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Generalized Suzuki (ψ, ϕ)-contraction in complete metric spaces

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Abstract

In this paper, we introduce a new class of mappings called the (ψ, ϕ)-Suzuki-type mapping and (ψ, ϕ)-Jungck-Suzuki contraction type mappings and we establish the existence, uniqueness and coincidence results for (ψ, ϕ)-Suzuki-type mapping and (ψ, ϕ)-Jungck-Suzuki contraction mappings in the frame work of complete metric spaces. Furthermore, we applied our results to the existence and uniqueness of solutions of a differential equation. Our results improve, extend and generalize some known results in the literature.

Keywords: (ψ, ϕ)-Suzuki-type mapping; fixed point; (ψ, ϕ)-Jungck-Suzuki; coincidence point; metric space.
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1. Introduction and Premilinaries

Banch contraction principle [3] can be seen as the pivot of the theory of fixed point and its applications. The theory of fixed point plays an important role in nonlinear functional analysis and its very useful for showing the existence and uniqueness theorems for nonlinear differential and integral equations. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. We recall that a mapping $T : X \to X$ is said to be an α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall \quad x, y \in X. \tag{1.1}$$

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Thereafter, the Banach contraction theorem have been extended and generalized by researchers in this area. Researchers in this area generalizes the well celebrated Banach contraction principle by considering a class of nonlinear mappings and spaces which are more general than the class of a contraction mappings and metric spaces (see [1, 4, 16, 17, 18, 19, 21, 22, 25, 27, 28, 33] and the references therein). For example, in 1973, Geraghty [9] introduced a generalized contraction mapping called Geraghty-contraction and established the fixed point theorem for this class of contraction mappings in the frame work of metric spaces. We recall the following definition and result from [9].

Definition 1.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is called a Geraghty-contraction mapping if and only if*

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y) \tag{1.2}$$

for all $x, y \in X$, where $\phi : \mathbb{R}^+ \to [0, 1)$ satisfies the following condition:

$$\phi(t_n) \to 1 \quad as \quad n \to \infty \Rightarrow t_n \to 0 \quad as \quad n \to \infty.$$

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a self map that satisfies condition (1.2). Then T has a unique fixed point $x^* \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = x^*$.*

In [10] Goebel generalized (1.1) by introducing a continuous mapping S in place of the identity mapping ($Ix = x$), such that S commute with T and $T(X) \subset S(X)$. More precisely, he introduced the following definition.

Definition 1.3. *Let $S, T : Y \to X$ be two mappings, T is called a Jungck-contraction if there exists a real number $\delta \in [0, 1)$ such that*

$$d(Tx, Ty) \leq \delta d(Sx, Sy) \tag{1.3}$$

for all $x, y \in X$.

In 1976, Jungck [7], proved a common fixed point theorem for commuting maps under the condition that $X = Y$. The result is as follows:

Theorem 1.4. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : X \to X$ satisfies condition (1.3) such that (T, S) are commuting pair, $T(X) \subseteq S(X)$ and S is continuous. Then T and S have a unique common fixed point say $p \in X$.*

Remark 1.5. *Clearly, if we take $Sx = x$ in (1.3) for all $x \in X$, we obtain condition (1.1).*

Definition 1.6. *[8] Let X be a nonempty set and $S, T : X \to X$ be any two mappings.*

1. *A point $x \in X$ is called:*
   (a) *coincidence point of S and T if $Sx = Tx$,*
   (b) *common fixed point of S and T if $x = Sx = Tx$.*
2. *If $y = Sx = Tx$ for some $x \in X$, then y is called the point of coincidence of S and T.*
3. *A pair (S, T) is said to be:*
   (a) *commuting if $TSx = STx$ for all $x \in X$,*
   (b) *weakly compatible if they commute at their coincidence points, that is $STx = TSx$, whenever $Sx = Tx$.*

In 1984, Khan, Swaleh and Sessa in [15] introduced the concept of alternating distance function, which is defined as follows: A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an alternating distance function if the following conditions are satisfied:

1. $\psi(0) = 0$,
2. ψ is monotonically nondecreasing,
3. ψ is continuous.

They established the following result.

Theorem 1.7. *Let (X, d) be a complete metric space, let ψ be an altering distance function, and let $T : X \to X$ be a self mapping which satisfies the following condition*

$$\psi(d(Tx, Ty)) \leq \delta\psi(d(x, y))$$

for all $x, y \in X$, where $\delta \in (0, 1)$. Then T has a unique fixed point.

