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Shrinking approximation method for solution of split monotone variational inclusion and fixed point problems in Banach spaces

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Abstract

In this paper, we investigate a shrinking algorithm for finding a solution of split monotone variational inclusion problem which is also a common fixed point problem of relatively nonexpansive mapping in uniformly convex real Banach spaces which are also uniformly smooth. The iterative algorithm employed in this paper is design in such a way that it does not require prior knowledge of operator norm. We prove a strong convergence result for approximating the solutions of the aforementioned problems and give applications of our main result to split convex minimization problem. The result present in this paper extends and complements many related results in literature.

Keywords: Maximal monotone operators; relatively nonexpansive mapping; shrinking iterative scheme; split feasibility problem; fixed point problem.

2010 MSC: 47H06, 47H09, 47J05, 47J25

1. Introduction

The monotone inclusion problem is to find an element $x \in H$ such that $0 \in B(x)$, where $B : H \rightarrow 2^H$ is a multi-valued operator and H is a real Hilbert space. This problem is very important in many areas such as convex optimization and monotone variational inequalities. It is worth mentioning that every monotone operator on Hilbert spaces can be regularized into single-valued, nonexpansive, Lipschitz continuous monotone operator by means of Yosida approximation notion. The inclusion

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problem can also be defined in terms of sum of two monotone operators M and B , where one of these operators is α -inverse strongly monotone which is $\frac{1}{\alpha}$ -Lipschitz continuous.

Let E be a real Banach space with $\|\cdot\|$, with dual space E^* and $\langle f, x \rangle$ the value of $f \in E^*$ at $x \in E$. Let $B : E \rightarrow 2^{E^*}$ be a maximal monotone operator and $f : E \rightarrow E^*$ be an α -inverse strongly monotone operator. The Monotone Variational Inclusion Problem (MVIP) is find $x \in E$ such that

$$0 \in (B + f)x. \quad (1.1)$$

We denote by $(M + B)^{-1}(0)$ the solution set of (1.1).

Based on a series of studies in the past years, the splitting method has been known to be a popular method for solving (1.1). The splitting methods for linear equations was introduced by Peaceman and Rashford [34]. Extensions to nonlinear equations in Hilbert spaces were carried out by Lions and Mercier [24]. Since then, many authors have considered approximating solutions of variational inclusion (1.1) using this method, (see [2, 3, 14, 41] and the references contained in).

Recently, Zhang and Jiang [52] proved the following strong convergence theorem for approximating solutions for a common zero point of the sum of two monotone operators which is also a fixed point of a family of countable quasi-nonexpansive mapping in the framework of Hilbert spaces as follows:

Theorem 1.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H , $A : C \rightarrow H$ be an α -inverse strongly monotone operator and B be a maximal monotone operator on H such that $\text{Dom}(B)$ is included in C . Let $\{S_n\} : C \rightarrow C$ be a family of countable quasi-nonexpansive mappings which are uniformly closed. Assume that $\Gamma := F(S_n) \cap (A + B)^{-1}(0) \neq \emptyset$. Let $\{r_n\}$ be a positive real number sequence and $\{\alpha_n\}$ be a real number sequence in $[0, 1)$. Let $\{x_n\}$ be a sequence of C generated by*

$$\begin{cases} x_1 \in C_1 = C, \text{ chosen arbitrarily;} \\ z_n = J_{r_n}(x_n - r_n A x_n); \\ y_n = \alpha_n z_n + (1 - \alpha_n) S_n z_n; \\ C_{n+1} = \{z \in C_n : \|z_n - z\| \leq \|y_n - z\| \leq \|x_n - z\|\}; \\ x_{n+1} = P_{C_{n+1}} x_1, \quad n \geq 1; \end{cases}$$

where $J_{r_n} = (I + r_n B)^{-1}$, $\liminf_{n \rightarrow \infty} r_n > 0$, $r_n \leq 2\alpha$ and $\limsup_{n \rightarrow \infty} \alpha_n < 1$. Then the sequence $\{x_n\}$ converges strongly to $q = P_\Gamma x_0$.

The Split Feasibility Problem (SFP) introduced by Censor and Elfving [12] is to find

$$x^* \in C \text{ such that } Ax^* \in Q, \quad (1.2)$$

where C and Q are nonempty, closed and convex subsets of real Banach spaces E_1 and E_2 respectively, and $A : E_1 \rightarrow E_2$ is a bounded linear operator. The SFP arises from phase retrievals and in medical image reconstruction to mention a few. For more details on SFP, we refer readers to (see [13, 32, 47, 25] and other references therein). In 2018, Ma et. al. [25] introduced an iterative algorithm to solve the SFP (1.2) and fixed point problem of quasi- ϕ -nonexpansive mappings in Banach spaces. They proved a strong convergence result to a common solution of the aforementioned problems and apply their result to convexly constrained inverse problem and split null point problem. Motivated by SFP (1.2), Censor et al. [11] introduced a new class of problem known as the Split Variational Inequality Problem (SVIP) as follows: Find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0, \forall x \in C, \quad (1.3)$$

and such that $y^* = Ax^* \in Q$ solves

$$\langle g(y^*), y - y^* \rangle \geq 0, \quad \forall y \in Q, \tag{1.4}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \rightarrow H_2$ is a bounded linear operator, $f : H_1 \rightarrow H_1$ and $g : H_2 \rightarrow H_2$ are two given operators. Based on the work of Censor et al. [11], Moudafi [31] studied and introduced a new type of split problem called the Split Monotone Variational Inclusion Problem (SMVIP) which is to find

$$x^* \in H_1 \text{ such that } 0 \in f(x^*) + F(x^*), \tag{1.5}$$

and such that $y^* = Ax^* \in H_2$ solves

$$0 \in g(y^*) + G(y^*), \tag{1.6}$$

where $F : H_1 \rightarrow 2^{H_1}$ and $G : H_2 \rightarrow 2^{H_2}$ are multivalued mappings, $A : H_1 \rightarrow H_2$ is a bounded linear operator, $f : H_1 \rightarrow H_1$ and $g : H_2 \rightarrow H_2$ are single-valued operators.

