ON FINITE FAMILY OF MONOTONE VARIATIONAL INCLUSION PROBLEMS IN REFLEXIVE BANACH SPACE

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The main purpose of this paper is to study monotone variational inclusion problems in a reflexive real Banach space. We propose a Halpern-type algorithm and prove that the sequence generated by it converges strongly to a common solution of a finite family of monotone vairiational inclusion problems in a reflexive real Banach space. We then apply our results to solve a finite family of variational inequality problems and convex feasibility problem.

Keywords: Monotone variational inclusion problem, maximal monotone mappings, Bregman inverse strongly monotone mappings, resolvent operators, anti-resolvent operators, Bregman firmly nonexpansive mapping.

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1. Introduction

Let C be a nonempty closed and convex subset of a real Banach space X and X^* be the dual space of X. A point $x \in C$ is called a fixed point of T if $Tx = x$. We say that x is an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to x and $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$. Throughout this paper, we shall denote the set of fixed points and asymptotic fixed points of T by $F(T)$ and $\hat{F}(T)$ respectively. We shall also denote by 0^*, the zero element of the dual space X^* (see [41]).

The theory of monotone mappings is one of the most important areas of research in nonlinear and convex analysis due to the role it plays in optimization theory, variational inequalities, semi group theory, evolution equations, among others (see [6, 20, 23, 25, 26, 27, 28, 40, 41, 52]). An important problem in this area of research is the following Monotone Inclusion Problem (MIP), also known as the null point problem:

$$\text{Find } x \in X \text{ such that } 0^* \in Bx, \tag{1.1}$$

where $B : X \to 2^{X^*}$ is a monotone mapping. The solution set of Problem (**1**.1) is denoted by $B^{-1}(0^*)$. Problem (**1**.1) describes the equilibrium or stable state of an evolution system governed by the monotone mapping, which is very important in ecology, physics, economics, among others (see [7, 18, 20, 37, 50] and the references therein). Also, many optimization (and other related mathematical) problems can be modeled as Problem (**1**.1). Thus, MIP is of central importance in the theory of monotone mappings.

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A popular method for solving Problem (**1**.1), known as the Proximal Point Algorithm (PPA) was introduced in Hilbert spaces by Martinet [34] and was later developed by Rockafeller [44], Bruck and Reich [12], see also [5]. These authors prove that the PPA which generates a sequence:

$$x_{n+1} = J_\lambda^B x_n, \tag{1.2}$$

where $J_\lambda^B = (I + \lambda B)^{-1}$ is the resolvent operator of the maximal monotone mapping B, converges weakly to a solution of (**1**.1). Since then, many authors have also studied the MIP in Hilbert spaces (see [29, 36] and the references therein). The study of Problem (**1**.1) was extended to real Banach spaces. For instance, Butnariu and Resmerita [14] studied Problem (**1**.1) in a reflexive real Banach space when B is an inverse-monotone mapping from X to X^* (see [14, Section 5]). Later, Riech and Sabach [41] introduced the following algorithm for approximating a finite family of MIPs:

$$\begin{cases} x_0 \in X; \\ y_n^i = \mathrm{Res}^f_{\lambda_n^i B_i}(x_n + e_n^i); \\ C_n^i = \{z \in X : D_f(z, y_n^i) \le D_f(z, x_n + e_n^i)\}; \\ C_n := \cap_{i=1}^N C_n^i; \\ Q_n = \{z \in X : \langle \nabla f(x_0) - \nabla f(x_n), z - x_n\rangle \le 0\}; \\ x_{n+1} = P^f_{C_{n+1}}(x_0),\ n \ge 0, \end{cases} \tag{1.3}$$

where $\mathrm{Res}^f_{\lambda_n^i B_i}$ is the resolvent associated with the maximal monotone mappings B_i, $i = 1, 2, \dots, N$ and $P^f_{C_{n+1}}$ is the Bregman projection of X onto C_{n+1} (we shall define these terms in the next section). By using the technique of Bregman distance, they obtained a strong convergence result for Algorithm (**1**.3).

A very important generalization of Problem (**1**.1) is the following Monotone Variational Inclusion Problem (MVIP): Find $x \in X$ such that

$$0^* \in A(x) + B(x), \tag{1.4}$$

where $A : X \to X^*$ is a single-valued monotone mapping and $B : X \to 2^{X^*}$ is a multivalued monotone mapping. The solution set of Problem (**1**.4) is denoted by $(A+B)^{-1}(0^*)$. MVIP is generally known to be an important tool for solving problems arising from mechanics, optimization, nonlinear programming, machine learning, linear inverse problems, economics, finance, applied sciences, among others (see for example [1, 2, 21, 22, 24, 48, 49, 51] and the references therein).

The classical method for solving MVIP (**1**.4) is the following forward-backward splitting method (which is more general than the PPA) introduced by Lions and Mercier [31] (and independently by Passty [38]):

$$\begin{cases} x_1 \in X, \\ x_{n+1} = J_\lambda^B (I - \lambda A)x_n,\ n \ge 1, \end{cases} \tag{1.5}$$

where $\lambda > 0$. This method has been used by many authors to solve Problem (**1**.4) in real Hilbert spaces when B and A are monotone mappings (see [1, 2, 45, 22, 21]). The study of MVIP has recently been extended from the framework of Hilbert spaces to general Banach spaces. For example, Lopez *et. al.* [32] introduced and studied an Halpern-type forward-backward splitting method for approximating solutions of MVIP in a uniformly convex and q-uniformly smooth Banach spaces when $B : X \to 2^X$ and $A : X \to X$ are m-accretive and inverse strongly accretive mappings respectively. Inspired by the results of Lopez *et. al.* [32], Cholamjiak [19], proposed and studied a viscosity-type forward-backward splitting method for approximating solutions of MVIP in a uniformly convex and q-uniformly smooth Banach spaces when $B : X \to 2^X$ and $A : X \to X$ are m-accretive and inverse strongly accretive mappings respectively. Also, Wei and Duan [53] extended the results of Lopez *et. al.* [32] from uniformly convex and q-uniformly smooth Banach spaces to uniformly smooth and uniformly convex Banach spaces. Furthermore, Shehu and Cai [47] extended the results of Cholamjiak [19] from uniformly convex and q-uniformly smooth Banach spaces to uniformly smooth and uniformly convex Banach spaces.

