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SUZUKI-TYPE FIXED POINT RESULTS IN G_b-METRIC SPACES.

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Abstract:	In this paper, we prove some fixed point theorems for a new type of generalized contractive mappings involving C-class function, α_s^δ-admissible type mapping and Suzuki type mappings in the frame work of complete G_b-metric spaces. The results obtained in this work generalizes and improves some fixed point results in the literature.

SUZUKI-TYPE FIXED POINT RESULTS IN G_b-METRIC SPACES.

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Abstract. In this paper, we prove some fixed point theorems for a new type of generalized contractive mappings involving C-class function, α_s^δ-admissible type mapping and Suzuki type mappings in the frame work of complete G_b-metric spaces. The results obtained in this work generalizes and improves some fixed point results in the literature.

1. Introduction and Premilinaries

Banach contraction principle [4] can be seen as the pivot of the theory of fixed point and its applications. The theory of fixed point plays an important role in nonlinear functional analysis and it is very useful for showing the existence and uniqueness theorems for nonlinear differential and integral equations. The importance of the Banach contraction principle cannot be over emphasized in the study of fixed point theory and its applications. The Banach contraction principle have been extended and generalized by researchers in this area by considering classes of nonlinear mappings and spaces which are more general than the class of a contraction mappings and metric spaces (see [1, 8, 18, 28, 25] and the references therein). For example, Geraghty [12] introduced a generalized contraction mapping called Geraghty-contraction and established the fixed point theorem for this class of contraction mappings in the frame work of metric spaces. We recall that for a metric space (X, d), a mapping $T : X \to X$ is said to be an α-contraction if there exists $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y), \quad \forall \quad x, y \in X. \tag{1.1}$$

Definition 1.1. [12] Let (X, d) be a metric space. A mapping $T : X \to X$ is called a Geraghty-contraction mapping if

$$d(Tx, Ty) \leq \phi(d(x, y))d(x, y) \tag{1.2}$$

for all $x, y \in X$, where $\phi : \mathbb{R}^+ \to [0, 1)$ satisfies the following condition:

$$\phi(t_n) \to 1 \quad \text{as} \quad n \to \infty \Rightarrow t_n \to 0 \quad \text{as} \quad n \to \infty.$$

The following is the result of Geraghty [12].

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ be a self map that satisfies condition (1.2). Then T has a unique fixed point $x^* \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = x^*$.*

Jaggi [14] introduced a class of contraction mappings involving rational expressions and proved some fixed point results for this class of mappings. Khan et al. [17] introduced the concept of alternating distance function, which is defined as follows: A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an alternating distance function if the following conditions are satisfied:

1. (1) $\psi(0) = 0$,
2. (2) ψ is monotonically nondecreasing,
3. (3) ψ is continuous.

They established the following result.

Theorem 1.3. *Let (X, d) be a complete metric space, ψ an altering distance function, and $T : X \to X$ be a self mapping which satisfies the following condition*

$$\psi(d(Tx, Ty)) \leq \delta\psi(d(x, y)) \tag{1.3}$$

Key words and phrases. Suzuki mapping; fixed point; G_b-metric space, α_s^δ-admissble mapping, triangular α_s^δ-admissble mapping, C-class function.

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for all $x, y \in X$, where $\delta \in (0,1)$. Then T has a unique fixed point.

Remark 1.4. Clearly, if we take $\psi(x) = x$, for all $x \in X$ in (1.3), we obtain condition (1.1).

Using the concept of alternating distance function Rhoades [23], Dutta et al. [11] and Doric [10] established some fixed points results for weak contraction and generalized contraction mappings in the frame work of metric spaces. We recall that for a metric space (X, d), a mapping $T : X \to X$ is said to be weakly contractive if for all $x, y \in X$

$$d(Tx, Ty) \leq d(x, y) - \psi(d(x, y)),$$

$\psi : [0, \infty) \to [0, \infty)$ is continuous and nondecreasing such that $\psi(t) = 0$ if and only if $t = 0$.

Theorem 1.5. [23] *Let (X, d) be a complete metric space and T a weakly contractive map. Then T has a unique fixed point.*

Theorem 1.6. [11] *Let (X, d) be a complete metric space. Suppose the mappings $T : X \to X$ satisfying*

$$\psi(d(Tx, Ty)) \leq \psi(d(x, y)) - \phi(d(x, y)) \tag{1.4}$$

for all $x, y \in X$, where ψ, ϕ are alternating distance functions. Then T has a fixed point.

Theorem 1.7. [10] *Let X be a complete metric space and $T : X \to X$ be a mapping satisfying the inequality*

$$\psi(d(Tx, Ty)) \leq \psi(M(x, y)) - \phi(M(x, y)), \tag{1.5}$$

where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$, ψ an alternating distance function and $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continous function with $\phi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

In 2008, Suzuki [31] introduced the concept of mappings satisfying condition (C) which is also known as Suzuki-type generalized nonexpansive mapping and he proved some fixed point theorems for such class of mappings.

Definition 1.8. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (C) if for all $x, y \in X$,

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \Rightarrow d(Tx, Ty) \leq d(x, y).$$

Theorem 1.9. *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping satisfying condition (C) for all $x, y \in X$. Then T has a unique fixed point.*

Samet et al. [26] introduced the notion of α-admissible mapping and obtain some fixed point results for this class of mappings.