Remark 1.2. *As observed by Moudafi, setting $F = N_C$ and $G = N_Q$ in SMVIP (1.5)-(1.6), where N_C and N_Q are the normal cones of C and Q respectively, then we recover SVIP (1.3)-(1.4). In summary, SMVIP can be seen as an important generalization of SFP, SVIP, MVIP and other related problems in the literature.*

Recently, Ezeora and Izuchukwu [17] introduced the following iterative algorithm to approximate solution of the following problem: $\Delta := \{z \in (F + f)^{-1}(0) : Az \in \text{Fix}(S)\} \neq \emptyset$. For arbitrary $x_1, u \in H_1$

$$\begin{cases} u_n = (1 - \beta_n)x_n + \beta_n u \\ y_n = P_C(u_n - \gamma_n A^*(I - T_\gamma)Au_n) \\ x_{n+1} = J_\lambda^M(I - \lambda f)y_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where $T_\gamma = \gamma I + (1 - \gamma)S$ with $\gamma \in [\mu, 1)$, $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{\|A\|^2}\right)$, $F : H_1 \rightarrow 2^{H_1}$ is a multivalued maximal monotone mapping, $f : H_1 \rightarrow H_1$ is an α -ism and $S : H_2 \rightarrow H_2$ being μ -strictly pseudocontractive mapping with T_γ being a nonexpansive mapping. They proved that the sequence $\{x_n\}$ converges strongly to an element of Γ .

Remark 1.3. *It is well-known that stepsizes play essential roles in the convergence properties of iterative methods, since the efficiency of the methods depends heavily on it. When the stepsize depends on the knowledge of either the operator norm or the coefficient of an operator, it usually slows down the convergence rate of the method. Moreover, in many practical cases, the operator norm or the coefficient of a given operator may not be known or may be difficult to estimate, thus, making the applicability of such method to be questionable. Therefore, iterative methods that does not depend on any of these, are more applicable in practice. It is easy to see in Algorithm 1.7 that*

$$a, b \in \left(0, \frac{1}{\|A\|^2}\right),$$

thus this condition makes the iterative algorithm not applicable to real life problems.

Question: It is natural to ask if we can further generalize problem $\Delta := \{z \in (F + f)^{-1}(0) : Az \in \text{Fix}(S)\} \neq \emptyset$ and makes Algorithm (1.7) more effective and applicable to real life problem.

Motivated by the works of Zhang and Jiang [52], and Ma et al. [25] and Ezeora and Izuchukwu [17], we introduced a shrinking iterative algorithm for finding zeros of the sum of two maximal monotone operators which is also a common fixed point of relatively nonexpansive mapping in Banach spaces. We prove a strong convergence result for approximating solutions of the aforementioned problems and give applications of our main result to split convex minimization problem. In simple and clear terms, the proposed method of this paper has the following features:

1. The problem and our iterative algorithm considered in this article generalizes the ones in [17, 25, 52] and so on.
2. As mentioned above in [17], they considered computing with the help of an operator norm ($\|A\|$) as this gives difficulties in computation. In our article, the stepsize employed is independent of an operator norm as this yield easy implementation of our algorithm in practice. Also, we were able to dispense with the compactness condition during the course of obtaining a strong convergence result.
3. As seen in different algorithms that certain conditions are needed to be imposed on algorithm before obtaining a strong convergence result, we obtain a strong convergence result by imposing a very minimal condition.
4. It is crucial to study the SMVIP because of its potential application to mathematical models whose constraints can be expressed as SMVIP. This happens, in particular, in practical problems in signal recovery, image processing, and network resource allocation. It is also important to study the SMVIP because of its generalization on other optimization problems like SFP, MVIP, SVIP, variational inequality problem and other related problems in literature.

The result present in this paper extends the result of Ma et al. [25], Zhang and Jiang [52] and other related results in literature.

2. Preliminaries

We give some definitions and important results which will be useful in establishing our main results. In the sequel, we denote strong and weak convergence by " \rightarrow " and " \rightharpoonup ", respectively.

Throughout this paper, we assume C to be a nonempty, closed and convex subset of a real Banach space with norm $\|\cdot\|$, $J : E \rightarrow 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2, \quad \forall x \in E\},$$

Consider the Lyapunov functional $\phi : E \times E \rightarrow [0, \infty)$ defined as in [4, 5] by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E.$$

Alber [4] introduced a generalized projection operator $\Pi_C : E \rightarrow C$ which is an analogue of the metric projection defined as follows:

$$\Pi_C(x) = \operatorname{argmin}_{y \in C} \phi(y, x), \quad x \in E.$$

That is, $\Pi_C(x) = \bar{x}$, where \bar{x} is the unique solution to the minimization problem $\phi(\bar{x}, x) = \inf_{y \in C} \phi(y, x)$. In real Hilbert space, we observe that $\Pi_C(x) \equiv P_C(x)$ and $\phi(x, y) = \|x - y\|^2$. It is obvious from the definition of the functional ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2. \quad (2.1)$$

Apart from inequality (2.1), the Lyapunov functional ϕ also satisfy the following inequalities:

$$A_1. \phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle;$$

$$A_2. 2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w);$$

$$A_3. \phi(x, y) \leq \|x\| \|Jx - Jy\| + \|y\| \|x - y\|.$$

Note: If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\phi(x, y) = 0$ if and only if $x = y$, see [46]. We are also concerned with the functional $V : E \times E^* \rightarrow \mathbb{R}$ which is defined by

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^* \rangle + \|x^*\|^2 \tag{2.2}$$

for all $x \in E$ and $x^* \in E^*$. Observe that, $V(x, x^*) = \phi(x, J^{-1}x^*)$, if E is a reflexive, strictly convex and smooth Banach space and

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^* \rangle \tag{2.3}$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [38].

Let C be a closed and convex subset of E and $T : C \rightarrow C$ be a mapping a point $x \in C$ is called a fixed point of T , if $x = Tx$. We denote the set of fixed points of T by $Fix(T)$. A point $p \in C$ is called an asymptotic fixed point of T , if C contains a sequence $\{x_n\}$ such that $x_n \rightarrow p$ and $\|x_n - Tx_n\| \rightarrow 0$ as $n \rightarrow \infty$. We denote by $\widehat{Fix}(T)$ the set of asymptotic fixed points of T . A mapping $T : C \rightarrow C$ is said to be relatively nonexpansive (see [26]) if the following conditions are satisfied:

- (L1) $Fix(T) \neq \emptyset$;
- (L2) $\phi(p, Tx) \leq \phi(p, x), \forall x \in C, p \in Fix(T)$;
- (L3) $Fix(T) = \widehat{Fix}(T)$.