Remark 1.8. *Clearly, if we take $\psi(x) = x$, for all $x \in X$, we obtain condition (1.1).*

In what follows, we present some examples of an alternating distance functions.

Example 1.9. *A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by*

1. $\psi(t) = nt$, *for all* $n \in \mathbb{N}$,
2. $\psi(t) = t^n$, *for all* $n \in \mathbb{N}$,
3. $\psi(t) = \cosh(t) - 1$.

In 1997, Alber and Guerre-Delabriere [2] introduced a generalization of Banach contraction mapping called weakly contraction mapping in the frame work of Hilbert space. They established some fixed point results. We recall that a mapping $T : H \to H$ is a weakly contractive mapping if

$$d(Tx, Ty) \leq d(x, y) - \phi(d(x, y)),$$

for all $x, y \in X$, where H is an Hilbert space and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nondecreasing function such that $\phi(t) = 0$ if and only if $t = 0$.

Remark 1.10. *Clearly, if we take $\phi(x) = (1 - \alpha)x$, were $0 \leq \alpha < 1$ for all $x \in X$, we obtain condition (1.1).*

Rhoades [26] extend the concepts of weakly contraction mapping to metric spaces and he established the following result.

Theorem 1.11. *Let (X, d) be a complete metric space. Suppose the mapping $T : X \to X$ is a weakly contractive, then T have a unique fixed point.*

Using the concept of alternating distance function, Doric [5], Dutta and Choudhury [6], Harjani and Sadarangani [11, 12] established some fixed points results for weak contraction and generalized contrac tion mappings in the frame work of partially ordered metric spaces. They established the following results.

Theorem 1.12. *[6] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for all $x, y \in X$ where $\phi, \psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that $\psi(t) > 0, \phi(t) > t$, for $t > 0$ and $\phi(0) = \psi(0) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Theorem 1.13. *[5] Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y))$$

for all $x, y \in X$ where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, $\phi, \psi : [0, \infty) \to [0, \infty)$ such that ψ is continuous and nondecreasing function with $\psi(t) > 0$, for $t > 0$ and $\phi(0) = 0$ if and only if $t = 0$ and ϕ a lower semi-continuous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Theorem 1.14. *[11] Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$$

for $x \geq y$, where $\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing function such that $\psi(t) > 0, \psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

Theorem 1.15. *[12] Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and nondecreasing mapping such that*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y))$$

for $x \geq y$, where ψ, ϕ are alteranting distance function. If there exists $x_0 \in X$ with $x_0 \leq Tx_0$, then T has a fixed point.

In 2008, Suzuki in [29] introduced the concept of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems.

Definition 1.16. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

The concept of alternating distance functions, weakly contractive mappings and mappings satisfying condition (C) have been explored by researchers in this area. For some notable works (see [11, 12, 14, 21, 29, 30, 31, 32] and the references therein).

Motivated by the above works and the research in this direction. The purpose of this paper is to introduce a new class of mappings called the (ψ, ϕ)-Suzuki-type mapping and (ψ, ϕ)-Jungck-Suzuki contraction type mappings and we establish the existence, uniqueness and coincidence results for (ψ, ϕ)-Suzuki-type mapping and (ψ, ϕ)-Jungck-Suzuki contraction mappings in the frame work of complete metric spaces. Furthermore, we applied our results to the existence and uniqueness of solutions of a differential equation. Our results improve, extend and generalize some known results in the literature.

2. Existence and Uniqueness of Fixed Point of (ψ, ϕ)-Suzuki type Mappings

In this section, we introduce the notion of (ψ, ϕ)-Suzuki type mapping and established the existence and uniqueness result for this class of mappings.

Definition 2.1. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be (ψ, ϕ)-Suzuki type, if for all $x, y \in X$,*

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow \psi(d(Tx, Ty)) \leq \psi(M(x, y)) - k\phi(N(x, y)) + L\phi(N_1(x, y)), \tag{2.1}$$

where $0 < k \leq 1, L \geq 0, M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$,
$N(x, y) = \max\{d(x, y), d(y, Ty)\}, N_1(x, y) = \min\{d(x, Ty), d(x, Tx), d(y, Tx)\}$ and ψ, ϕ are alternating distance functions.