Definition 1.10. [26] Let $\alpha : X \times X \to [0, \infty)$ be a function. We say that a self mapping $T : X \to X$ is α-admissible if for all $x, y \in X$,

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1.$$

Definition 1.11. [16] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be mappings. We say that T is a triangular α-admissible if

(1) T is α-admissible and
(2) $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1$ for all $x, y, z \in X$.

Theorem 1.12. [26] *Let (X, d) be a complete metric space and $T : X \to X$ be an α-admissible mapping. Suppose that the following conditions hold:*

(1) *for all $x, y \in X$, we have $\alpha(x, y)d(Tx, Ty) \leq \psi(d(x, y))$, where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$;*
(2) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(3) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$.*

Then T has a fixed point.

One of the interesting generalization of metric spaces is the concept of b-metric spaces introduced by Czerwik in [9]. He established the Banach contraction principle in this frame work with the fact that b need not to be continuous. Thereafter, several results has been extended from metric spaces to b-metric spaces, more so, a lot of results on the fixed point theory of various classes of mappings in the frame work of b-metric spaces has been established by different researchers in this area (see[7, 9, 35] and the references therein). For example in [30], Sintunavarat introduced the concept of α-admissible mapping type S as a generalization of α-admissible mapping. He further established some proved fixed point theorems based on his new types of α-admissibility in the frame work of b-metric spaces

Definition 1.13. [30] Let X be a nonempty set and $s \geq 1$ be a given real number. Let $\alpha : X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. The mapping T is said to be an α-admissible mapping type S if for all $x, y \in X$

$$\alpha(x, y) \geq s \Rightarrow \alpha(Tx, Ty) \geq s.$$

Remark 1.14. Clearly, if $s = 1$, we obtain Defintion 1.10.

Mustafa and Sims [19] introduced, the concept of generalized metric space ($G -$ metric) and they established some fixed point theorem in the frame work of complete G-metric spaces.

Definition 1.15. Let X be a nonempty set and $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties

(1) $G(x, y, z) = 0$ if and only if $x = y = z$,
(2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(4) $G(x, x, y) = G(x, z, y) = G(y, z, x) = \cdots$, (symmetry in all the three variables),
(5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

The function G is called a G-metric on X and the pair (X, G) is called a G-metric space.

Motivated by the concept of b-metric spaces [9], Aghajani et al. in [2], introduced the notion of generalized b-metric space ($G_b -$ metric spaces), presented some properties of G_b-metric spaces and prove some coupled coincidence fixed point theorems for (ψ, φ)-weakly contractive mappings in the frame work of partially ordered G_b-metric spaces.

Definition 1.16. [2] Let X be a nonempty set and $s \geq 1$ be a given real number.Suppose that $G_b : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following properties

(1) $G_b(x, y, z) = 0$ if and only if $x = y = z$,
(2) $0 < G_b(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(3) $G_b(x, x, y) \leq G_b(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,
(4) $G_b(x, x, y) = G_b(p\{x, z, y\})$, where p is a permutation of x, y, z (symmetry),
(5) $G_b(x, y, z) \leq sG_b(x, a, a) + sG_b(a, y, z)$ for all $x, y, z, a \in X$.

The function G_b is called a generalized b-metric and the pair (X, G_b) is called a generalized b-metric space ($G_b -$ metric space).

Example 1.17. Let $X = \mathbb{R}$ and $d(x, y) = |x - y|^2$. It is well known that (X, d) is a b-metric space with $s = 2$. Let $G_b(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, it is easy to see that (X, G_b) is not G_b-metric space. However, if we define $G_b(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ is a G_b-metric space.

Definition 1.18. [2] A G_b-metric space is said to be symmetric if $G_b(x, y, y) = G_b(y, x, x)$ for all $x, y \in X$.

Proposition 1.19. [2] *Let X be a G_b-metric space. Then for each $x, y, z, a \in X$, it follows that*

(1) $G_b(x, y, z) = 0$ *then* $x = y = z$,
(2) $G_b(x, y, z) \leq sG_b(x, x, y) + sG_b(x, x, z)$,
(3) $G_b(x, y, y) \leq 2sG_b(y, x, x)$,
(4) $G_b(x, y, z) \leq sG_b(x, a, z) + sG_b(a, y, z)$.

Definition 1.20. [2] Let X be a G_b-metric space. A sequence $\{x_n\}$ in X is said to be;

(1) G_b-Cauchy if for each $\epsilon > 0$ there exists a positive integer n_0 such that for all $m, n, l \geq n_0, G_b(x_n, x_m, x_l) < \epsilon$;

(2) G_b-convergent to a point $x \in X$, if for $\epsilon > 0$ there exists a positive integer n_0 such that for all $m, n \geq n_0, G_b(x_n, x_m, x) < \epsilon$. That is $\lim_{n,m\to\infty} G_b(x_n, x_m, x) = 0$. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Definition 1.21. [2] A G_b-metric space is called G_b-complete, if every G_b-Cauchy sequence is G_b-convergent in X.

Proposition 1.22. [2] *Let (X, G_b) be a G_b-metric space. The following statement are equivalent*

(1) x_n *is G_b-convergent to* x;
(2) $G_b(x_n, x_n, x) \to 0$ *as* $n \to \infty$;
(3) $G_b(x_n, x, x) \to 0$ *as* $n \to \infty$;
(4) $G_b(x_n, x_m, x) \to 0$ *as* $m, n \to \infty$.

