y = 4$, we have that

$$\frac{1}{2}d(x, Tx) = 0 \leq 4 = d(x, y)$$

but

$$\psi(d(Tx, Ty)) = \psi(8) = 64 \geq 16 - \phi(4) = \psi(M(x, y)) - \phi(M(x, y))$$

for any $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. More so,

$$d(Tx, Ty) = 8 > 4 = d(x, y).$$

It is also easy to see that Theorem 1.3 is not applicable. For any δ, $x = 0$ and $y = 4$, we have that

$$\psi(d(Tx, Ty)) = \psi(8) = 64 > \delta(16) = \delta\psi(4) = \delta d(x, y).$$

3. Conclusion

In this paper, we introduce the notion of cyclic (α, β)-admissible mapping with respect to φ, modified TAC-Suzuki-Berinde type F-contraction and modified TAC-(ψ, ϕ)-Suzuki type rational mappings in the frame work of complete metric spaces. We also established and obtained fixed point theorems in such spaces. More so, we present some examples to support our main theorems.

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