Theorem 2.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying condition (2.1). Then T has a unique fixed point.*

Proof . Let $x_0 \in X$. We define the sequence $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$. If we suppose that $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, we have $x_n = Tx_n$, which is the required result. So, suppose $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Now observe that

$$\begin{aligned} &\frac{1}{2}d(x_n, Tx_n) \\ &= \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) \\ \Rightarrow &\psi(d(x_{n+1}, x_{n+2})) = \psi(d(Tx_n, Tx_{n+1})) \leq \psi(M(x_n, x_{n+1})) - k\phi(N(x_n, x_{n+1})) + L\phi(N_1(x_n, x_{n+1})), \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} M(x_n, x_{n+1}) &= \max\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(x_n, x_{n+1})\} = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\ N(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, Tx_{n+1})\} = \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \\ N_1(x_n, x_{n+1}) &= \min\{d(x_n, Tx_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_n)\} = 0. \end{aligned}$$

If we take $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_{n+1}, x_{n+2})$ for some $n \in \mathbb{N}$, then (2.2) becomes

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &\leq \psi(d(x_{n+1}, x_{n+2})) - k\phi(d(x_{n+1}, x_{n+2})) \\ &\Rightarrow \phi(d(x_{n+1}, x_{n+2})) \leq 0. \end{aligned}$$

Using the fact that $\phi(t) > 0$ and $\phi(t) = 0$ if $t = 0$, we have that $d(x_{n+1}, x_{n+2}) = 0$, which implies that $x_{n+1} = x_{n+2}$, which is a contradiction. Thus, we have that the $\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} = d(x_n, x_{n+1})$ and so

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq d(x_n, x_{n+1}), \\ d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_n) \end{aligned} \tag{2.3}$$

and (2.2) becomes

$$\psi(d(x_{n+1}, x_{n+2})) \leq \psi(d(x_n, x_{n+1})) - k\phi(d(x_n, x_{n+1})). \tag{2.4}$$

From (2.3), we have that $\{d(x_{n+1}, x_{n+2})\}$ is a nonincreasing sequence. Thus, there exists $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = c,$$
$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c. \tag{2.5}$$

Taking the limit as $n \to \infty$ in (2.4), we have

$$\begin{aligned}&\psi(c) \leq \psi(c) - k\phi(c)\\ &\Rightarrow \phi(c) \leq 0 \quad \text{(using the definition of } \phi)\\ &\Rightarrow c = 0.\end{aligned}$$

And so, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_{n+2}) = 0,$$
$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

We now show that $\{x_n\}$ is a Cauchy sequence in X. Assume on contrary that the sequence $\{x_n\}$ is not Cauchy. Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \quad \text{and} \quad d(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{2.7}$$

Then, we have

$$\begin{aligned}\epsilon \leq d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{n_k-1}) + d(x_{n_k-1}, x_{n_k})\\ &< \epsilon + d(x_{n_k-1}, x_{n_k}).\end{aligned} \tag{2.8}$$

Setting $k \to \infty$ and using (2.6), we have

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{2.9}$$

Also, using (2.6) and (2.9), we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{m_k+1}) + d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$$

and

$$d(x_{m_k+1}, x_{n_k+1}) \leq d(x_{m_k+1}, x_{m_k}) + d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

Setting $k \to \infty$, we have

$$\lim_{n\to\infty} d(x_{m_{k+1}}, x_{n_{k+1}}) = \epsilon. \tag{2.10}$$

More so, using (2.6) and (2.9), we have

$$d(x_{m_k}, x_{n_k}) \leq d(x_{m_k}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k})$$

and

$$d(x_{m_k}, x_{n_{k+1}}) \leq d(x_{m_k}, x_{n_k}) + d(x_{n_k}, x_{n_k+1}).$$

Setting $k \to \infty$, we have

$$\lim_{n\to\infty} d(x_{m_k}, x_{n_k+1}) = \epsilon. \tag{2.11}$$

Furthermore, using (2.6) and (2.10), we have

$$d(x_{m_k+1}, x_{n_k+1}) \leq d(x_{m_k+1}, x_{n_k}) + d(x_{n_k}, x_{n_k+1})$$

and

$$d(x_{m_k+1}, x_{n_k}) \leq d(x_{m_k+1}, x_{n_k+1}) + d(x_{n_k+1}, x_{n_k}).$$

Setting $k \to \infty$, we have

$$\lim_{n\to\infty} d(x_{m_k+1}, x_{n_k}) = \epsilon. \tag{2.12}$$

In addition, for the $\epsilon > 0$, the convergence of the sequence $\{d(x_n, x_{n+1})\}$ implies that there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+1}) < \epsilon$ for all $n \geq n_0$. Let $\mathbb{N}_1 = \max\{m_i, n_0\}$. Then, for all $m_k > n_k \geq \mathbb{N}_1$, we have