Proposition 1.23. [2] *Let (X, G_b) be a G_b-metric space. The following statement are equivalent:*

(1) $\{x_n\}$ *is G_b-Cauchy sequence.*
(2) $G_b(x_m, x_n, x_n) \to 0$ *as* $n, m \to \infty$.

Definition 1.24. Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$ be mappings. Then T is called α-admissible if for all $x, y, z \in X$ with $\alpha(x, y, z) \geq 1$ implies $\alpha(Tx, Ty, Tz) \geq 1$.

Definition 1.25. Let X be a nonempty set, $T : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$ be mappings. Then T is called triangular α-admissible if

(1) T is α-admissible,
(2) $\alpha(x, a, a) \geq 1$ and $\alpha(a, y, z) \geq 1$ implies $\alpha(x, y, z) \geq 1$,

for all $x, y, z, a \in X$.

Definition 1.26. Let X be a nonempty set with $s \geq 1$ a given real number. $\alpha : X \times X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. We say that T is α-admissible type S if for all $x, y, z \in X$ with $\alpha(x, y, z) \geq s$ implies $\alpha(Tx, Ty, Tz) \geq s$.

Definition 1.27. Let X be a nonempty set with $s \geq 1$ a given real number. $T : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$ be mappings. We say that T is called triangular α-admissible type S if

(1) T is α-admissible type S,
(2) $\alpha(x, a, a) \geq s$ and $\alpha(a, y, z) \geq s$ implies $\alpha(x, y, z) \geq s$,

for all $x, y, z, a \in X$.

In 2014, Ansari [3] introduced the notion of C-class function, he proved some fixed point results using the concept of C-class function and also established that the C-class function is a generalization of a whole lot of contractive conditions.

Definition 1.28. [3] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continous and the following axioms holds:

(1) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;
(2) $F(s, t) = s$ implies either $s = 0$ or $t = 0$.

We denote $\mathcal{C}$ the family of C-class functions. For details about C-class function see [3].

Example 1.29. The following functions $F : [0, \infty)^2 \to \mathbb{R}$ defined for all $s, t \in [0, \infty)$ by

(1) $F(s, t) = s - t$, $F(s, t) = s$ implies $t = 0$;
(2) $F(s, t) = ms$, $0 < m < 1$, $F(s, t) = s$ implies $s = 0$;
(3) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$ is a continuous function, $F(s, t) = s$ implies $s = 0$.

Motivated by the research works described above, our purpose in this paper is to introduce the notion of α_s^δ-admissible type mapping, triangular α_s^δ-admissible type mapping and using the concept of C-class function, we prove some fixed point results for α_s^δ-Suzuki type rational contraction mappings in the frame work of complete G_b-metric spaces.

2. MAIN RESULT

In this section, we introduce the notion of α_s^δ-Suzuki contraction type mappings and established the existence and uniqueness results of the fixed point for this class of mappings.

We start by establishing some results that will be used in the proof of our main result.

Definition 2.1. Let X be a nonempty set with $s, \delta \geq 1$ a given real number. $\alpha : X \times X \times X \to [0, \infty)$ and $T : X \to X$ be mappings. We say that T is α_s^δ-admissible type mapping if for all $x, y, z \in X$ with $\alpha(x, y, z) \geq s^\delta$ implies $\alpha(Tx, Ty, Tz) \geq s^\delta$.

Definition 2.2. Let X be a nonempty set with $s \geq 1$ and $\delta \geq 1$ a given real number. $T : X \to X$ and $\alpha : X \times X \times X \to [0, \infty)$ be mappings. We say that T is called triangular α_s^δ-admissible type mapping if

(1) T is α_s^δ-admissible type mapping,
(2) $\alpha(x, a, a) \geq s^\delta$ and $\alpha(a, y, z) \geq s^\delta$ implies $\alpha(x, y, z) \geq s^\delta$,

for all $x, y, z, a \in X$.

Remark 2.3. If $s = 1$, we recovery Definition 1.24 and 1.25 in both cases. More so, if $\delta = 1$, we recovery Definition 1.26 and 1.27 in both cases.

Lemma 2.4. *Let X be a nonempty set and T be a triangular α_s^δ-admissible mapping. Assume that there exists $x_0 \in X$, such that $\alpha(x_0, Tx_0, Tx_0) \geq s^\delta$. Suppose the sequence $\{x_n\}$ is defined by $x_{n+1} = Tx_n$, then $\alpha(x_m, x_n, x_n) \geq s^\delta$ for all $m, n \in \mathbb{N}$.*

Proof. Since T is triangular α_δ-admissible mapping and there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq s^\delta$, we then have that $\alpha(x_1, x_2, x_2) = \alpha(Tx_0, Tx_1, Tx_1) \geq s^\delta$, continuing the process we have that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s^\delta$ for all $n \in \mathbb{N} \cup \{0\}$. Now, suppose that $m < n$ for all $m, n \in \mathbb{N}$, since $\alpha(x_m, x_{m+1}, x_{m+1}) \geq s^\delta$ and $\alpha(x_{m+1}, x_{m+2}, x_{m+2}) \geq s^\delta$, we have that $\alpha(x_m, x_{m+2}, x_{m+2}) \geq s^\delta$. More so, since $\alpha(x_m, x_{m+2}, x_{m+2}) \geq s^\delta$ and $\alpha(x_{m+2}, x_{m+3}, x_{m+3}) \geq s^\delta$, we have that $\alpha(x_m, x_{m+3}, x_{m+3}) \geq s^\delta$. Continuing this process, we have that

$$\alpha(x_m, x_n, x_n) \geq s^\delta.$$

□

Lemma 2.5. *Let (X, G_b) be a G_b-metric space with coefficient $s \geq 1$ and suppose that $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = 0$. If $\{x_n\}$ is not a G_b-Cauchy sequence, then there exists $\epsilon > 0$ and two sequences say $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of positive integer such that for the following cases $G_b(x_{m_k}, x_{n_k}, x_{n_k}), G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}), G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}})$ and $G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k})$, we have that*