It is worth mentioning that in the works of Lopez *et. al.* [32], Cholamjiak [19], Wei and Duan [53], Shehu and Cai [47], and other related works in this direction, B and A are assumed to be accretive mappings from X to X. Therefore, their results cannot be used to solve Problem (**1**.4) where B and A are required to be monotone mappings from X to X^* (which is a more general problem).

Motivated by this, we generalize the results of Lopez *et. al.* [32], Cholamjiak [19], Wei and Duan [53], Shehu and Cai [47] from uniformly smooth and uniformly convex Banach spaces to the framework of reflexive Banach spaces. We prove that the sequence generated by our proposed algorithm converges strongly to a common solution of a finite family of MVIP (**1**.4) when B and A are monotone mappings from X to X^*. Furthermore, we applied our results to solve a finite family of variational inequality problems and convex feasibility problem. Our results also generalize the results of Riech and Sabach [41] from solving Problem (**1**.1) to solving Problem (**1**.4).

2. Preliminaries

Let X be a reflexive real Banach space and C be a nonempty, closed and convex subset of X. Let $f : X \to (-\infty, +\infty]$ be a function. Then, the domain of f is defined as

$$\text{dom}f := \{x \in X : f(x) < +\infty\}.$$

The function $f : X \to (-\infty, +\infty]$ is called convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)\ \forall x, y \in X,\ \lambda \in (0,1).$$

f is called proper, if $\text{dom}f \neq \emptyset$. The function $f : \text{dom}f \subseteq X \to (-\infty, \infty]$ is said to be lower semicontinuous at a point $x \in \text{dom}f$ if

$$f(x) \le \liminf_{n\to\infty} f(x_n), \tag{2.1}$$

for each sequence $\{x_n\}$ in $\text{dom}f$ such that $\lim\limits_{n\to\infty} x_n = x$. f is said to be lower semicontinuous on $\text{dom}f$ if it is lower semicontinuous at any point in $\text{dom}f$. Throughout this paper, $f : X \to (-\infty, +\infty]$ is a proper convex and lower semicontinuous function.

Definition 2.1. *(see* [11, 15]*). The bifunction $D_f : domf \times int(domf) \to [0, +\infty)$, which is defined by*

$$D_f(x, y) := f(x) - f(y) - \langle \nabla f(y), x - y\rangle, \tag{2.2}$$

is called the Bregman distance.

It is generally known that the Bregman distance does not satisfy the properties of a metric, however, it has an important property, called the three point identity. That is, for any $x \in \text{dom}f$ and $y, z \in \text{int dom}f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{2.3}$$

The Fenchel conjugate of f is the function $f^* : X^* \to (-\infty, \infty]$, defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}.$$

Let $x \in \text{int dom } f$, then for any $y \in X$, we define the right-hand derivative of f at x by

$$f'(x, y) := \lim_{t\to 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{2.4}$$

The function f is said to be Gâteaux differentiable at x if the limit in (**2**.4) exists as $t \to 0$ for each $y \in X$. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle := f'(x, y)$ for all $y \in X$. f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in$ int dom$f$. If the limit in (**2**.4) is attained uniformly for any $y \in X$ with $||y|| = 1$, we say that f is Fréchet differentiable at x. Whenever, the limit in (**2**.4) is attained uniformly for any $x \in C$ and for any $y \in X$ with $||y|| = 1$, then we say that the function f is uniformly Fréchet differentiable on subset C of X. The function f is called Legendre if the following two conditions hold:

(i) f is Gâteaux differentiable, int dom$f \neq \emptyset$ and dom∇f = int(dom$f$);

(ii) f^* is Gâteaux differentiable and int dom$f^* \neq \emptyset$ and dom∇f^* = int(domf^*).

It has been shown that $\nabla f = (\nabla f^*)^{-1}$ in reflexive Banach spaces (see [9, 30]). Thus combining this fact with conditions (i) and (ii) above, we have that
ran ∇f = dom ∇f^* = int dom f^* and $ran\nabla f^*$ = dom ∇f = int dom$f$.
We also know that f is Legendre if and only if f^* is Legendre (see [8, Corollary 5.5]) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains. The function f is called totally convex at a point $x \in$ int(dom$f$) if its modulus of total convexity at x, that is, v_f : int(dom$f$) $\times [0, +\infty) \to [0, +\infty]$, defined by

$$v_f(x, t) := \inf\{D_f(y, x) : y \in \text{dom}f,\ ||y - x|| = t\}, \tag{2.5}$$

is positive whenever $t > 0$ (see [10, 13, 14]). f is said to be totally convex whenever it is totally convex on every point $x \in$ int(dom$f$). In addition, the function f is called totally convex on bounded sets if $v_f(C, t)$ is positive for any nonempty bounded subset C of X and for any $t > 0$, where the modulus of totally convexity of the function f on the set C is the function v_f : int(dom$f$) $\times [0, +\infty) \to [0, +\infty]$ defined by

$$v_f(C, t) := \inf\{v_f(x, t) | x \in C \cap \text{dom}f\}.$$

We know that the function f is totally convex on bounded subsets if and only if f is uniformly convex on bounded subsets (see [14, Theorem 2.10]).

Definition 2.2. *(see* [14]*). The function $f : X \to \mathbb{R}$ is called sequentially consistent, if for any sequences $\{x_n\}$ and $\{y_n\}$ in int(domf) and domf respectively, such that $\{x_n\}$ is bounded and*

$$\lim_{n\to\infty} D_f(y_n, x_n) = 0 \implies \lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 2.1. [13]. *The function $f : X \to \mathbb{R}$ is totally convex on bounded sets if and only if it is sequentially consistent.*

Definition 2.3. [41] *Let $B : X \to 2^{X^*}$ be a multivalued mapping. Then B is called monotone, if for any $x, y \in domB$, we have*

$$\langle u - v, x - y\rangle \geq 0 \ \forall u \in Bx \text{ and } v \in By. \tag{2.6}$$

B is called maximal monotone, if B is monotone and the graph of B is not properly contained in the graph of any other monotone mapping.