If T satisfies (L1) and (L2), then T is said to be relatively quasi-nonexpansive. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings. Many authors have considered the relative quasi-nonexpansive mappings, (see [43, 48]).

Definition 2.1. Let C be a nonempty, closed and convex subset of a real Banach spaces E . A mapping $T : C \rightarrow C$ is said to be strongly relatively nonexpansive, see [21] if the following conditions are satisfied.

1. T is relatively nonexpansive.
2. If $\{x_n\}$ is a bounded sequence in C such that

$$\lim_{n \rightarrow \infty} (\phi(p, x_n) - \phi(p, Tx_n)) = 0,$$

for some $p \in Fix(T)$, then $\lim_{n \rightarrow \infty} \phi(Tx_n, x_n) = 0$.

Definition 2.2. Let $X \subset E$ be a nonempty subset. Then a mapping $A : X \rightarrow E^*$ is called (i) monotone on X if

$$\langle Ax - Ay, x - y \rangle \geq 0, \forall x, y \in X;$$

Below is an example of a monotone operator in quantum mechanics.

Example 2.3. [41] Let the operator

$$Au := -b^2 \Delta u + (f(x) + c)u(x) + u(x) \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} dy,$$

where $\Delta := \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in \mathbb{R}^3 , b and c are constants, $f(x) = f_0(x) + f_1(x)$, where $f_0(x) \in L^\infty(\mathbb{R}^3)$ and $f_1(x) \in L^2(\mathbb{R}^3)$. Let $A := L + B$, where the operator L which is the schrödinger operator is the linear part of A and B defined by the last term. It is known that B is a monotone operator on $L^2(\mathbb{R}^3)$, (see p. 23 of [6]) which also implies that $A : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is also a monotone operator.

Definition 2.4. A multi-valued operator $B : E \rightarrow 2^{E^*}$ with domain $Dom(B) = \{x \in E : Bx \neq \emptyset\}$ and the range $R(B) = \{Bx : x \in D(B)\}$ is said to be monotone if for $x, y \in D(B), a \in Bx, b \in By$, the following inequality holds:

$$\langle x - y, a - b \rangle \geq 0.$$

A monotone operator B is said to be maximal if its graph $Gra(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone operator.

If E is a strictly convex, reflexive and smooth Banach space and $B : E \rightarrow 2^{E^*}$ is a maximal monotone operator.

For a maximal monotone operator F , the metric resolvent of F of parameter $\lambda > 0$, denoted by K_λ^F is the operator, (see [36])

$$K_\lambda^F := (I + \lambda J^{-1}F)^{-1} : E \rightarrow dom(F).$$

The metric resolvent is known to satisfy the following property (see [8]).

$$\langle K_\lambda^F x - p, J(x - K_\lambda^F x) \rangle \geq 0, \forall x \in E, p \in Fix(K_\lambda^F).$$

It is also known that K_λ^F is nonexpansive and $0 \in Fx$ if and only if $K_\lambda^F x = x$, (see [36]).

Also, we know that If E is a smooth, strictly convex and reflexive Banach space, then F is maximal monotone if and only if $R(J + \lambda F) = E^*$. The resolvent of E with parameter $\lambda > 0$, denoted by L_λ^F , is the operator

$$L_\lambda^F := (J + \lambda F)^{-1}J : E \rightarrow dom(F).$$

It is clear from [21] that L_λ^F satisfies the following properties.

1. $L_\lambda^F : E \rightarrow dom(F)$ is a single-valued mapping.
2. $0 \in Fx$ if and only if $L_\lambda^F x = x$ for each λ .
3. L_λ^F is strongly relatively nonexpansive.

Definition 2.5. A mapping $f : E \rightarrow E^*$ is called a single-valued λ -inverse strongly monotone (ism) (see [20]) if for any $x, y \in E$, there exists $\lambda > 0$ such that

$$\langle J^{-1}(Jx - \lambda fx) - J^{-1}(Jx - \lambda fy), fx - fy \rangle \geq 0.$$

Following [39], the anti-resolvent $A_\lambda^f : E \rightarrow E$ associated with $f : E \rightarrow E^*$ and $\lambda > 0$ defined as

$$A_\lambda^f := J^{-1} \circ (J - \lambda f) : E \rightarrow E.$$

It has been shown in [45] that A_λ^f is strongly relatively nonexpansive. More so, assuming $f^{-1}(0) \neq \emptyset$, it is easy to see that $f^{-1}(0) = Fix(A_\lambda^f)$.

For a real Banach space E , the modulus of convexity of E is the function $\delta_E : [0, 2] \rightarrow [0, 1]$ defined as

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : \|x\| = \|y\| = 1, \|x - y\| \geq \epsilon\}. \tag{2.4}$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\frac{\|x + y\|}{2} < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. Also, E is p -uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p \epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x + ty\| - \|x - ty\|) - 1 : \|x\| = \|y\| = 1\}. \tag{2.5}$$

E is said to be uniformly smooth if $\lim_{t \rightarrow 0} \frac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$, then E is q -uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p -uniformly convex if and only if E^* is q -uniformly smooth, where $p^{-1} + q^{-1} = 1$. It is also known that every q -uniformly smooth Banach space is uniformly smooth. It is also widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E . The following are some important and useful properties of duality mapping J , for further details, see [46]:

- For every $x \in E$, Jx is nonempty, closed, convex and bounded subset of E^* .
- If E is smooth or E^* is strictly convex, then J is single valued. Also, If E is reflexive, then J is onto.
- If E is strictly convex, then J is strictly monotone, that is

$$\langle x - y, Jx - Jy \rangle > 0, x \neq y \quad \forall x, y \in E.$$

- If E is smooth, strictly convex and reflexive and $J^* : E^* \rightarrow 2^E$ is the normalized duality mapping on E^* , then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mappings on E and E^* respectively.
- If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^* = J^{-1}$ is also uniformly norm-to-norm continuous on bounded subsets of E^* .