(1) $\epsilon \leq \liminf_{k\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq \limsup_{k\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_k}), \leq s\epsilon$,
(2) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k}) \leq \limsup_{k\to\infty} G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k}) \leq s^2\epsilon$,
(3) $\frac{\epsilon}{s} \leq \liminf_{k\to\infty} G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) \leq s^2\epsilon$,
(4) $\frac{\epsilon}{s^2} \leq \liminf_{k\to\infty} G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}}) \leq s^3\epsilon$.

Proof. Suppose $\{x_n\}$ is not a G_b-Cauchy sequence, then there exists $\epsilon > 0$ and two sequences say $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of positive integers such that $n_k > m_k \geq k$,

$$G_b(x_{m_k}, x_{n_{k-1}}, x_{n_{k-1}}) < \epsilon \qquad \text{and} \qquad G_b(x_{m_k}, x_{n_k}, x_{n_k}) > \epsilon. \tag{2.1}$$

Using the fact that $\lim_{n\to\infty} G_b(x_n, x_{n+1}, x_{n+1}) = 0$ and (2.1), we have that

$$\begin{aligned}\epsilon \leq G_b(x_{m_k}, x_{n_k}, x_{n_k}) &\leq sG_b(x_{m_k}, x_{n_{k-1}}, x_{n_{k-1}}) + sG_b(x_{n_{k-1}}, x_{n_k}, x_{n_k})\\ &\leq s\epsilon + sG_b(x_{n_{k-1}}, x_{n_k}, x_{n_k}),\end{aligned}$$

clearly, we have that

$$\epsilon \leq \liminf_{n\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq \limsup_{n\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq s\epsilon.$$

More so, we have that

$$G_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq sG_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}) + s^2G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}}) + s^2G_b(x_{n_{k+1}}, x_{n_k}, x_{n_k})$$

and

$$G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}}) \leq sG_b(x_{m_{k+1}}, x_{m_k}, x_{m_k}) + s^2G_b(x_{m_k}, x_{n_k}, x_{n_k}) + s^2G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}).$$

We also have that

$$\frac{\epsilon}{s^2} \leq \liminf_{n\to\infty} G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}}) \leq \limsup_{n\to\infty} G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}}) \leq s^3\epsilon.$$

Furthermore, we have that

$$G_b(x_{m_k}, x_{n_k}, x_{n_k}) \leq sG_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) + sG_b(x_{n_{k+1}}, x_{n_k}, x_{n_k})$$

and

$$G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) \leq sG_b(x_{m_k}, x_{n_k}, x_{n_k}) + sG_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}).$$

We also have that

$$\frac{\epsilon}{s} \leq \liminf_{n\to\infty} G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) \leq \limsup_{n\to\infty} G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) \leq s^2\epsilon.$$

Using similar approach, we obtain that

$$\frac{\epsilon}{s^2} \leq \liminf_{n\to\infty} G_b(x_{m_{k+1}}, x_{n_k}, x_{n_k}) \leq \limsup_{n\to\infty} G_b(x_{m_k}, x_{n_k}, x_{n_{k+1}}) \leq s^2\epsilon.$$

□

We now establish our main result.

Definition 2.6. Let (X, G_b) be a G_b-metric space with $s, \delta \geq 1$ a given real number, $\alpha : X \times X \times X \to [0, \infty)$ be a function and T be a self map on X. The mapping T is said to be α_s^δ-Suzuki type rational contraction mapping, if

$$\alpha(x, y, z) \geq s^\delta \text{ and } \frac{1}{3s^2} G_b(x, Tx, Tx) \leq G_b(x, y, z)$$
$$\Rightarrow \psi(s^3 G_b(Tx, Ty, Tz)) \leq F(\psi(M(x, y, z)), \phi(M(x, y, z))) + L\psi(N(x, y)) \tag{2.2}$$

for all $x, y, z \in X$, where $L \geq 0$, ψ, ϕ are alternating distance functions, $F \in \mathcal{C}$, $M(x, y, z) = \max\{G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Tz), \frac{G_b(x,Tx,Tx)G_b(y,Ty,Tz)}{s+G_b(x,y,z)}, \frac{G_b(y,z,Tx)[1+G_b(x,Tx,Tx)]}{s+G_b(x,y,z)}\}$ and $N(x, y, z) = \min\{G_b(x, Ty, Ty), G_b(x, Tx, Tx), G_b(y, Tx, Tx)\}$.

Theorem 2.7. *Let (X, G_b) be a G_b-complete metric space and $T : X \to X$ be an α_s^δ-Suzuki type rational contraction mapping. Suppose the following conditions hold:*

1. *T is a triangular α_s^δ-admissible type mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq s^\delta$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s^\delta$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq s^\delta$.*

Then T has a fixed point.