Let $B : X \to 2^{X^*}$ be a multivalued mapping. Then the resolvent associated with B and λ for any $\lambda > 0$, is the operator $\text{Res}^f_{\lambda B} : X \to 2^X$ defined by

$$\text{Res}^f_{\lambda B} = (\nabla f + \lambda B)^{-1} \circ \nabla f. \tag{2.7}$$

Lemma 2.2. *(see* [41]*). Let $B : X \to 2^{X^*}$ be a maximal monotone mapping such that $B^{-1}(0^*) \neq \emptyset$. Then*

$$D_f\left(u, Res^f_{\lambda B}(x)\right) + D_f\left(Res^f_{\lambda B}(x), x\right) \leq D_f(u, x), \tag{2.8}$$

for all $\lambda > 0$, $u \in B^{-1}(0^*)$ *and* $x \in X$. *Furthermore,* $B^{-1}(0^*) = F(Res^f_{\lambda B})$ *and* $Res^f_{\lambda B}$ *is singlevalued.*

Definition 2.4. *Let* C *be a nonempty closed and convex subset of a reflexive Banach space* X. *Then the mapping* $A : X \to 2^{X^*}$ *is called Bregman Inverse Strongly Monotone (BISM) on the set* C, *if*

$$C \cap (domf) \cap (int\ domf) \neq \emptyset \tag{2.9}$$

and for any $x, y \in C \cap int(domf)$, $u \in Ax$ *and* $v \in Ay$, *we have*

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0. \tag{2.10}$$

Remark 2.1. *The BISM class of mappings is more general than the class of firmly nonexpansive operators in Hilbert spaces (see* [30]*).*

The anti-resolvent $A^f_\lambda : X \to 2^X$ associated with a mapping $A : X \to 2^{X^*}$ and $\lambda > 0$ is defined by

$$A^f_\lambda := \nabla f^* \circ (\nabla f - \lambda A). \tag{2.11}$$

Lemma 2.3. [30] *Let* $f : X \to (-\infty, +\infty]$ *be a Legendre function and let* $A : X \to 2^{X^*}$ *be a BISM mapping such that* $A^{-1}(0^*) \neq \emptyset$. *Then for any* $\lambda > 0$, *we have the following:*

(i) $A^{-1}(0^*) = F(A^f_\lambda)$ *and* A^f_λ *is singlevalued.*

(ii) *For any* $u \in A^{-1}(0^*)$ *and* $x \in$*(dom*A^f_λ*), we have*

$$D_f(u, A^f x) + D_f(A^f x, x) \leq D_f(u, x).$$

Remark 2.2. *It follows easily from* (**2**.7) *and* (**2**.11) *that*

$$(A + B)^{-1}(0^*) = F(Res^f_{\lambda B} \circ A^f_\lambda), \tag{2.12}$$

where A *and* B *are singlevalued and multivalued mappings respectively. If in addition,* A *and* B *are BISM and maximal monotone mappings respectively, then it follows from Lemma* **2**.*2 and Lemma* **2**.*3 that the composition* $Res^f_{\lambda B} \circ A^f_\lambda$ *is also singlevalued for any* $\lambda > 0$.

Let C be a nonempty closed and convex subset of int(dom$f$) and T be a mapping on C. The mapping T is called

(i) *Bregman Firmly Nonexpansive (BFNE)* if

$$\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \forall x, y \in C, \tag{2.13}$$

(ii) *Quasi-Bregman Firmly Nonexpansive (QBFNE)* if $F(T) \neq \emptyset$ and

$$\langle \nabla f(x) - \nabla f(Tx), Tx - y\rangle \geq 0\ \forall x \in C,\ y \in F(T), \tag{2.14}$$

(ii) *Quasi-Bregman Nonexpansive (QBNE)* if $F(T) \neq \emptyset$ and

$$D_f(y, Tx) \leq D_f(y, x)\ \forall x \in C,\ y \in F(T),$$

(iii) *Bregman Strongly Nonexpansive (BSNE)* with $\hat{F}(T) \neq \emptyset$ if

$$D_f(y, Tx) \leq D_f(y, x)\ \forall x \in C,\ y \in \hat{F}(T)$$

and for any bounded sequence $\{x_n\}_{n\geq 1} \subset C$,

$$\lim_{n\to\infty} (D_f(y, x_n) - D_f(y, Tx_n)) = 0$$

implies

$$\lim_{n\to\infty} D_f(Tx_n, x_n) = 0.$$

Remark 2.3. *(see* [30]*). It is known that if* T *is BFNE and* $f : X \to \mathbb{R}$ *is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of* X, *then* $F(T) = \hat{F}(T)$ *and* $F(T)$ *is closed and convex.*

Remark 2.4. *(see* [30][40]*).*

(i) *It is easy to see from the definition of Bregman distance that* (**2**.13) *and*

$$D_f(Tx,Ty)+D_f(Ty,Tx)+D_f(Tx,x)+D_f(Ty,y)\le D_f(Tx,y)+D_f(Ty,x)$$

are equivalent.

(ii) *Also, it is not difficult to see that* (**2**.14) *and* $D_f(y,Tx)+D_f(Tx,x)\le D_f(y,x)$ *are equivalent.*

(iii) *We can easily see that if* $F(T)\neq\emptyset$*, then BFNE* $\subset$ *QBFNE* $\subset$ *QBNE. If in addition,* $\hat{F}(T)=F(T)\neq\emptyset$*, then QBFNE* $\subset$ *BSNE.*

Lemma 2.4. *(*[30]*): Assume that* $f:X\to\mathbb{R}$ *is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of* X*. Let* $\{T_i:\ 1\le i\le N\}$ *be BSNE operators which satisfy* $\hat{F}(T_i)=F(T_i)$ *for each* $1\le i\le N$ *and let* $T:=T_NT_{N-1}\dots T_1$*. If*

$$\cap\{F(T_i):1\le i\le N\}$$

is nonempty, then T *is also BSNE with* $F(T)=\hat{F}(T)$*.*

Definition 2.5. [11] *Let* X *be a reflexive real Banach space and* C *be a nonempty closed and convex subset of* X*. A Bregman projection of* $x\in int(domf)$ *onto* $C\subset int(domf)$ *is the unique vector* $P_C^f(x)\in C$ *satisfying*

$$D_f\left(P_C^f(x),x\right)=\inf\{D_f(y,x):y\in C\}. \tag{2.15}$$

Lemma 2.5. [14] *Let* C *be a nonempty closed and convex subset of* X *and* $x\in X$*. Let* $f:X\to\mathbb{R}$ *be a Gáteaux differentiable and totally convex function. Then,*