We now state the following results which will be useful to prove our main result.

Lemma 2.6. [9] Let $\frac{1}{p} + \frac{1}{q} = 1$, for $p, q > 1$. The space E is q -uniformly smooth if and only if its dual space E^* is p -uniformly convex.

Lemma 2.7. [49] Let E be a 2-uniformly smooth Banach space with the best smoothness constant $k > 0$. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, Jx \rangle + 2\|ky\|^2, \quad \forall x, y \in E.$$

Lemma 2.8. [49] Given a number $r > 0$, a real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$ such that

$$\|\lambda x + (1 - \lambda)y\|^2 \leq \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)g(\|x - y\|);$$

for all $x, y \in E$ with $\|x\| \leq r$ and $\|y\| \leq r$ and $\lambda \in [0, 1]$.

Lemma 2.9. [5] Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty closed convex subset of E . Then, the following conclusions hold:

- (i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y), \forall x \in C, y \in E$.
- (ii) If $x \in E$ and $z \in C$, then $z = \Pi_C x$ iff $\langle z - y, Jx - Jz \rangle \geq 0, \forall y \in C$.
- (iii) For $x, y \in E, \phi(x, y) = 0$, iff $x = y$.

Lemma 2.10. [23] Let E be a uniformly convex and smooth Banach space and let $\{x_n\}, \{y_n\}$ be two sequences of E . If $\phi(x_n, y_n) \rightarrow 0$ and either of $\{x_n\}$ or $\{y_n\}$ is bounded. Then, $\|x_n - y_n\| \rightarrow 0$.

Lemma 2.11. [7] Let E be a real uniformly convex, smooth Banach space. Then, the following identities hold:

- (i) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \forall x, y \in E$;
- (ii) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy \rangle, \forall x, y \in E$.

Lemma 2.12. [46] Let E be a smooth, strictly convex, and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conclusions hold:

- (i) $z = P_C x_1$,
- (ii) $\langle z - y, J(x_1 - z) \rangle \geq 0, \forall y \in C$.

Lemma 2.13. [19] Let C be a nonempty, closed and convex subset of a uniformly convex Banach space E and $S : C \rightarrow E$ be a nonexpansive mapping. Then, $I - S$ is demiclosed on C , i.e., if $\{x_n\}$ is a sequence of C such that $x_n \rightarrow x$ and $\|(I - S)x_n\| \rightarrow 0$, then $x = Sx$.

Lemma 2.14. [41] Let $B : E \rightarrow 2^{E^*}$ be a maximal monotone operator and $M : E \rightarrow E^*$ be an operator. Define an operator

$$T_\lambda x := J_\lambda^B \circ J^{-1}(J - \lambda M), x \in E, \lambda > 0.$$

Then $\text{Fix}(T_\lambda) = (M + B)^{-1}(0)$.

Remark 2.15. It can be seen from the above Lemma that $(F + f)^{-1}(0) = \text{Fix}(L_\lambda^F \circ A_\lambda^f)$.

3. Main result

Lemma 3.1. Suppose $F : E \rightarrow 2^{E^*}$ is a maximal monotone operator and $f : E \rightarrow E^*$ is a λ -ism mapping with $\lambda > 0$ such that $(F + f)^{-1}(0) \neq \emptyset$. Then

$$\phi(u, L_\lambda^F \circ A_\lambda^f(x)) + \phi(L_\lambda^F \circ A_\lambda^f(x), x) \leq \phi(u, x),$$

for any $u \in (F + f)^{-1}(0)$ and $x \in E$.

The proof of the Lemma stated above is similar to the one in [33].

Lemma 3.2. Let E be a real Banach space, $T : E \rightarrow E$ be a relatively nonexpansive mapping and $F : E \rightarrow 2^{E^*}$ be a maximal monotone operator. Suppose $f : E \rightarrow E^*$ is a λ -ism mapping for $\lambda > 0$ and $(f + F)^{-1}(0) \neq \emptyset$, then

$$\text{Fix}(T(L_\lambda^F \circ A_\lambda^f)) = \text{Fix}(T) \cap \text{Fix}(L_\lambda^F \circ A_\lambda^f).$$

Proof . Clearly, $Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f) \subseteq Fix(T(L_\lambda^F \circ A_\lambda^f))$. We only need to prove that $Fix(T(L_\lambda^F \circ A_\lambda^f)) \subseteq Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$. Let $p \in Fix(T(L_\lambda^F \circ A_\lambda^f))$ and $q \in Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$, then

$$\begin{aligned} \phi(q, p) &= \phi(q, T(L_\lambda^F \circ A_\lambda^f)x) \\ &\leq \phi(q, (L_\lambda^F \circ A_\lambda^f)x). \end{aligned} \tag{3.1}$$

Now by applying Lemma 3.2 and (3.1), we get

$$\begin{aligned} \phi(p, (L_\lambda^F \circ A_\lambda^f)) &= \phi(q, p) - \phi(q, (L_\lambda^F \circ A_\lambda^f)x) \\ &\leq \phi(q, p) - \phi(q, p) \\ &= 0. \end{aligned}$$

Hence, $p \in Fix(L_\lambda^F \circ A_\lambda^f)$.

Next, we show that $p \in Fix(T)$ since $p \in Fix(T(L_\lambda^F \circ A_\lambda^f))$, we obtain

$$\begin{aligned} \phi(p, Tp) &= \phi(p, (T(L_\lambda^F \circ A_\lambda^f)p)) \\ &= \phi(p, Tp) \\ &= 0. \end{aligned}$$

Hence $p \in Fix(T)$. This implies that $p \in Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$. Therefore, we conclude that $Fix(T(L_\lambda^F \circ A_\lambda^f)) = Fix(T) \cap Fix(L_\lambda^F \circ A_\lambda^f)$. \square