Proof. Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0, Tx_0) \geq s^\delta$. We define the sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N} \cup \{0\}$. If we suppose that $x_{n+1} = x_n$, for some $n \in \mathbb{N} \cup \{0\}$, we obtain the desired result. Now, suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since T is triangular α_δ-admissible type S mapping and $\alpha(x_0, x_1, x_1) = \alpha(x_0, Tx_1, Tx_1) \geq s^\delta$, we have that $\alpha(x_1, x_2, x_2) = \alpha(Tx_0, Tx_1, Tx_1) \geq s^\delta$, continuing this process, we obtain that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s^\delta$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s^\delta$ and $\frac{1}{3s^2} G_b(x_n, Tx_n, Tx_n) = \frac{1}{3s^2} G_b(x_n, x_{n+1}, x_{n+1}) < G_b(x_n, x_{n+1}, x_{n+1})$, we have

$$\psi(G_b(x_{n+1}, x_{n+2}, x_{n+2})) \leq \psi(s^3 G_b(Tx_n, Tx_{n+1}, Tx_{n+1}))$$
$$\leq F(\psi(M(x_n, x_{n+1}, x_{n+1})), \phi(M(x_n, x_{n+1}, x_{n+1}))) + LN(x_n, x_{n+1}, x_{n+1}), \tag{2.3}$$

where,

$$M(x_n, x_{n+1}, x_{n+1}) = \max\Big\{G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n+1}, x_{n+2}, x_{n+2}),$$
$$\frac{G_b(x_n, x_{n+1}, x_{n+1})G_b(x_{n+1}, x_{n+2}, x_{n+2})}{s + G_b(x_n, x_{n+1}, x_{n+1})}, \frac{G_b(x_{n+1}, x_{n+1}, x_{n+1})G_b(x_n, x_{n+1}, x_{n+1})}{s + G_b(x_n, x_{n+1}, x_{n+1})}\Big\}$$
$$= \max\Big\{G_b(x_n, x_{n+1}, x_{n+1}), G_b(x_{n+1}, x_{n+2}, x_{n+2}), \frac{G_b(x_n, x_{n+1}, x_{n+1})G_b(x_{n+1}, x_{n+2}, x_{n+2})}{s + G_b(x_n, x_{n+1}, x_{n+1})}\Big\}.$$

Since, $\frac{G_b(x_n,x_{n+1},x_{n+1})}{s+G_b(x_n,x_{n+1},x_{n+1})} < 1$, clearly, $\frac{G_b(x_n,x_{n+1},x_{n+1})G_b(x_{n+1},x_{n+2},x_{n+2})}{s+G_b(x_n,x_{n+1},x_{n+1})} < G_b(x_{n+1},x_{n+2},x_{n+2})$. So that

$$M(x_n,x_{n+1},x_{n+1}) = \max\left\{G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n+1},x_{n+2},x_{n+2})\right\}.$$

Also, we have that

$$N(x_n,x_{n+1},x_{n+1}) = \min\{G_b(x_n,x_{n+2},x_{n+2}), G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n+1},x_{n+1},x_{n+1})\} = 0.$$

If we suppose that

$$M(x_n,x_{n+1},x_{n+1}) = \max\left\{G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n+1},x_{n+2},x_{n+2})\right\} = G_b(x_{n+1},x_{n+2},x_{n+2}),$$

then (2.3) becomes

$$\begin{aligned}\psi(G_b(x_{n+1}x_{n+2},x_{n+2})) &\leq \psi(s^3G_b(Tx_nTx_{n+1},Tx_{n+1}))\\ &\leq F(\psi(G_b(x_{n+1},x_{n+2},x_{n+2})),\phi(G_b(x_{n+1},x_{n+2},x_{n+2})))\\ &\leq \psi(G_b(x_{n+1},x_{n+2},x_{n+2})),\end{aligned} \tag{2.4}$$

which implies that

$$\psi(G_b(x_{n+1}x_{n+2},x_{n+2})) = \psi(G_b(x_{n+1}x_{n+2},x_{n+2}))$$

so that $F(\psi(G_b(x_{n+1},x_{n+2},x_{n+2})),\phi(G_b(x_{n+1},x_{n+2},x_{n+2}))) = \psi(G_b(x_{n+1}x_{n+2},x_{n+2}))$ and by definition of F, we must have that $\psi(G_b(x_{n+1},x_{n+2},x_{n+2})) = 0$ or $\phi(G(x_{n+1},x_{n+2},x_{n+2})) = 0$. Using the propeties of ψ and phi, we have that $G_b(x_{n+1},x_{n+2},x_{n+2}) = 0$ which implies that $x_{n+1} = x_{n+2}$ which is a contradiction. Thus we must have that

$$M(x_n,x_{n+1},x_{n+1}) = \max\left\{G_b(x_n,x_{n+1},x_{n+1}), G_b(x_{n+1},x_{n+2},x_{n+2})\right\} = G_b(x_n,x_{n+1},x_{n+1}),$$

which implies that

$$G_b(x_{n+1},x_{n+2},x_{n+2}) \leq G_b(x_n,x_{n+1},x_{n+1}). \tag{2.5}$$

Thus, we have that

$$\begin{aligned}\psi(G_b(x_{n+1}x_{n+2},x_{n+2})) &\leq \psi(s^3G_b(Tx_nTx_{n+1},Tx_{n+1}))\\ &\leq F(\psi(G_b(x_n,x_{n+1},x_{n+1})),\phi(G_b(x_n,x_{n+1},x_{n+1})))\\ &\leq \psi(G_b(x_n,x_{n+1},x_{n+1})),\end{aligned} \tag{2.6}$$