(i) $z=P_C^f(x)$ *if and only if* $\langle\nabla f(x)-\nabla f(z),y-z\rangle\le 0,\ \forall y\in C.$

(ii) $D_f(y,P_C^f(x))+D_f(P_C^f(x),x)\le D_f(y,x)\ \forall y\in C.$

Lemma 2.6. [42]*. If* $f:X\to\mathbb{R}$ *is uniformly Fréchet differentiable and bounded on bounded subsets of* X*, then* ∇f *is uniformly continuous on bounded subsets of* X *from the strong topology of* X *to the strong topology of* X^**.*

Lemma 2.7. [39] *Let* $f:X\to(-\infty,+\infty]$ *be a proper, convex and lower semicontinuous function, then* $f^*:X\to(-\infty,+\infty]$ *is a proper convex weak* lower semicontinuous function. Thus, for all* $z\in X$*, we have*

$$D_f\left(z,\nabla f^*\left(\sum_{i=1}^N t_i\nabla f(x_i)\right)\right)\le\sum_{i=1}^N t_iD_f(z,x_i), \tag{2.16}$$

where $\{x_i\}\subseteq X$ *and* $\{t_i\}\subset(0,1)$ *with* $\sum\limits_{i=1}^N t_i=1$*.*

Lemma 2.8. [35] *Let* $f:X\to\mathbb{R}$ *be a Gâteaux differentiable function on int(domf) such that* ∇f^* *is bounded on bounded subset of* $domf^*$*. Let* $x^*\in X$ *and* $\{x_n\}\subset int(X)$*. If* $\{D_f(x,x_n)\}$ *is bounded, so is the sequence* $\{x_n\}$*.*

Let $f:X\to\mathbb{R}$ be a Legendre and Gâteaux differentiable function. Then, the function $V_f:X\times X^*\to[0,+\infty)$ associated with f is defined by (see [4, 15, 46])

$$V_f(x,x^*)=f(x)-\langle x,x^*\rangle+f^*(x^*),\ \forall x\in X,\ x^*\in X^*. \tag{2.17}$$

The function V_f is nonnegative and

$$V_f(x,x^*)=D_f(x,\nabla f(x^*)),\ \forall x\in X,\ x^*\in X^*. \tag{2.18}$$

Furthermore (see [43]), V_f satisfies

$$V_f(x,x^*)+\langle y^*,\nabla f^*(x^*)-x\rangle\le V_f(x,x^*+y^*),\ \forall x\in X,x^*,y^*\in X^*. \tag{2.19}$$

Lemma 2.9. [54]. *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\delta_n + \gamma_n,\ n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:
(i) $\{\alpha_n\} \subset [0,1]$, $\Sigma_{n=0}^{\infty}\alpha_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\gamma_n \geq 0 (n \geq 0)$, $\Sigma_{n=0}^{\infty}\gamma_n < \infty$.
Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.10. [33]. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1}$ $\forall j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ when the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$.

3. Main Results

Lemma 3.1. *Let X be a reflexive real Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of X. Let T_i, $i = 1, 2, \ldots, N$ be QBFNE on X and $F_N = T_N \circ T_{N-1} \circ \cdots \circ T_1$. Assume that $\cap_{i=1}^{N} F(T_i) \neq \emptyset$, then $F(F_N) = \cap_{i=1}^{N} F(T_i)$.*

Proof. Clearly, $\cap_{i=1}^{N} F(T_i) \subseteq F(F_N)$. Thus, we will only have to show that $F(F_N) \subseteq \cap_{i=1}^{N} F(T_i)$. Since $\cap_{i=1}^{N} F(T_i) \neq \emptyset$, we have that $F(F_N) \neq \emptyset$. Thus, for any $x \in F(F_N)$ and $y \in \cap_{i=1}^{N} F(T_i) \neq \emptyset$, we have that

$$D_f(y, x) = D_f(y, F_N x). \tag{3.1}$$

Since T_i is QBFNE for each $i = 1, 2, \ldots, N$, we obtain from Remark **2**.4 (ii), (iii) and (**3**.1) that

$$\begin{aligned} D_f(T_N(F_{N-1}x), F_{N-1}x) &\leq D_f(y, F_{N-1}x) - D_f(y, T_N(F_{N-1}x)) \\ &\leq D_f(y, F_{N-2}x) - D_f(y, T_N(F_{N-1}x)) \\ &\leq D_f(y, x) - D_f(y, T_N(F_{N-1}x)) \\ &\leq D_f(y, F_N x) - D_f(y, F_N x) = 0, \end{aligned}$$

which implies that

$$F_N x = F_{N-1}x. \tag{3.2}$$

Note that $F_{N-1} = T_{N-1} \circ T_{N-2} \circ \cdots \circ T_1$. Thus, by similar argument, we can show that

$$F_{N-1}x = F_{N-2}x. \tag{3.3}$$

By repeating the same process, we obtain

$$F_N x = F_{N-1}x = F_{N-2}x = F_{N-3}x = \cdots = F_2 x = F_1 x = x. \tag{3.4}$$

From (**3**.4), we obtain

$$x = T_1 x. \tag{3.5}$$

From (**3**.4) and (**3**.5), we obtain

$$x = F_2 x = T_2(T_1 x) = T_2 x. \tag{3.6}$$

As in (**3**.5)-(**3**.6), we can show that

$$x = T_1 x = T_2 x = \cdots = T_{N-1}x = T_N x. \tag{3.7}$$

Thus, we obtain that $F(F_N) \subseteq \cap_{i=1}^{N} F(T_i)$. □

Lemma 3.2. *Let X be a reflexive real Banach space and $f : X \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of X. Let $B : X \to 2^{X^*}$ be a maximal monotone mapping and T be a QBFNE mapping on X. Suppose that $F(Res^f_{\lambda B}) \cap (T) \neq \emptyset$, then $Res^f_{\lambda B} \circ T$ is also a QBFNE mapping.*