$$d(x_{n_k}, x_{n_{k+1}}) < \epsilon \leq d(x_{n_k}, x_{m_k}),$$

where $m_k > n_k$ and so

$$\frac{1}{2} d(x_{n_k}, x_{n_{k+1}}) \leq d(x_{n_k}, x_{m_k})$$

which implies that

$$\psi(d(x_{n_{k+1}}, x_{m_{k+1}})) = \psi(d(Tx_{n_k}, Tx_{m_k})) \leq \psi(M(x_{n_k}, x_{m_k})) - k\phi(N(x_{n_k}, x_{m_k})) + L\phi(N_1(x_{n_k}, x_{m_k})), \tag{2.13}$$

where

$$\begin{aligned}
M(x_{n_k}, x_{m_k}) &= \max\{d(x_{n_k}, Tx_{n_k}), d(x_m, Tx_{m_k}), d(x_{n_k}, x_{m_k})\} \\
&= \max\{d(x_{n_k}, x_{n_k+1}), d(x_m, x_{m+1}), d(x_{n_k}, x_{m_k})\} \\
N(x_{n_k}, x_{m_k}) &= \max\{d(x_{n_k}, x_{m_k}), d(x_{m_k}, Tx_{m_k})\} \\
&= \max\{d(x_{n_k}, x_{m_k}), d(x_m, x_{m+1})\} \\
N_1(x_{n_k}, x_{m_k}) &= \min\{d(x_{n_k}, Tx_{m_k}), d(x_{n_k}, Tx_{n_k}), d(x_{m_k}, Tx_{n_k})\} \\
&= \min\{d(x_{n_k}, x_{m_k+1}), d(x_{n_k}, x_{n_k+1}), d(x_{m_k}, x_{n_k+1})\}.
\end{aligned}$$

Setting $k \to \infty$ and using (2.6), (2.9), (2.10), (2.11) and (2.12), (2.13) becomes

$$\begin{aligned}
&\psi(\epsilon) \leq \psi(\epsilon) - k\phi(\epsilon) + L\phi(0) \\
&\quad \Rightarrow \phi(\epsilon) \leq 0 \quad \text{(using the definition of } \phi) \\
&\quad \Rightarrow \epsilon = 0.
\end{aligned}$$

This contradict our assumption that $\epsilon > 0$. Thus $\{x_n\}$ is Cauchy. Since X is complete, then there exists say $y \in X$ such that $\lim_{n\to\infty} x_n = y$. Now, suppose that for every $n \in \mathbb{N}$, we have

$$d(x_n, y) < \frac{1}{2}d(x_n, x_{n+1})$$

and

$$d(x_{n+1}, y) < \frac{1}{2}d(x_{n+1}, x_{n+2}).$$

Now, observe that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq d(x_n, y) + d(y, x_{n+1}) \\ &< \frac{1}{2}d(x_n, x_{n+1}) + \frac{1}{2}d(x_{n+1}, x_{n+2}) \\ &\Rightarrow d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2}). \end{aligned}$$

The above inequality has been shown as a contradiction in (2.2). Hence, we have that

$$\frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, y)$$

and

$$\frac{1}{2}d(x_{n+1}, x_{n+2}) \leq d(x_{n+1}, y).$$

It then follows that

$$\frac{1}{2}d(x_n, Tx_n) = \frac{1}{2}d(x_n, x_{n+1}) \leq d(x_n, y),$$

which implies that

$$\psi(d(x_{n+1}, Ty)) = \psi(d(Tx_n, Ty)) \leq \psi(M(x_n, y)) - k\phi(N(x_n, y)) + L\phi(N_1(x_n, y)), \tag{2.14}$$

where

$$\begin{aligned} M(x_n, y) &= \max\{d(x_n, Tx_n), d(y, Ty), d(x_n, y)\} \\ N(x_n, y) &= \max\{d(x_n, y), d(y, Ty)\} \\ N_1(x_n, y) &= \min\{d(x_n, Ty), d(x_n, Tx_n), d(y, Tx_n)\}. \end{aligned}$$

Setting $n \to \infty$, we have (2.14) becomes

$$\begin{aligned} \psi(d(y, Ty)) &\leq \psi(d(y, Ty)) - k\phi(d(y, Ty)) + L\phi(0) \\ &\Rightarrow \phi(d(y, Ty)) \leq 0 \quad \text{(using the definition of } \phi) \\ &\Rightarrow d(y, Ty) = 0 \\ &\Rightarrow y = Ty. \end{aligned}$$