which implies that $\psi(G_b(x_{n+1}x_{n+2},x_{n+2})) \leq \psi(G_b(x_n,x_{n+1},x_{n+1}))$, using the property of ψ, we have that

$$G_b(x_{n+1}x_{n+2},x_{n+2}) \leq G_b(x_n,x_{n+1},x_{n+1}).$$

Using similar approach, we also have that

$$G_b(x_n,x_{n+1},x_{n+1}) \leq G_b(x_{n-1},x_n,x_n).$$

Therefore, $\{G_b(x_n,x_{n+1},x_{n+1})\}$ is a nonincreasing sequence and bounded below. Thus there exists $c \geq 0$ such that

$$\lim_{n\to\infty} G_b(x_n,x_{n+1},x_{n+1}) = c. \tag{2.7}$$

Now, suppose that $c > 0$, taking the limit as $n \to \infty$ of (2.6), we have that $\psi(c) = \psi(c)$ so that $F(\psi(c)),\phi(c)) = \psi(c)$ and by definition of F, we must have that $\psi(c) = 0$ or $\phi(c) = 0$. Using the propeties of ψ and psi, we have that $c = 0$. Thus, we have that

$$\lim_{n\to\infty} G_b(x_n,x_{n+1},x_{n+1}) = 0. \tag{2.8}$$

Now, we shall show that $\{x_n\}$ is G_b-Cauchy sequence. Suppose that $\{x_n\}$ is not a G_b-Cauchy sequence, then by Lemma 2.5, there exists an $\epsilon > 0$ and sequences of positive integers $\{n_k\}$ and $\{m_k\}$ with $n_k > m_k \geq k$ such that $G_b(m_k,n_k,n_k) \geq \epsilon$. For each $k > 0$, corresponding to m_k, we can choose n_k to be the smallest positive integer such that $G_b(m_k,n_k,n_k) \geq \epsilon, G_b(m_k,n_{k-1},n_{k-1}) < \epsilon$ and $(1)-(4)$. Using Lemma 2.4, we have that $\alpha(x_{m_k},x_{n_k},x_{n_k}) \geq s^\delta$ and we can choose $n_0 \in \mathbb{N}\cup\{0\}$ such that

$$\frac{1}{3s^2}G_b(x_{m_k},Tx_{m_k},Tx_{m_k}) < \frac{\epsilon}{3s^2} < \epsilon \leq G_b(x_{m_k},x_{n_k},x_{n_k}).$$

Hence, for all $k \geq n_0$, we have

$$\begin{aligned}\psi(G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}})) &\leq \psi(s^3 G_b(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}))\\ &\leq F(\psi(M(x_{m_k}, x_{n_k}, x_{n_k})), \phi(M(x_{m_k}, x_{n_k}, x_{n_k}))) + L\psi(N(x_{m_k}, x_{n_k}, x_{n_k})),\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}M(x_{m_k}, x_{n_k}, x_{n_k}) = \max\Big\{ & G_b(x_{m_k}, x_{n_k}, x_{n_k}), G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}),\\ & \frac{G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}})G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}})}{s + G_b(x_{m_k}, x_{n_k}, , x_{n_k})}, \frac{G_b(x_{n_k}, x_{n_k}, x_{m_{k+1}})[1 + G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}})]}{s + G_b(x_{m_k}, x_{n_k}, , x_{n_k})}\Big\}\end{aligned}$$

$$N(x_{m_k}, x_{n_k}, x_{n_k}) = \min\{G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}), G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), G_b(x_{n_k}, x_{m_{k+1}}, x_{m_{k+1}})\}.$$

Using Lemma 2.5 and (2.8), we have that

$$\begin{aligned}\epsilon \leq \limsup_{n\to\infty} M(x_{m_k}, x_{n_k}, x_{n_k}) = \max\Big\{ & G_b(x_{m_k}, x_{n_k}, , x_{n_k}), G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}),\\ & \frac{G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}})G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}})}{s + G_b(x_{m_k}, x_{n_k}, , x_{n_k})}, \frac{G_b(x_{n_k}, x_{n_k}, x_{m_{k+1}})[1 + G_b(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}})]}{s + G_b(x_{m_k}, x_{n_k}, x_{n_k})}\Big\}\\ = \{ & s\epsilon, 0, 0, 0, \frac{s\epsilon}{1+\epsilon}\} = s\epsilon\end{aligned}$$

$$\epsilon \leq \limsup_{n\to\infty} N(x_{m_k}, x_{n_k}, x_{n_k}) = \min\{G_b(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}), G_b(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), G_b(x_{n_k}, x_{m_{k+1}}, x_{m_{k+1}})\} = 0.$$

So that (2.9) becomes

$$\begin{aligned}\psi(s\epsilon) = \psi(s^3 \frac{\epsilon}{s^2}) &\leq \psi(s^3 \limsup_{n\to\infty} G_b(x_{m_{k+1}}, x_{n_{k+1}}, x_{n_{k+1}})) = \psi(s^3 \limsup_{n\to\infty} G_b(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}))\\ &= \limsup_{n\to\infty} \psi(s^3 G_b(Tx_{m_k}, Tx_{n_k}, Tx_{n_k}))\\ &\leq F(\psi(s\epsilon), \phi(s\epsilon)) \leq \psi(s\epsilon)\end{aligned}$$

we obtain $\psi(s\epsilon) \leq \phi(s\epsilon)$ which implies that so that $F(\psi(s\epsilon), \phi(s\epsilon)) = \psi(s\epsilon)$ and by definition of F, we must have that $\psi(s\epsilon) = 0$ or $\phi(s\epsilon) = 0$. Using the propeties of ψ and ϕ, we have that $s\epsilon = 0$. Since $s > 0$, we must have that $\epsilon = 0$ and this contradicts the assumption that $\epsilon > 0$. We therefore have that $\{x_n\}$ is G_b-Cauchy. Since (X, G_b) is G_b-complete, it follows that there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Suppose that T is continuous, we have that

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T \lim_{n\to\infty} x_n = Tx.$$

Thus, T has a fixed point.