Proof. From (**2**.3), we obtain for all $x \in X$, $y \in F(\mathrm{Res}^f_{\lambda B}) \cap (T) \subseteq F(\mathrm{Res}^f_{\lambda B} \circ T)$ and $\lambda > 0$ that

$$\begin{aligned} D_f(y, \mathrm{Res}^f_{\lambda B}(Tx)) & \\ + \; & D_f(\mathrm{Res}^f_{\lambda B}(Tx), x) - D_f(y, x) \\ = \; & \langle \nabla f(x) - \nabla f(\mathrm{Res}^f_{\lambda B}(Tx)), y - \mathrm{Res}^f_{\lambda B}(Tx)\rangle \\ = \; & -\lambda \langle 0^* - \frac{1}{\lambda}\left(\nabla f(x) - \nabla f(\mathrm{Res}^f_{\lambda B}(Tx))\right), \\ y - \mathrm{Res}^f_{\lambda B}(Tx)\rangle. & \end{aligned} \tag{3.8}$$

Now, since $y \in F(Res^f_{\lambda B})$, it follows from Lemma **2**.2 that $0^* \in By$. Also, from (**2**.7), we obtain that $\frac{1}{\lambda}\left(\nabla f(x) - \nabla f\left(\mathrm{Res}^f_{\lambda B}(Tx)\right)\right) \in B\left(\mathrm{Res}^f_{\lambda B}(Tx)\right)$. Using these facts in (**3**.8), we obtain by the monotonicity of B that

$$D_f(y, (Res^f_{\lambda B}(Tx)) + D_f((Res^f_{\lambda B}(Tx), x) - D_f(y, x) \leq 0,$$

which implies by Remark **2**.4 (ii) that $\mathrm{Res}^f_{\lambda B} \circ T$ is QBFNE. □

Proposition 3.1. *If $f : X \to \mathbb{R}$ is a Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X, and $A^{-1}(0^*) \cap B^{-1}(0^*) \neq \emptyset$, then $(A + B)^{-1}(0^*)$ is closed and convex.*

Indeed, since $\mathrm{Res}^f_{\lambda B}$ and A^f_λ are BFNE mappings (see [41] and [30] respectively), we have from Remark **2**.3 that $F(\mathrm{Res}^f_{\lambda B})$ and $F(A^f_\lambda)$ are closed and convex. Also, by Remark **2**.4 (iii), we have that $\mathrm{Res}^f_{\lambda B}$ and A^f_λ are QBFNE mappings. Thus, by Remark **2**.2 and Lemma **3**.1, we have that $(A+B)^{-1}(0^*) = F(\mathrm{Res}^f_{\lambda B} \circ A^f_\lambda) = F(\mathrm{Res}^f_{\lambda B}) \cap F(A^f_\lambda)$ is closed and convex.

Remark 3.1. *Set $T^i_\lambda = Res^f_{\lambda B_i} \circ A^f_{i\lambda}$, where $i = 1, 2, \ldots, N$ and $\lambda > 0$. If $\left(\cap^N_{i=1} F(Res^f_{\lambda B_i})\right) \cap \left(\cap^N_{i=1} F(A^f_{i\lambda})\right)$ is nonempty for each $i = 1, 2, \ldots, N$, then by Lemma **3**.2, we obtain that T^i_λ is QBFNE for each $i = 1, 2, \ldots, N$. Thus, by Lemma **3**.1, we obtain that*

$$F(T^N_\lambda \circ T^{N-1}_\lambda \circ \cdots \circ T^1_\lambda) = \cap^N_{i=1} F(T^i_\lambda). \tag{3.9}$$

Theorem 3.1. *Let X be a reflexive real Banach space and X^* be its dual space. For $i = 1, 2, \ldots, N$, let $A_i : X \to X^*$ be a finite family of BISM mappings and $B_i : X \to 2^{X^*}$ be a finite family of maximal monotone mappings such that $\left(\cap^N_{i=1} A^{-1}(0^*)\right) \cap \left(\cap^N_{i=1} B^{-1}(0^*)\right) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = T^N_\lambda \circ T^{N-1}_\lambda \circ \cdots \circ T^1_\lambda x_n, \\ w_n = \nabla f^* \left(\frac{\beta_n}{1-\alpha_n} \nabla f(x_n) + \frac{\gamma_n}{1-\alpha_n} \nabla f(y_n)\right), \\ x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(w_n)\right), \; n \geq 1, \end{cases} \tag{3.10}$$

where $T^i_\lambda = Res^f_{\lambda B_i} \circ A^f_{i\lambda}$, $i = 1, 2, \ldots, N$, $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $z = P^f_\Gamma u$, where $\Gamma := \cap^N_{i=1} (A_i + B_i)^{-1}(0^*)$.

Proof. Since $\left(\cap_{i=1}^N A^{-1}(0^*)\right) \cap \left(\cap_{i=1}^N B^{-1}(0^*)\right) \neq \emptyset$, it follows from Lemma **2**.2, Lemma **2**.3, Remark **2**.2 and Lemma **3**.1 that $\Gamma \neq \emptyset$. Also, by Proposition **3**.1, Γ is closed and convex subset of X. Now, let $z = P_\Gamma^f u \subset \Gamma$, then by Lemma **2**.7, we obtain that

$$\begin{aligned} D_f(z, x_{n+1}) &\leq \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z, w_n) \\ &\leq \alpha_n D_f(z,u) + \beta_n D_f(z, x_n) + \gamma_n D_f(z, y_n) \\ &\vdots \\ &\leq \alpha_n D_f(z,u) + (1-\alpha_n) D_f(z, x_n) \\ &\leq \max\{D_f(z,u), D_f(z,x_n)\} \\ &\vdots \\ &\leq \max\{D_f(z,u), D_f(z,x_1)\},\ n \geq 1. \end{aligned}$$