To show that the fixed point is unique, we suppose on the contrary that there exists another fixed point say $z \in X$ such that $z = Tz$ and $y \neq z$.

$$\frac{1}{2}d(y, Ty) = 0 \leq d(y, z),$$

which implies that

$$\psi(d(y,z)) = \psi(d(Ty,Tz)) \leq \psi(M(y,z)) - k\phi(N(y,z)) + L\phi(N_1(y,z),$$

we obtain

$$\begin{aligned}\psi(d(y,z)) \leq \psi(d(y,z)) - k\phi(d(y,z))\\ &\Rightarrow \phi(d(y,z)) \leq 0 \quad \text{(using the definition of } \phi)\\ &\Rightarrow d(y,z) = 0\\ &\Rightarrow y = z.\end{aligned}$$

Hence, the fixed point is unique. □

Example 2.3. *Let $X = \{0,1,2,3,4,\cdots\}$. We define*

$$d(x,y) = \begin{cases} x+y+4 & if \quad x \neq y\\ 0 & if \quad x = y,\end{cases}$$

$T : X \to X$ *by*

$$Tx = \begin{cases} x-1 & if \quad x \geq 1\\ 0 & if \quad x = 0,\end{cases}$$

and $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(x) = x^2$ and $\phi(x) = x$. Clearly, (X,d) is a complete metric space (see, [4]). Then for any $k \in (0,1]$ and $L \geq 0$, T is a (ψ,ϕ)-Suzuki type mapping.

Proof . To establish that T is a (ψ,ϕ)-Suzuki type mapping, we consider the following cases.
Case 1: If $x = y$, we consider the following sub-cases.
Case 1(a): If $x = y = 0$. We have that $d(Tx,Ty) = 0$ and $\psi(0) = 0$. Clearly, we have that

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$$

Case 1(b): If $x = y \neq 0$. We have

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(x,x-1) = \frac{2x+3}{2} > 0 = d(x,y).$$

Thus, we have nothing to show.
Case 2: If $x > y$ and $y = 1$.

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(x,x-1) = \frac{2x+3}{2} < x+5 = d(x,y),$$

and

$$\begin{aligned}d(Tx,Ty) &= d(x-1,y-1) = d(x-1,0) = x+3\\ d(x,y) &= d(x,1) = x+5\\ d(x,Tx) &= d(x,x-1) = 2x+3\\ d(y,Ty) &= d(1,0) = 5.\end{aligned}$$

Therefore, we have that

$$M(x,y) = 2x+3, \quad N(x,y) = x+5$$

and

$$(x+3)^2 \leq (2x+3)^2 - (x+5),$$

we have

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$$

Case 3: If $x > y$ and $y = 0$. We consider the following sub-cases.
Case 3(a): If $x = 1$ and $y = 0$.

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(1,0) = 5 = d(x,y),$$

and

$$d(Tx,Ty) = d(x-1,0) = d(0,0) = 0.$$

Therefore, we have that

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$$

Case 3(b): If $x > 1$ and $y = 0$.

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(x,x-1) = \frac{2x+3}{2} < x+4 = d(x,y),$$

and

$$\begin{aligned} d(Tx,Ty) &= d(x-1,0) = x+3 \\ d(x,y) &= d(x,0) = x+4 \\ d(x,Tx) &= d(x,x-1) = 2x+3 \\ d(y,Ty) &= d(0,0) = 0. \end{aligned}$$

Therefore, we have that

$$M(x,y) = 2x+3, \quad N(x,y) = x+4$$

and

$$(x+3)^2 \leq (2x+3)^2 - (x+4),$$

we have

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$$

Case 4: If $x > y$ and $y \geq 2$. We consider the following sub-cases.
Case 4(a): If $x = y + 1$.

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(y+1,y) = \frac{2y+5}{2} < 2y+5 = d(y+1,y) = d(x,y),$$

and

$$\begin{aligned} d(Tx,Ty) &= d(x-1,y-1) = d(y,y-1) = 2y+3 \\ d(x,y) &= (y+1,y) = 2y+5 \\ d(x,Tx) &= d(y+1,y) = 2y+5 \\ d(y,Ty) &= 2y+3. \end{aligned}$$

Therefore, we have that

$$M(x,y) = 2x+3, \quad N(x,y) = x+4$$

and

$$(2y+3)^2 \leq (2y+5)^2 - (2y+5),$$

we have

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$$

Case 4(b): If $x > y + 1$.