More so, using the condition that $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s^\delta$ for all $n \in \mathbb{N} \cup \{0\}$, we obtain that $\alpha(x_n, x, x) \geq s^\delta$. We establish that T has a fixed point. Now suppose that

$$G_b(x_n, x, x) < \frac{1}{3s^2} G_b(x_n, x_{n+1}, x_{n+1})$$

or

$$G_b(x_{n+1}, x, x) < \frac{1}{3s^2} G_b(x_{n+1}, x_{n+2}, x_{n+2}).$$

Then using the fact that $G_b(x_{n+1}, x_{n+2}, x_{n+2}) \leq G_b(x_n, x_{n+1}, x_{n+1})$, we have

$$\begin{aligned}G_b(x_n, x_{n+1}, x_{n+1}) &\leq sG_b(x_n, x, x) + sG_b(x, x_{n+1}, x_{n+1})\\ &\leq sG_b(x_n, x, x) + 2s^2 G_b(x_{n+1}, x, x)\\ &< \frac{1}{3s} G_b(x_n, x_{n+1}, x_{n+1}) + \frac{2}{3} G_b(x_{n+1}, x_{n+2}, x_{n+2})\\ &\leq (\frac{1}{3s} + \frac{2}{3}) G_b(x_n, x_{n+1}, x_{n+1})\\ &\leq G_b(x_n, x_{n+1}, x_{n+1})\end{aligned}$$

The above inequality is a contradiction, thus, we must have that

$$G_b(x_n, x, x) \geq \frac{1}{3s^2} G_b(x_n, x_{n+1}, x_{n+1}) \quad \text{or} \quad G_b(x_{n+1}, x, x) \geq \frac{1}{3s^2} G_b(x_{n+1}, x_{n+2}, x_{n+2}).$$

Hence, we have

$$\psi(G_b(x_{n+1},Tx,Tx)) \leq \psi(s^3 G_b(Tx_n,Tx,Tx))$$
$$\leq F(\psi(M(x_n,x,x)),\phi(M(x_n,x,x))) + L\psi(N(x_n,x,x)), \tag{2.10}$$

where

$$M(x_n,x,x) = \max\left\{G_b(x_n,x,x), G_b(x_n,Tx_n,Tx_n), G_b(x,Tx,Tx), \frac{G_b(x_n,Tx_n,Tx_n)G_b(x,Tx,Tx)}{s+G_b(x_n,x,x)}, \frac{G_b(x,x,Tx_n)[1+G_b(x_n,Tx_n,Tx_n)]}{s+G_b(x_n,x,x)}\right\}$$
$$N(x_n,x,x) = \min\{G_b(x_n,Tx,Tx), G_b(x_n,Tx_n,Tx_n), G_b(x,Tx_n,Tx_n)\}.$$

Using the properties of ψ,ϕ and taking limit as $n\to\infty$, (2.10) becomes

$$\psi(G_b(x,Tx,Tx)) \leq \psi(G_b(x,Tx,Tx)),$$

which implies that $F(\psi(G_b(x,Tx,Tx),\phi(G_b(x,Tx,Tx))) = \psi(G_b(x,Tx,Tx)$ and by definition of F, we must have that $\psi(G_b(x,Tx,Tx) = 0$ or $\phi(G_b(x,Tx,Tx)) = 0$. Using the propeties of ψ and phi, we have that $G(x,Tx,Tx)=0$. which implies that

$$x = Tx.$$

Hence, T has a fixed point. □

Theorem 2.8. *Suppose that the hypothesis of Theorem 2.7 holds and in addition suppose $\alpha(x,y,y)\geq s^\delta$ for all $x,y\in F(T)$, where $F(T)$ is the set of fixed point of T. Then T has a unique fixed point.*

Proof. Let $x,y\in F(T)$, that is $Tx=x$ and $Ty=y$ such that $x\neq y$. Using our hypothesis that $\alpha(x,y,y)\geq s^\delta$ and $\frac{1}{3s^2s}G_b(x,Tx,Tx) = 0 \leq G_b(x,y,y)$, we have

$$\psi(G_b(x,y,y)) \leq \psi(s^3G_b(Tx,Ty,Ty)) \leq F(\psi(M(x,y,y)),\phi(M(x,y,y)) + L\psi(N(x,y,y)), \tag{2.11}$$

where

$$M(x,y,y) = \max\left\{G_b(x,y,y), G_b(x,Tx,Tx), G_b(y,Ty,Ty), \frac{G_b(x,Tx,Tx)G_b(y,Ty,Ty)}{s+G_b(x,y,y)}, \frac{G_b(y,y,Tx)[1+G_b(x,Tx,Tx)]}{1+G_b(x,y,y)}\right\}$$
$$= G(x,y,y)$$
$$N(x,y,y) = \min\{G_b(x,Ty,Ty), G_b(x,Tx,Tx), G_b(y,Tx,Tx)\} = 0.$$