Thus, $\{D_f(z, x_n)\}$ is bounded. It follows from Lemma **2**.8 that $\{x_n\}$ is bounded. Consequently, $\{w_n\}$ and $\{y_n\}$ are also bounded.
Now, from (**2**.19), we obtain that

$$\begin{aligned} D_f(z, x_{n+1}) &= D_f\left(z, \nabla f^*(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n))\right) \\ &= V_f\left(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n)\right) \\ &\leq V_f\left(z, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n) - \alpha_n(\nabla f(u) - \nabla f(z)\right) \\ &\quad + \langle \alpha_n(\nabla f(u) - \nabla f(z)), \nabla f^*(\alpha_n \nabla f(u) \\ &\quad + (1-\alpha_n)\nabla f(w_n)) - z\rangle \\ &= V_f\left(z, \alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(w_n)\right) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &= D_f\left(z, \nabla f^*(\alpha_n \nabla f(z) + (1-\alpha_n)\nabla f(w_n)\right) \\ &\quad + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &\leq (1-\alpha_n) D_f\left(z, w_n\right) + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle \\ &\leq (1-\alpha_n) D_f\left(z, x_n\right) + \alpha_n \langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle. \end{aligned} \quad (\mathbf{3}.11)$$

Again, from (**3**.10), we obtain that

$$D_f(w_n, x_{n+1}) \leq \alpha_n D_f(w_n, u) \to 0, \text{ as } n \to \infty, \quad (\mathbf{3}.12)$$

which implies by Lemma **2**.1 that

$$\lim_{n\to\infty} ||w_n - x_{n+1}|| = 0. \quad (\mathbf{3}.13)$$

Thus, by Lemma **2**.6, we have

$$\lim_{n\to\infty} ||\nabla f(w_n) - \nabla f(x_{n+1})|| = 0. \quad (\mathbf{3}.14)$$

Also, since f is uniformly Fréchet differentiable on bounded subsets of X, we have that f is uniformly continuous on bounded subsets of X. Thus, we obtain from (**3**.13) that

$$\lim_{n\to\infty} ||f(w_n) - f(x_{n+1})|| = 0. \quad (\mathbf{3}.15)$$

We now consider two cases for the remaining part of our proof:
Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(z, x_n)\}$ is monotone decreasing for all $n \geq n_0$. Then, we get that $\{D_f(z, x_n)\}$ is convergent and hence

$$\lim_{n\to\infty} \left(D_f(z, x_n) - D_f(z, x_{n+1})\right) = 0. \quad (\mathbf{3}.16)$$

From (**3**.10) and (**3**.11), we obtain that

$$\begin{aligned} D_f(z,x_{n+1}) &\leq (1-\alpha_n)D_f\,(z,w_n)+\alpha_n\langle\nabla f(u)-\nabla f(z),x_{n+1}-z\rangle \\ &\leq (1-\alpha_n)\left[\frac{\beta_n}{(1-\alpha_n)}D_f(z,x_n)+\frac{\gamma_n}{(1-\alpha_n)}D_f(z,y_n)\right] \\ &\quad +\alpha_n\langle\nabla f(u)-\nabla f(z),x_{n+1}-z\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} D_f(z,x_n)-D_f(z,x_{n+1}) &\geq (1-\beta_n)D_f(z,x_n)-\gamma_nD_f(z,y_n) \\ &\quad -\alpha_n\langle\nabla f(u)-\nabla f(z),x_{n+1}-z\rangle \\ &= (1-\beta_n)\left(D_f(z,x_n)-D_f(z,y_n)\right) \\ &\quad -\alpha_n\left[\langle\nabla f(u)-\nabla f(z),x_{n+1}-z\rangle+D_f(z,y_n)\right], \end{aligned}$$

which further implies that

$$\begin{aligned} (1-\beta_n)\left(D_f(z,x_n)-D_f(z,y_n)\right)\leq D_f(z,x_n)-D_f(z,x_{n+1}) \\ +\alpha_n\left[\langle\nabla f(u)-\nabla f(z),x_{n+1}-z\rangle+D_f(z,y_n)\right]. \end{aligned}$$

Thus, by the condition on α_n and β_n, and by (**3**.16), we obtain that

$$\begin{aligned} &\lim_{n\to\infty}\left(D_f(z,x_n)-D_f(z,T_\lambda^N\circ T_\lambda^{N-1}\circ\cdots\circ T_\lambda^1x_n)\right) \\ &\quad =\lim_{n\to\infty}\left(D_f(z,x_n)-D_f(z,y_n)\right)=0. \end{aligned} \tag{3.17}$$

Now, since f is a Legendre function which is bounded and uniformly Fréchet differentiable on bounded subsets of X, we have for each $i=1,2,\dots,N$ that $\text{Res}^f_{\lambda B_i}$ and $A^f_{i\lambda}$ are both BSNE satisfying $F(\text{Res}^f_{\lambda B_i})=\hat{F}(\text{Res}^f_{\lambda B_i})$ and $F(A^f_{i\lambda})=\hat{F}(A^f_{i\lambda})$ respectively (see [40, Lemma 1.3.2]). Thus, it follows from Lemma **2**.4 that $T^i_\lambda=\text{Res}^f_{\lambda B_i}\circ A^f_{i\lambda}$ is also BSNE with $F(T^i_\lambda)=\hat{F}(T^i_\lambda)$, for each $i=1,2,3,\dots,N$. Again, since T^i_λ is BSNE mapping for each $i=1,2,3\dots,N$, then by similar argument, we obtain that the composition $T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda$ is also a BSNE mapping with $F(T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda)=\hat{F}(T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda)$. Thus, it follows from (**3**.17) that

$$\lim_{n\to\infty}D_f(x_n,T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda x_n)=\lim_{n\to\infty}D_f(x_n,y_n)=0, \tag{3.18}$$

which implies from Lemma **2**.1 that

$$\lim_{n\to\infty}||x_n-T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda x_n||=\lim_{n\to\infty}||x_n-y_n||=0. \tag{3.19}$$

From (**3**.10) and (**3**.18), we obtain that

$$D_f(x_n,w_n)\leq\frac{\gamma_n}{(1-\alpha_n)}D_f(x_n,y_n)\to 0,\ \text{as } n\to\infty,$$

which implies from Lemma **2**.1 that

$$\lim_{n\to\infty}||x_n-w_n||=0. \tag{3.20}$$

From (**3**.13) and (**3**.20), we obtain that

$$\lim_{n\to\infty}||x_n-x_{n+1}||=0. \tag{3.21}$$

Since X is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ weakly converges to $v\in X$, and

$$\limsup_{n\to\infty}\langle\nabla f(u)-\nabla f(z),x_n-z\rangle=\lim_{k\to\infty}\langle\nabla f(u)-\nabla f(z),x_{n_k}-z\rangle. \tag{3.22}$$

Thus, by (**3**.19), we obtain that $v\in\hat{F}(T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda)=F(T^N_\lambda\circ T^{N-1}_\lambda\circ\cdots\circ T^1_\lambda)$, which implies by (**3**.9) and Remark **2**.2 that $v\in\cap_{i=1}^NF(T^i_\lambda)=\cap_{i=1}^NF(\text{Res}^f_{\lambda B_i}\circ A^f_{i\lambda})=\Gamma$.