$$\frac{1}{2}d(x,Tx) = \frac{1}{2}d(x,x-1) = \frac{2x+3}{2} < x+y+4 = d(x,y),$$

and

$$\begin{aligned} d(Tx,Ty) &= d(x-1,y-1) = x+y+2 \\ d(x,y) &= d(x,y) = x+y+4 \leq 2x+3 \\ d(x,Tx) &= d(x,x-1) = 2x+3 \\ d(y,Ty) &= d(y,y-1) = 2y+3. \end{aligned}$$

Therefore, we have that

$$M(x,y) = 2x+3, \quad N(x,y) = x+y+4$$

and

$$(x+y+2)^2 \leq (2x+1)^2 \leq (2x+3)^2 - (2x+3),$$

we have

$$\frac{1}{2}d(x,Tx) \leq d(x,y) \Rightarrow \psi(d(Tx,Ty)) \leq \psi(M(x,y)) - k\phi(N(x,y)) + L\phi(N_1(x,y)).$$

Hence, T is a (ψ,ϕ)-Suzuki type mapping with a unique fixed point 0. □

3. Coincidence Point Theorem for (ψ, ϕ)-Jungck-Suzuki type Mappings

In this section, we introduce the concept of (ψ, ϕ)- Jungck-Suzuki type mapping and established the existence of a coincidence point for this class of mappings.

Definition 3.1. *Let (X, d) be a metric space, Y an arbitrary nonempty set and $S, T : Y \to X$ be two mappings. A mapping T is said to be (ψ, ϕ)-Jungck-Suzuki type mapping, if for all $x, y \in Y$,*

$$\begin{aligned}&\frac{1}{2}d(Sx, Tx) \leq d(Sx, Sy)\\ &\qquad \Rightarrow \psi(d(Tx, Ty)) \leq \psi(M(Sx, Sy)) - k\phi(N(Sx, Sy)) + L\phi(N_1(Sx, Sy)),\end{aligned} \tag{3.1}$$

where $0 < k \leq 1, L \geq 0, M(Sx, Sy) = \max\{d(Sx, Sy), d(Sx, Tx), d(Sy, Ty)\}$,
$N(x, y) = \max\{d(Sx, Sy), d(Sy, Ty)\}, N_1(Sx, Sy) = \min\{d(Sx, Ty), d(Sx, Tx), d(Sy, Tx)\}$
and ψ, ϕ are alternating distance functions.

Theorem 3.2. *Let (X, d) be a complete metric space. Suppose the mappings $S, T : Y \to X$ satisfying condition (3.1) such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of X, then T and S have a coincidence point.*

Proof . For every $x_0 \in Y$, there exists $x_1 \in Y$ such that $Sx_1 = Tx_0$, since $T(Y) \subseteq S(Y)$. Using this fact, for any $x_n \in Y$, there exists x_{n+1} such that $Sx_{n+1} = Tx_n$. Now observe that

$$\begin{aligned}&\frac{1}{2}d(Sx_n, Tx_n)\\ &= \frac{1}{2}d(Sx_n, Sx_{n+1}) \leq d(Sx_n, Sx_{n+1})\\ \Rightarrow &\psi(d(Sx_{n+1}, Sx_{n+2})) = \psi(d(Tx_n, Tx_{n+1}))\\ &\leq \psi(M(Sx_n, Sx_{n+1})) - k\phi(N(Sx_n, Sx_{n+1})) + L\phi(N_1(Sx_n, Sx_{n+1})),\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}M(Sx_n, Sx_{n+1}) &= \max\{d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_{n+1}), d(Sx_n, Sx_{n+1})\}\\ &= \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2})\}\\ N(Sx_n, Sx_{n+1}) &= \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Tx_{n+1})\}\\ &= \max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2})\}\\ N_1(Sx_n, Sx_{n+1}) &= \min\{d(Sx_n, Tx_{n+1}), d(Sx_n, Tx_n), d(Sx_{n+1}, Tx_n)\} = 0.\end{aligned}$$

Using similar approach as in Theorem 2.2, we have that the $\max\{d(Sx_n, Sx_{n+1}), d(Sx_{n+1}, Sx_{n+2})\} = d(Sx_n, Sx_{n+1})$ and so

$$\begin{aligned}d(Sx_{n+1}, Sx_{n+2}) &\leq d(Sx_n, Sx_{n+1}),\\ d(Sx_n, Sx_{n+1}) &\leq d(Sx_{n-1}, Sx_n)\end{aligned} \tag{3.3}$$

and (3.2) becomes

$$\psi(d(Sx_{n+1}, Sx_{n+2})) \leq \psi(d(Sx_n, Sx_{n+1})) - k\phi(d(Sx_n, Sx_{n+1})). \tag{3.4}$$