Theorem 3.3. Let E_1, E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ and duals E_1^*, E_2^* , respectively. Let Q be a nonempty, closed and convex subset of E_2 , $T : E_1 \rightarrow E_1$ and $S : E_2 \rightarrow E_2$ be relatively nonexpansive mappings respectively. Suppose that $A : E_1 \rightarrow E_2$ is a bounded linear operator with adjoint A^* , $F : E_1 \rightarrow 2^{E_1^*}$ and $G : E_2 \rightarrow 2^{E_2^*}$ are maximal monotone operators. Let $f : E_1 \rightarrow E_1^*$ and $g : E_2 \rightarrow E_2^*$ be single-valued λ, μ -ism operators with $R_\lambda^F \circ B_\lambda^f := (J + \lambda F)^{-1} \circ (J - \lambda f) : E_1 \rightarrow dom F$ for $\lambda > 0$ and $R_\mu^G \circ B_\mu^g := (J + \mu G)^{-1} \circ (J - \mu g) : E_2 \rightarrow dom G$ for $\mu > 0$, respectively. Assume that $\Gamma := \{x^* \in Fix(T) \cap (F + f)^{-1}(0) \text{ and } Ax^* \in Fix(S) \cap (G + g)^{-1}(0)\} \neq \emptyset$, then $\{x_n\}_{n=0}^\infty$ is generated iteratively by $x_1 \in E_1$ and $C_1 = E_1$ with

$$\begin{cases} w_n = J_1^{-1}(J_1x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n); \\ u_n = J_1^{-1}[(1 - \beta_n)J_1w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}x_1; n \geq 1; \end{cases} \tag{3.2}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1} . Suppose $\{\beta_n\}_{n=1}^\infty$ is a sequence in $(0, 1)$ such that $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, and the step size γ_n is chosen in such a way that $\gamma_n = \frac{\rho_n \|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2}{\|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2}$, for $Ax_n \neq (S(R_\mu^G \circ B_\mu^g))Ax_n$, where $0 < d \leq \rho_n \leq e < 1$ for $d, e \in \mathbb{R}$, otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Then, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$, where $\bar{x} = \Pi_\Gamma x_1$.

We divide our proof into several steps:

Step 1: We prove using Theorem 3.3 that C_n is closed and convex for each $n \geq 1$.

Proof . We obtain from Theorem 3.3 that $C_1 = E_1$, therefore C_1 is closed and convex. Now assume that C_n is closed and convex, then

$$\begin{aligned}
 \phi(v, u_n) &\leq \phi(v, x_n) \\
 &\Leftrightarrow \|v\|^2 - 2\langle v, J_1 u_n \rangle + \|u_n\|^2 \\
 &\leq \|v\|^2 - 2\langle v, J_1 x_n \rangle + \|x_n\|^2 \\
 &\Leftrightarrow 2\langle v, J_1 x_n - J_1 u_n \rangle \leq \|x_n\|^2 - \|u_n\|^2.
 \end{aligned}
 \tag{3.3}$$

We have from (3.3) that C_{n+1} is closed and convex subset of E_1 . Therefore, $\Pi_{C_{n+1}}$ is well defined. \square
 Step 2: We show that $\Gamma \subseteq C_n$ for all $n \geq 1$.

Proof . Let $x^* \in \Gamma \subseteq C_n$, for $n \geq 1$ then we have from (3.2) and Lemma 2.8 that

$$\begin{aligned}
 \phi(x^*, u_n) &= \phi(x^*, J_1^{-1}((1 - \beta_n)J_1 w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n)) \\
 &= \|x^*\|^2 - 2\langle x^*, (1 - \beta_n)J_1 w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n \rangle \\
 &\quad + \|(1 - \beta_n)J_1 w_n + \beta_n J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|^2 \\
 &\leq \|x^*\|^2 - 2(1 - \beta_n)\langle x^*, J_1 w_n \rangle - 2\beta_n \langle x^*, J_1(T(R_\lambda^F \circ B_\lambda^f))w_n \rangle \\
 &\quad + (1 - \beta_n)\|w_n\|^2 + \beta_n\|(T(R_\lambda^F \circ B_\lambda^f))w_n\|^2 - \beta_n(1 - \beta_n)g(\|J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) \\
 &= (1 - \beta_n)\phi(x^*, w_n) + \beta_n\phi(x^*, (T(R_\lambda^F \circ B_\lambda^f))w_n) - \beta_n(1 - \beta_n)g(\|J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) \\
 &\leq (1 - \beta_n)\phi(x^*, w_n) + \beta_n\phi(x^*, w_n) - \beta_n(1 - \beta_n)g(\|J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) \\
 &= \phi(x^*, w_n) - \beta_n(1 - \beta_n)g(\|J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) \\
 &\leq \phi(x^*, w_n).
 \end{aligned}
 \tag{3.4}$$

$$\tag{3.5}$$

Also, we obtain from (3.2) that

$$\begin{aligned}
 \phi(x^*, w_n) &= \phi(x^*, J_1^{-1}(J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n))) \\
 &= \|x^*\|^2 - 2\langle x^*, J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle \\
 &\quad + \|J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n)\|^2 \\
 &= \|x^*\|^2 - 2\langle x^*, J_1 x_n - \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle \\
 &\quad + \|x_n - J_1^{-1}\gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n)\|^2 \\
 &\leq \|x^*\|^2 - 2\langle x^*, J_1 x_n \rangle + 2\gamma_n \langle x^*, A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle \\
 &\quad + 2\|kx_n\|^2 - 2\langle x_n, \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle + \gamma_n^2 \|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n)\|^2 \\
 &\leq \|x^*\|^2 - 2\langle x^*, J_1 x_n \rangle + 2\gamma_n \langle x^*, A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle \\
 &\quad + \|x_n\|^2 - 2\langle x_n, \gamma_n A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle + \gamma_n^2 \|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n)\|^2 \\
 &= \phi(x^*, x_n) - 2\gamma_n \langle x_n - x^*, A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle + \gamma_n^2 \|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n)\|^2 \\
 &= \phi(x^*, x_n) - 2\gamma_n \langle Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n) \rangle + \gamma_n^2 \|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g))Ax_n)\|^2.
 \end{aligned}
 \tag{3.6}$$

Applying Lemma 2.7, we get

$$\begin{aligned}
 & \langle Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n \rangle \\
 &= \langle Ax_n - (S(R_\mu^G \circ B_\mu^g))Ax_n + (S(R_\mu^G \circ B_\mu^g))Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n \rangle \\
 &= \|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2 + \langle S(R_\mu^G \circ B_\mu^g)Ax_n - Ax^*, J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n \rangle \\
 &\geq \|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2 + \frac{1}{2}(\|Ax_n - Ax^*\|^2 - \|(I - (S(R_\mu^G \circ B_\mu^g)))\|^2 \\
 &\quad - \|(S(R_\mu^G \circ B_\mu^g))Ax_n - Ax^*\|^2) \\
 &\geq \frac{1}{2}\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2.
 \end{aligned} \tag{3.7}$$