We now show that $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$.
From (**3**.21), (**3**.22) and Lemma **2**.5 (i), we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_{n+1} - z\rangle &= \limsup_{n\to\infty}\langle \nabla f(u) - \nabla f(z), x_n - z\rangle \\ &= \langle \nabla f(u) - \nabla f(z), v - z\rangle \leq 0.\end{aligned}$$

Using this, and applying Lemma **2**.9 in (**3**.11), we obtain that $D_f(z, x_n) \to 0$, as $n \to \infty$. Thus, by Lemma **2**.1, we obtain that $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$.
Case 2: Suppose that $\{D_f(z, x_n)\}$ is not monotone decreasing sequence. Then, there exists a subsequence $\{D_f(z, x_{n_i})\}$ of $\{D_f(z, x_n)\}$ such that $D_f(z, x_{n_i}) < D_f(z, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma **2**.10, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$

$$D_f(z, x_{m_k}) \leq D_f(z, x_{m_k+1}) \text{ and } D_f(z, x_k) \leq D_f(z, x_{m_k+1}) \ \forall k \in \mathbb{N}.$$

Thus, we have

$$\begin{aligned}0 &\leq \lim_{k\to\infty}\left(D_f(z, x_{m_k+1}) - D_f(z, x_{m_k})\right) \\ &\leq \limsup_{n\to\infty}\left(D_f(z, x_{n+1}) - D_f(z, x_n)\right) \\ &\leq \limsup_{n\to\infty}\left(\alpha_n D_f(z, u) + (1-\alpha_n) D_f(z, x_n) - D_f(z, x_n)\right) \\ &\leq \limsup_{n\to\infty}\alpha_n\left(D_f(z, u) - D_f(z, x_n)\right) = 0,\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\left(D_f(z, x_{m_k+1}) - D_f(z, x_{m_k})\right) = 0. \qquad (\mathbf{3}.23)$$

Following the same line of argument as in Case 1, we can verify that

$$\limsup_{k\to\infty}\langle \nabla f(u) - \nabla f(z), x_{m_k+1} - z\rangle \leq 0. \qquad (\mathbf{3}.24)$$

Also from (**3**.11), we have

$$D_f(z, x_{m_k+1}) \leq (1-\alpha_{m_k}) D_f(z, x_{m_k}) + \alpha_{m_k}\langle \nabla f(u) - \nabla f(z), x_{m_k+1} - z\rangle.$$

Since $D_f(z, x_{m_k}) \leq D_f(z, x_{m_k+1})$, we have

$$D_f(z, x_{m_k}) \leq \langle \nabla f(u) - \nabla f(z), x_{m_k+1} - z\rangle,$$

which implies from (**3**.24) that

$$\lim_{k\to\infty} D_f(z, x_{m_k}) = 0. \qquad (\mathbf{3}.25)$$

Since $D_f(z, x_k) \leq D_f(z, x_{m_k+1})$, we obtain from (**3**.25) and (**3**.23) that $\lim_{k\to\infty} D_f(z, x_k) = 0$. Thus, from Case 1 and Case 2, we conclude that $\{x_n\}$ converges to z. □

By setting $N = 1$ in Theorem **3**.1, we obtain the following new result.

Corollary 3.1. *Let X be a reflexive real Banach space and X^* be its dual space. Let $A : X \to X^*$ be a BISM mapping and $B : X \to 2^{X^*}$ be a maximal monotone mapping such that $A^{-1}(0^*) \cap B^{-1}(0^*) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = Res_{\lambda B}^f \circ A_\lambda^f x_n, \\ w_n = \nabla f^*\left(\frac{\beta_n}{1-\alpha_n}\nabla f(x_n) + \frac{\gamma_n}{1-\alpha_n}\nabla f(y_n)\right), \\ x_{n+1} = \nabla f^*\left(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(w_n)\right), \ n \geq 1, \end{cases} \qquad (\mathbf{3}.26)$$

where $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$, where $\Gamma := (A+B)^{-1}(0^*)$.

By setting $A \equiv 0$, we obtain the following result.

Corollary 3.2. *Let X be a reflexive real Banach space and X^* be its dual space. For $i = 1, 2, \dots, N$, let $B_i : X \to 2^{X^*}$ be a finite family of maximal monotone mappings such that $\left(\cap_{i=1}^N B^{-1}(0^*)\right) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = T_\lambda^N \circ T_\lambda^{N-1} \circ \cdots \circ T_\lambda^1 x_n, \\ w_n = \nabla f^* \left(\frac{\beta_n}{1-\alpha_n} \nabla f(x_n) + \frac{\gamma_n}{1-\alpha_n} \nabla f(y_n) \right), \\ x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(w_n) \right), \ n \geq 1, \end{cases} \tag{3.27}$$

where $T_\lambda^i = Res_{\lambda B_i}^f$, $i = 1, 2, \dots, N$, $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$, where $\Gamma := \cap_{i=1}^N B_i^{-1}(0^*)$.

Also, by setting $B \equiv 0$, we obtain the following corollary.

Corollary 3.3. *Let X be a reflexive real Banach space and X^* be its dual space. For $i = 1, 2, \dots, N$, let $A_i : X \to X^*$ be a finite family of BISM mappings such that $\left(\cap_{i=1}^N A^{-1}(0^*)\right) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = T_\lambda^N \circ T_\lambda^{N-1} \circ \cdots \circ T_\lambda^1 x_n, \\ w_n = \nabla f^* \left(\frac{\beta_n}{1-\alpha_n} \nabla f(x_n) + \frac{\gamma_n}{1-\alpha_n} \nabla f(y_n) \right), \\ x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(w_n) \right), \ n \geq 1, \end{cases} \tag{3.28}$$

where $T_\lambda^i = A_{i\lambda}^f$, $i = 1, 2, \dots, N$, $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$, where $\Gamma := \cap_{i=1}^N (A_i)^{-1}(0^*)$.