From (3.3), we have that $\{d(Sx_{n+1}, Sx_{n+2})\}$ is a nonincreasing sequence. Thus, there exists $c \geq 0$ such that

$$\begin{aligned}&\lim_{n\to\infty} d(Sx_{n+1}, Sx_{n+2}) = c,\\ &\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = c.\end{aligned} \tag{3.5}$$

Taking the limit as $n \to \infty$ in (3.4), we have

$$\begin{aligned}&\psi(c) \leq \psi(c) - k\phi(c)\\ &\Rightarrow \phi(c) \leq 0 \quad \text{(using the definition of } \phi)\\ &\Rightarrow c = 0.\end{aligned}$$

$$\begin{aligned}&\lim_{n\to\infty} d(Sx_{n+1}, Sx_{n+2}) = 0,\\ &\lim_{n\to\infty} d(Sx_n, Sx_{n+1}) = 0.\end{aligned} \tag{3.6}$$

Using similar approach as in Theorem 2.2, it is easy to show that $\{Sx_n\}$ is Cauchy. Since $S(Y)$ is complete, then there exists say $x \in S(Y)$ such that $\lim_{n\to\infty} Sx_n = x$. More so, we can find $y \in Y$ such that $Sy = x$. Now suppose that for every $n \in \mathbb{N}$, we have

$$d(Sx_n, y) < \frac{1}{2} d(Sx_n, Sx_{n+1})$$

and

$$d(Sx_{n+1}, y) < \frac{1}{2} d(Sx_{n+1}, Sx_{n+2}).$$

Now, observe that

$$\begin{aligned}d(Sx_n, Sx_{n+1}) &\leq d(Sx_n, y) + d(y, Sx_{n+1})\\ &< \frac{1}{2} d(Sx_n, Sx_{n+1}) + \frac{1}{2} d(Sx_{n+1}, Sx_{n+2})\\ &\Rightarrow d(Sx_n, Sx_{n+1}) < d(Sx_{n+1}, Sx_{n+2}).\end{aligned}$$

Which is a contradiction. Hence, we have that

$$\frac{1}{2} d(Sx_n, Sx_{n+1}) \leq d(Sx_n, y) \quad \text{and} \quad \frac{1}{2} d(Sx_{n+1}, Sx_{n+2}) \leq d(Sx_{n+1}, y).$$

It then follows that

$$\frac{1}{2} d(Sx_n, Tx_n) = \frac{1}{2} d(Sx_n, Sx_n) \leq d(Sx_n, y),$$

which implies that

$$\psi(d(Sx_{n+1}, Ty)) = \psi(d(Tx_n, Ty)) \leq \psi(M(Sx_n, Sy)) - k\phi(N(Sx_n, Sy)) + L\phi(N_1(Sx_n, Sy)), \tag{3.7}$$

where

$$\begin{aligned}M(Sx_n, Sy) &= \max\{d(Sx_n, Tx_n), d(Sy, Ty), d(Sx_n, Sy)\} = \max\{d(Sx_n, Sx_{n+1}), d(x, Ty), d(Sx_n, x)\}\\ N(Sx_n, Sy) &= \max\{d(Sx_n, Sy), d(Sy, Ty)\} = \max\{d(Sx_n, x), d(x, Ty)\}\\ N_1(Sx_n, Sy) &= \min\{d(Sx_n, Ty), d(Sx_n, Tx_n), d(Sy, Tx_n)\} = \min\{d(Sx_n, Ty), d(Sx_n, Sx_{n+1}), d(x, Sx_{n+1})\}.\end{aligned}$$

Setting $n \to \infty$, we have (3.7) becomes

$$\begin{aligned}\psi(d(x,Ty)) \leq \psi(d(x,Ty)) - k\phi(d(x,Ty)) + L\phi(0)\\ \Rightarrow &\phi(d(x,Ty)) \leq 0 \quad \text{(using the definition of } \phi)\\ \Rightarrow &d(x,Ty) = 0\\ \Rightarrow &x = Ty.\end{aligned}$$

It follows that

$$\lim_{n\to\infty} d(Sx_n, Ty) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Sx_n, Sy) = 0$$