On substituting (3.7) into (3.6), we obtain

$$\phi(x^*, w_n) \leq \phi(x^*, x_n) - \gamma_n(\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2 + \gamma_n\|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2) \tag{3.8}$$

$$\leq \phi(x^*, x_n). \tag{3.9}$$

Hence, we conclude from (3.4) and (3.8) that

$$\begin{aligned}
 \phi(x^*, u_n) &\leq \phi(x^*, w_n) \\
 &\leq \phi(x^*, x_n).
 \end{aligned} \tag{3.10}$$

Therefore, we conclude that $x^* \in C_{n+1}$. This implies that $\Gamma \subseteq C_n$ for all $n \geq 1$.

Hence, (3.2) is well-defined. \square Step 3: We show that $\{x_n\}$ is a Cauchy sequence.

Proof . Let $x^* \in \Gamma$, by using the definition of C_n , we have that $x_n = \Pi_{C_n}x_1$ for all $n \geq 1$. It follows from Lemma 2.9, we have that

$$\begin{aligned}
 \phi(x_n, x_1) &= \phi(\Pi_{C_n}x_1, x_1) \leq \phi(x^*, x_1) - \phi(x^*, \Pi_{C_n}x_1) \\
 &\leq \phi(x^*, x_1), \quad \forall n \geq 1.
 \end{aligned}$$

This implies that $\{\phi(x_n, x_1)\}$ is bounded.

More so, since $x_n = \Pi_{C_n}x_1$ and $x_{n+1} = \Pi_{C_{n+1}}x_1 \in C_{n+1} \subseteq C_n$, we have that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1), \quad \forall n \geq 1. \tag{3.11}$$

Therefore, $\{\phi(x_n, x_1)\}$ is non-decreasing and hence bounded. So, the limit also exists.

From Lemma 2.9, we obtain that

$$\begin{aligned}
 \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n}x_1) \leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n}x_1, x_1) \\
 &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1),
 \end{aligned} \tag{3.12}$$

thus, we have that

$$\lim_{n \rightarrow \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.13}$$

Applying Lemma 2.10, we obtain that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0. \tag{3.14}$$

Suppose $x_n = \Pi_{C_n} x_1 \subseteq C_m$, for some positive integers m, n with $m \leq n$, then applying Lemma 2.9 and using the same approach as in (3.12), we obtain that

$$\begin{aligned} \phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n} x_1) \\ &\leq \phi(x_m, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_m, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.15}$$

Since $\lim_{n \rightarrow \infty} \phi(x_n, x_1)$ exists, it follows from (3.15) and Lemma 2.10 that $\lim_{n \rightarrow \infty} \|x_n - x_m\| = 0$. Hence, we conclude that $\{x_n\}$ is a Cauchy sequence. \square

Step 4: Let $\{x_n\}$ be a sequence generated by (3.2), then (i) $\lim_{n \rightarrow \infty} \|T(R_\lambda^F \circ B_\lambda^f)w_n - w_n\| = 0$.

(ii) $\lim_{n \rightarrow \infty} \|(I - S(R_\mu^G \circ B_\mu^g))Ax_n\| = 0$.

(iii) $\lim_{n \rightarrow \infty} \|A^*J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n\| = 0$.

Proof . Since $x_{n+1} = \Pi_{C_{n+1}} \in C_{n+1} \subseteq C_n$, by the definition of C_{n+1} , (3.11) and (3.13), we have that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n) \rightarrow 0, \quad (n \rightarrow \infty). \tag{3.16}$$

We have from Lemma 2.10 that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - u_n\| = 0. \tag{3.17}$$

Also, from (3.14) and (3.17), we have that

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{3.18}$$

From (3.4), (3.5) and (3.8), we have that

$$\begin{aligned} \phi(x^*, u_n) &\leq \phi(x^*, x_n) - \beta_n(1 - \beta_n)g(\|J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) \\ &\quad - \gamma_n(\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2 + \gamma_n\|A^*J_2(S(R_\mu^G \circ B_\mu^g))Ax_n\|^2). \end{aligned} \tag{3.19}$$

It then follows that

$$\begin{aligned} &\beta_n(1 - \beta_n)g(\|J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) \\ &\leq \phi(x^*, x_n) - \phi(x^*, u_n) \\ &= \|x^*\|^2 - 2\langle x^*, J_1x_n \rangle + \|x_n\|^2 - \|x^*\|^2 + 2\langle x^*, J_1u_n \rangle - \|u_n\|^2 \\ &= 2\langle x^*, J_1u_n - J_1x_n \rangle + \|x_n\|^2 - \|u_n\|^2 \\ &\leq 2\|x^*\| \|J_1u_n - J_1x_n\| + \|x_n - u_n\| (\|x_n\| + \|u_n\|). \end{aligned} \tag{3.20}$$

Since E_1 is 2-uniformly convex and uniformly smooth Banach space, J_1 is uniformly continuous from norm-to-norm. Then, we obtain from (3.18) that

$$\lim_{n \rightarrow \infty} \|J_1u_n - J_1x_n\| = 0. \tag{3.21}$$

By applying the condition $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$ and (3.21) in (3.20), we obtain that

$$\lim_{n \rightarrow \infty} g(\|J_1w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\|) = 0. \tag{3.22}$$

Using the property of g in Lemma 2.8, we have that

$$\lim_{n \rightarrow \infty} \|J_1 w_n - J_1(T(R_\lambda^F \circ B_\lambda^f))w_n\| = 0 \tag{3.23}$$

Since J_1^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n \rightarrow \infty} \|w_n - (T(R_\lambda^F \circ B_\lambda^f))w_n\| = 0. \tag{3.24}$$