4. Application to variational inequality and convex feasibility problems

In this section, we apply our results to solve a finite family of variational inequality problems and convex feasibility problem. Throughout this section, we assume that C is a nonempty closed and convex subset of a reflexive real Banach space X and X^* is the dual space of X. Recall that the subdifferential $\partial g : X \to 2^{X^*}$ of g, defined by

$$\partial g(x) = \begin{cases} \{x^* \in X^* : g(z) - g(x) \geq \langle x^*, z - x \rangle, \ \forall z \in X\}, & \text{if } x \in \text{dom} g, \\ \emptyset, & \text{otherwise} \end{cases} \tag{4.1}$$

is a maximal monotone mapping whenever $g : X \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function.
Furthermore, the indicator function $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases} \tag{4.2}$$

is a proper convex and lower semicontinuous function. Thus, the subdifferential of δ_C, given as

$$\partial\delta_C(x) = \begin{cases} \{x^* \in X^* : \langle x^*, z - x\rangle \leq 0 \ \forall z \in C\} \text{ if } x \in C, \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases} \tag{4.3}$$

is a maximal monotone mapping.
Let $A : X \to X^*$ be a BISM mapping. The Variational Inequality Problem (VIP) is to find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0 \ \forall y \in C. \tag{4.4}$$

The solution set of VIP (**4**.4) is denoted by $VIP(A, C)$. If $f : X \to (-\infty, +\infty]$ is a Legendre and totally convex function which satisfies the range condition ran $(\nabla f - A) \subset$ ran (∇f) (see [30, Proposition 12]), then

$$VIP(A, C) = F(P_C^f \circ A_\lambda^f). \tag{4.5}$$

Now, observe that

$$\begin{aligned} w = \text{Res}^f_{\lambda\partial\delta_C}(x) &\iff w = \left((\nabla f + \lambda\partial\delta_C)^{-1} \circ \nabla f\right)(x) \\ &\iff \frac{1}{\lambda}\left((\nabla f(x) - \nabla f(w)\right) \in \partial\delta_C(w) \\ &\iff \langle\nabla f(x) - \nabla f(w), z - w\rangle \leq 0 \ \forall y \in C \iff w = P_C^f(x). \end{aligned}$$

Thus, it follows that

$$(A + \partial\delta_C)^{-1}(0^*) = F(\text{Res}^f_{\lambda\partial\delta_C} \circ A_\lambda^f) = F(P_C^f \circ A_\lambda^f) = VIP(A, C).$$

Therefore, by setting $B_i = \partial\delta_{C_i}$ in Theorem **3**.1, we apply Theorem **3**.1 to approximate a common solution of a finite family of VIPs.

Theorem 4.1. *Let X be a reflexive real Banach space and X^* be its dual space. For $i = 1, 2, \ldots, N$, let $A_i : X \to X^*$ be a finite family of BISM mappings and $\partial\delta_{C_i}$ be as defined in (**4**.3) such that $\left(\cap_{i=1}^N A^{-1}(0^*)\right) \cap \left(\cap_{i=1}^N \partial\delta_{C_i}^{-1}(0^*)\right) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = T_\lambda^N \circ T_\lambda^{N-1} \circ \cdots \circ T_\lambda^1 x_n, \\ w_n = \nabla f^*\left(\frac{\beta_n}{1-\alpha_n}\nabla f(x_n) + \frac{\gamma_n}{1-\alpha_n}\nabla f(y_n)\right), \\ x_{n+1} = \nabla f^*\left(\alpha_n\nabla f(u) + (1-\alpha_n)\nabla f(w_n)\right), \ n \geq 1, \end{cases} \tag{4.6}$$

where $T_\lambda^i = Res^f_{\lambda\partial\delta_{C_i}} \circ A^f_{i\lambda}$, $i = 1, 2, \ldots, N$, $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$, where $\Gamma := \cap_{i=1}^N (A_i + \partial\delta_{C_i})^{-1}(0^*)$.

The Convex Feasibility Problem (CFP) is defined as:

$$\text{Find } x \in C \text{ such that } x \in \cap_{i=1}^N C_i, \tag{4.7}$$

where $C_i, i = 1, 2, \ldots, N$ is a finite family of nonempty closed and convex subsets of C such that $\cap_{i=1}^N C_i \neq \emptyset$. Now, observe that

$$(\partial\delta_{C_i})^{-1}(0^*) = F(\text{Res}^f_{\lambda\partial\delta_{C_i}}) = F(P^f_{C_i}) = C_i, \ i = 1, 2, \ldots, N,$$

which implies that $\cap_{i=1}^N \partial\delta_{C_i}^{-1}(0^*) = \cap_{i=1}^N C_i$. Thus, by setting $A \equiv 0$ in Theorem **4**.1, we obtain the following corollary for approximating a solution of the CFP (**4**.7).

Corollary 4.1. *Let X be a reflexive real Banach space and X^* be its dual space. For $i = 1, 2, \ldots, N$, let $\partial\delta_{C_i}$ be as defined in (**4**.3) such that $\left(\cap_{i=1}^N \partial\delta_{C_i}^{-1}(0^*)\right) \neq \emptyset$. Let $f : X \to \mathbb{R}$ be a strongly coercive Legendre function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of X. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = T_\lambda^N \circ T_\lambda^{N-1} \circ \cdots \circ T_\lambda^1 x_n, \\ w_n = \nabla f^* \left(\frac{\beta_n}{1-\alpha_n} \nabla f(x_n) + \frac{\gamma_n}{1-\alpha_n} \nabla f(y_n) \right), \\ x_{n+1} = \nabla f^* \left(\alpha_n \nabla f(u) + (1-\alpha_n) \nabla f(w_n) \right), \ n \geq 1, \end{cases} \tag{4.8}$$

where $T_\lambda^i = Res_{\lambda\partial\delta_{C_i}}^f$, $i = 1, 2, \ldots, N$, $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\alpha_n + \beta_n + \gamma_n = 1$ and $0 < a < \beta_n, \gamma_n < b < 1$. Then $\{x_n\}$ converges strongly to $z = P_\Gamma^f u$, where $\Gamma := \cap_{i=1}^N (\partial\delta_{C_i})^{-1}(0^*)$.

Declaration
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