Thus, we have $x = Sy = Ty$. This completes the proof. □

4. Application

In this section, we present an application on existence of a solution of a differential equation. Let $C(I)$ be the space of all continuous function defned on $I = [0, 1]$. Consider the following second order differential equation with the associated boundary conditions

$$\begin{aligned}\frac{-d^2x}{dt^2} = f(t, x(t)), \quad t \in I, x \in [0,\infty)\\ x(0) = x'(1) = 0.\end{aligned} \tag{4.1}$$

If $x \in C^2(I)$ is a zero of (4.1), then $x \in C(I)$ is also the zero of the following integral equation

$$x(t) = \int_0^1 G(t,\tau) f(\tau, x(\tau))d\tau \quad t \in I,$$

where $G(t, \tau)$ is the Green function defined by

$$G(t,\tau) = \begin{cases} t(1-\tau) & \text{if} \quad 0 \leq t \leq \tau \leq 1\\ \tau(1-t) & \text{if} \quad 0 \leq \tau \leq t \leq 1.\end{cases}$$

Theorem 4.1. *Considering the differential equation (4.1) and a mapping $f : I \times C(I) \to \mathbb{R}$ defined by*

$$f(t,x) - f(t,y) = \delta\sqrt{(x-y)^2 + A(x-y)}, \quad x \geq y,$$

where $\delta \in (0, 8], A = L - 1 > 0$. Suppose that f is weakly increasing with respect to the second variable, then there exists a unique nonnegative solution of the differential equation (4.1).

Proof . Suppose that $X = \{x \in C(I) : x(t) \geq 0\}$ and $d(x, y) = \sup_{t\in I}\{|x(t) - y(t)|\}$ for all $x, y \in X$. Clearly, (X, d) is a complete metric space. We define $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^1 G(t,\tau) f(\tau, x(\tau))d\tau. \tag{4.2}$$

If $x \in C(I)$ is a fixed point of T, then $x \in C^1(I)$ is a zero of (4.1). By our hypothesis that $x \geq y$, we have

$$Ty(t) = \int_0^1 G(t,\tau)f(\tau, y(\tau)) \leq \int_0^1 G(t,\tau)f(\tau, x(\tau)) = Tx(t).$$

More so, we have that

$$\sup_{t\in I}\{|Tx(t) - x(t)|\} \leq \sup_{t\in I}\{|x(t) - y(t)|\}$$

that is

$$\frac{1}{2}d(Tx, x) \leq d(Tx, x) \leq d(x, y),$$

which implies that

$$\begin{aligned} d(Tx, Ty) = \sup_{t\in I}\{|Tx(t) - Ty(t)|\} &= \sup_{t\in I}\int_0^1 G(t,\tau)[f(\tau, x(\tau)) - f(\tau, y(\tau))]d\tau \\ &= \sup_{t\in I}\int_0^1 G(t,\tau)\delta\sqrt{(x(\tau) - y(\tau))^2 + A(x(\tau) - y(\tau))}d\tau \qquad (4.3) \\ &= \delta\sqrt{d(x, y)^2 + A(d(x, y))}\sup_{t\in I}\int_0^1 G(t,\tau)d\tau. \end{aligned}$$

Clearly, the $\sup_{t\in I}\int_0^1 G(t,\tau)d\tau = \frac{1}{8}$. And so (4.3) becomes

$$\begin{aligned} d(Tx, Ty) = \delta\frac{1}{8}\sqrt{d(x, y)^2 + Ad(x, y)} &\leq \sqrt{d(x, y)^2 + Ad(x, y)} \\ \Rightarrow d(Tx, Ty)^2 \leq d(x, y)^2 + Ad(x, y) \ &\Rightarrow \ d(Tx, Ty)^2 \leq d(x, y)^2 + (L - 1)d(x, y), \end{aligned}$$

where $A = L - 1$. Suppose that $\psi(t) = t^2, \phi(t) = t$. Therefore, the last inequalities, becomes

$$\begin{aligned} \psi(d(Tx, Ty)) &\leq \psi(d(x, y)) - \phi(d(x, y)) + L\phi(d(x, y)) \\ &\leq \psi(M(x, y)) - \phi(N(x, y)) + L\phi(N_1(x, y)). \end{aligned}$$

Since all of the conditions of Theorem 2.2 are satisfied, as a result, the mapping T has a unique fixed point which is a solution of (4.2) and consequently a solution of (4.1). □

5. Conclusion

In this work, we have extend and improve various fixed point results in metric spaces. More so, the result obtain in this paper generalizes and complements many well-known results in Banach spaces.

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