Also, from (3.19) and following the same approach in (3.20), we have that

$$\begin{aligned} & \gamma_n (\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2 - \gamma_n \|A^* J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n\|^2) \\ & \leq \phi(x^*, x_n) - \phi(x^*, u_n) \\ & = \|x^*\|^2 - 2\langle x^*, J_1 x_n \rangle + \|x_n\|^2 - \|x^*\|^2 + 2\langle x^*, J_1 u_n \rangle - \|u_n\|^2 \\ & = 2\langle x^*, J_1 u_n - J_1 x_n \rangle + \|x_n\|^2 - \|u_n\|^2 \\ & \leq 2\|x^*\| \|J_1 u_n - J_1 x_n\| + \|x_n - u_n\| (\|x_n\| + \|u_n\|). \end{aligned}$$

Using (3.18) and (3.21), we have that

$$\lim_{n \rightarrow \infty} \gamma_n (\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2 - \gamma_n \|A^* J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n\|^2) = 0. \tag{3.25}$$

Applying the definition on γ_n and the fact that ρ_n is bounded from above and away from zero, (3.25) gives

$$\lim_{n \rightarrow \infty} \frac{\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^4}{\|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2} = 0. \tag{3.26}$$

Observe that

$$\begin{aligned} \|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\| & \leq \|A^*\| \|J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\| \\ & = \|A\| \|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|. \end{aligned} \tag{3.27}$$

Therefore, from (3.26), we get

$$\lim_{n \rightarrow \infty} \|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\| \leq \|A\| \lim_{n \rightarrow \infty} \frac{\|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^4}{\|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2} = 0. \tag{3.28}$$

It follows from (3.27) and (3.28) that

$$\lim_{n \rightarrow \infty} \|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\| = 0. \tag{3.29}$$

From (3.2) and (3.26), we have that

$$\begin{aligned} \|J_1 w_n - J_1 x_n\| & = \gamma_n \|A^* J_2(I - S(R_\mu^G \circ B_\mu^g))Ax_n\| \\ & \leq \frac{\rho_n \|(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|^2}{\|A^* J_2(I - (S(R_\mu^G \circ B_\mu^g)))Ax_n\|}. \end{aligned} \tag{3.30}$$

From (3.2), (3.30) and by uniform continuity of J_1 and J_1^{-1} on bounded subset, we obtain that

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0. \tag{3.31}$$

Also, from (3.2) and (3.23), we get

$$\|J_1 u_n - J_1 w_n\| \leq \beta_n \|J_1(T(R_\lambda^F \circ B_\lambda^f))w_n - J_1 w_n\| \rightarrow 0, \text{ as } n \rightarrow \infty. \tag{3.32}$$

More so, from (3.32) and by uniform continuity of J_1 and J_1^* on bounded subset, we obtain that

$$\lim_{n \rightarrow \infty} \|u_n - w_n\| = 0. \tag{3.33}$$

From (3.31) and (3.33), we get that

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \tag{3.34}$$

□

Step 4: We show that $\bar{x} \in \Gamma$.

Proof . Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\bar{x} \in E_1$ such that $x_{n_k} \rightharpoonup \bar{x}$. Now using (3.31) and (3.34), there exist subsequence $\{w_{n_k}\}$ of $\{w_n\}$ and $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{w_n\}$ and $\{u_n\}$ converges weakly to \bar{x} . From (3.24), the fact that T is relatively nonexpansive mapping and Lemma 3.1, we obtain that $\bar{x} \in \text{Fix}(T(R_\lambda^F \circ B_\lambda^f)) = \text{Fix}(T) \cap (F + f)^{-1}(0)$. Also, since A is a bounded linear operator, we have that $Ax_{n_k} \rightharpoonup A\bar{x}$. Thus from (3.28), the fact that S is a relatively nonexpansive mapping, the demiclosedness principle and Lemma 3.1, we have that $A\bar{x} \in \text{Fix}(S(R_\lambda^G \circ B_\lambda^g)) = \text{Fix}(S) \cap (G + g)^{-1}(0)$. Hence, we therefore conclude that $\bar{x} \in \Gamma$. □

Step 5: We prove that $\{x_n\} \rightarrow \bar{x}$

Proof . Let $\bar{x} = \Pi_\Gamma x_1$, $\bar{x} \in \Gamma$, from $x_n = \Pi_{C_n} x_1$ and $\bar{x} \in \Gamma \subseteq C_n$, we have

$$\phi(x_n, x_1) \leq \phi(\bar{x}, x_1), \tag{3.35}$$

which implies that

$$\phi(\bar{x}, x_1) \leq \liminf_{n \rightarrow \infty} \phi(x_n, x_1) \leq \phi(\bar{x}, x_1). \tag{3.36}$$

From the definition of $\bar{x} = \Pi_\Gamma x_1$, we have that $x^* = \bar{x}$. Hence $\liminf_{n \rightarrow \infty} x_n = \bar{x} = \Pi_C x_1$. We therefore conclude that $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$, where $\bar{x} = \Pi_\Gamma x_1$. □ In the following result, we considered only the SMVIP without the fixed point problems.

Corollary 3.4. *Let E_1, E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ and duals E_1^*, E_2^* , respectively. Let Q be a nonempty, closed and convex subset of E_2 . Suppose that $A : E_1 \rightarrow E_2$ is a bounded linear operator with adjoint A^* , $F : E_1 \rightarrow 2^{E_1^*}$ and $G : E_2 \rightarrow 2^{E_2^*}$ are maximal monotone operators. Let $f : E_1 \rightarrow E_1^*$ and $g : E_2 \rightarrow E_2^*$ be single-valued λ, μ -ism operators with $R_\lambda^F \circ B_\lambda^f := (J + \lambda F)^{-1} \circ (J - \lambda f) : E_1 \rightarrow \text{dom}F$ for $\lambda > 0$ and $R_\mu^G \circ B_\mu^g := (J + \mu G)^{-1} \circ (J - \mu g) : E_2 \rightarrow \text{dom}G$ for $\mu > 0$, respectively. Assume that $\Gamma := \{x^* \in (F + f)^{-1}(0) \text{ and } Ax^* \in (G + g)^{-1}(0)\} \neq \emptyset$, then $\{x_n\}_{n=0}^\infty$ is generated iteratively by $x_1 \in E_1$ and $C_1 = E_1$ with*

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n - \gamma_n A^* J_2(I - (R_\mu^G \circ B_\mu^g))Ax_n); \\ u_n = J_1^{-1}[(1 - \beta_n)J_1 w_n + \beta_n J_1(R_\lambda^F \circ B_\lambda^f)w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1; \quad n \geq 1; \end{cases} \tag{3.37}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1} . Suppose $\{\beta_n\}_{n=1}^\infty$ is a sequence in $(0,1)$ such that $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, and the step size γ_n is chosen in such a way that $\gamma_n = \frac{\rho_n \|(I - (R_\mu^G \circ B_\mu^g))Ax_n\|^2}{\|A^*J_2(I - (R_\mu^G \circ B_\mu^g))Ax_n\|^2}$, for $Ax_n \neq (R_\mu^G \circ B_\mu^g)Ax_n$, where $0 < d \leq \rho_n \leq e < 1$ for $d, e \in \mathbb{R}$, otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Then, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$, where $\bar{x} = \Pi_\Gamma x_1$.

Also, in the result discussed below, we considered the split common fixed point problem.

Corollary 3.5. Let E_1, E_2 be 2-uniformly convex and uniformly smooth real Banach spaces with smoothness constant k satisfying $0 < k \leq \frac{1}{\sqrt{2}}$ and duals E_1^*, E_2^* , respectively. Let Q be a nonempty, closed and convex subset of E_2 , $T : E_1 \rightarrow E_1$ and $S : E_2 \rightarrow E_2$ be relatively nonexpansive mappings respectively. Suppose that $A : E_1 \rightarrow E_2$ is a bounded linear operator with adjoint A^* . Assume that $\Gamma := \{x^* \in \text{Fix}(T) \text{ and } Ax^* \in \text{Fix}(S)\} \neq \emptyset$, then $\{x_n\}_{n=0}^\infty$ is generated iteratively by $x_1 \in E_1$ and $C_1 = E_1$ with

$$\begin{cases} w_n = J_1^{-1}(J_1x_n - \gamma_n A^* J_2(I - S)Ax_n); \\ u_n = J_1^{-1}[(1 - \beta_n)J_1w_n + \beta_n J_1(T)w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}}x_1; \quad n \geq 1; \end{cases} \tag{3.38}$$

where $\Pi_{C_{n+1}}$ is the generalized projection of E_1 onto C_{n+1} . Suppose $\{\beta_n\}_{n=1}^\infty$ is a sequence in $(0,1)$ such that $\liminf_{n \rightarrow \infty} \beta_n(1 - \beta_n) > 0$, and the step size γ_n is chosen in such a way that $\gamma_n = \frac{\rho_n \|(I - S)Ax_n\|^2}{\|A^*J_2(I - S)Ax_n\|^2}$, for $Ax_n \neq (S)Ax_n$, where $0 < d \leq \rho_n \leq e < 1$ for $d, e \in \mathbb{R}$, otherwise $\gamma_n = \gamma$ (γ being any nonnegative real number). Then, the sequence $\{x_n\}$ converges strongly to $\bar{x} \in \Gamma$, where $\bar{x} = \Pi_\Gamma x_1$.

Remark 3.6. The result discussed in this article generalizes many related results, most especially, results where SVIP and SMVIP were discussed in the framework of real Hilbert spaces. Our results holds for the classes of nonexpansive and pseudocontractive mappings in the framework of real Hilbert spaces.

4. Applications

1. Application to Split Convex Minimization Problem:

Definition 4.1. Let Q be a convex subset of a vector space X and $f : Q \rightarrow \mathbb{R} \cup \{+\infty\}$ be a map. Then,

(i) f is convex if for each $\lambda \in [0, 1]$ and $x, y \in Q$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);$$

(ii) f is called proper if there exists at least one $x \in Q$ such that

$$f(x) \neq +\infty;$$

(iii) f is lower semi-continuous at $x_0 \in Q$ if

$$f(x_0) \leq \liminf_{x \rightarrow x_0} f(x).$$

Let E_1 and E_2 be real Banach spaces. Let $M : E_1 \rightarrow \mathbb{R}$ and $N : E_2 \rightarrow \mathbb{R}$ are convex and differentiable functions and $F : E_1 \rightarrow (-\infty, +\infty]$ and $G : E_2 \rightarrow (-\infty, +\infty]$ are proper, convex and lower semi-continuous functions. It is clear that if ∇M and ∇N is $\frac{1}{\alpha}$ and $\frac{1}{\theta}$ - Lipschitz continuous, then it is α, θ -ism, where ∇M and ∇N are the gradient of M and N respectively. It is also known that the subdifferential ∂F and ∂G are maximal monotone. (see [40]). Moreover,

$$M(x^*) + F(x^*) = \min_{x \in E_1} [M(x) + F(x)] \Leftrightarrow 0 \in M(x^*) + \partial F(x^*).$$

and

$$N(x^*) + G(x^*) = \min_{x \in E_2} [N(x) + G(x)] \Leftrightarrow 0 \in N(x^*) + \partial G(x^*).$$

Our aim is to solve the following Split Convex Minimization and Fixed Point Problem, (in short, SCMFPP): find $x^* \in E_1$ such that

$$x^* \in \text{Fix}(T) \cap \text{argmin}_{x \in E_1} M(x) + F(x) \text{ and } y^* = Ax^* \in \text{Fix}(S) \cap \text{argmin}_{y \in E_2} N(y) + G(y). \quad (4.1)$$

Suppose the solution set of (4.1) is denoted by Θ , then by setting $F = \partial F, G = \partial G, f = \nabla M$ and $g = \nabla N$, (3.2) becomes

$$\begin{cases} w_n = J_1^{-1}(J_1 x_n - \gamma_n A^* J_2 (I - (S(R_\mu^{\partial G} \circ B_\mu^{\nabla N}))) A x_n); \\ u_n = J_1^{-1}[(1 - \beta_n) J_1 w_n + \beta_n J_1 (T(R_\lambda^{\partial F} \circ B_\lambda^{\nabla M})) w_n]; \\ C_{n+1} = \{v \in C_n : \phi(v, u_n) \leq \phi(v, x_n)\}; \\ x_{n+1} = \Pi_{C_{n+1}} x_1; \quad n \geq 1. \end{cases} \quad (4.2)$$

Assume that the conditions in (3.2) holds, then $\{x_n\}$ converges strongly to an element in Θ .

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