Using the properties of ψ,ϕ, (2.11) becomes

$$\psi(G_b(x,y,y)) \leq \psi(G_b(x,y,y)),$$

which implies that $F(\psi(G_b(x,y,y)),\phi(G_b(x,y,y))) = \psi(G_b(x,y,y))$ and by definition of F, we must have that $\psi(G_b(x,y,y)) = 0$ or $\phi(G_b(x,y,y)) = 0$. Using the propeties of ψ and phi, we have that $G_b(x,y,y)=0$. which implies that

$$x = y.$$

Thus, T has a unique fixed point. □

Using Remark 2.3 $L=0$ and we defined $F(s,t)=s-t$, we obtain the following results.

Corollary 2.9. *Let (X,G_b) be a G_b-complete metric space and $T:X\to X$ be a mapping satisfying the inequalities*

$$\alpha(x,y,z)\geq 1 \text{ and } \frac{1}{3s^2}G_b(x,Tx,Tx) \leq G_b(x,y,z)$$
$$\Rightarrow \psi(s^3G_b(Tx,Ty,Tz)) \leq \psi(M(x,y,z)) - \phi(M(x,y,z))) \tag{2.12}$$

for all $x,y,z\in X$, where ψ,ϕ are alternating distance functions, and $M(x,y,z) = \max\{G_b(x,y,z), G_b(x,Tx,Tx), G_b(y,Ty,Tz), \frac{G_b(x,Tx,Tx)G_b(y,Ty,Tz)}{s+G_b(x,y,z)}, \frac{G_b(y,z,Tx)[1+G_b(x,Tx,Tx)]}{s+G_b(x,y,z)}\}$. Suppose the following conditions hold:

1. *T is a triangular α-admissible type mapping,*
2. *there exists $x_0\in X$ such that $\alpha(x_0,Tx_0,Tx_0)\geq 1$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1},x_{n+1})\geq 1$ for all $n\geq 0$ and $x_n\to x$ as $n\to\infty$, then $\alpha(x_n,x,x)\geq 1$.*

Then T has a fixed point.

Corollary 2.10. *Let (X, G_b) be a G_b-complete metric space and $T : X \to X$ be a mapping satisfying the inequalities*

$$\alpha(x,y,z) \geq s \text{ and } \frac{1}{3s^2}G_b(x,Tx,Tx) \leq G_b(x,y,z) \Rightarrow \psi(s^3G_b(Tx,Ty,Tz)) \leq \psi(M(x,y,z)) - \phi(M(x,y,z))) \tag{2.13}$$

for all $x, y, z \in X$, where ψ, ϕ are alternating distance functions, and $M(x, y, z) = \max\{G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Tz), \frac{G_b(x,Tx,Tx)G_b(y,Ty,Tz)}{s+G_b(x,y,z)}, \frac{G_b(y,z,Tx)[1+G_b(x,Tx,Tx)]}{s+G_b(x,y,z)}\}$. Suppose the following conditions hold:

1. *T is a triangular α-admissible type S mapping,*
2. *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, Tx_0) \geq s$,*
3. *T is continuous,*
4. *if for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq s$ for all $n \geq 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha(x_n, x, x) \geq s$.*

Then T has a fixed point.

If, we suppose that $\alpha(x, y, z) = 1$, we obtain the following results.

Corollary 2.11. *Let (X, G_b) be a G_b-complete metric space and $T : X \to X$ be a mapping satisfying the inequalities*

$$\frac{1}{3s^2}G_b(x,Tx,Tx) \leq G_b(x,y,z) \Rightarrow \psi(s^3G_b(Tx,Ty,Tz)) \leq \psi(M(x,y,z)) - \phi(M(x,y,z))) \tag{2.14}$$

for all $x, y, z \in X$, where ψ, ϕ are alternating distance functions, and $M(x, y, z) = \max\{G_b(x, y, z), G_b(x, Tx, Tx), G_b(y, Ty, Tz), \frac{G_b(x,Tx,Tx)G_b(y,Ty,Tz)}{s+G_b(x,y,z)}, \frac{G_b(y,z,Tx)[1+G_b(x,Tx,Tx)]}{s+G_b(x,y,z)}\}$. Then T has a fixed point.

Example 2.12. Let $X = [0, \infty)$ with $G_b(x, y, z) = [|x - y| + |y - z| + |x - z|]^2$. Clearly, (X, G_b) is a complete G_b-metric space with $s = 2$. We defined $T : X \to X$ by

$$Tx = \begin{cases} \frac{x}{16} & \text{if} \quad x,y,z \in [0,1] \\ 5x & \text{if} \quad x,y,z \in (1,\infty), \end{cases}$$

$\alpha : X \times X \times X \to [0, \infty)$ by

$$\alpha(x,y,z) = \begin{cases} 3 & \text{if} \quad x,y,z \in [0,1] \\ 0 & \text{if} \quad x,y,z \in (1,\infty) \end{cases}$$

and $\phi, \psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 2t, \phi(t) = t, \delta = 1$ and $F(s, t) = s - t$. T is α_2^1-Suzuki type mapping and T satisfy conditions in Corollary 2.10 with a unique fixed point 0.

Proof. Clearly, for any $x, y, z \in [0, 1]$, we have that $\alpha(x, y, z) > 2$ and $Tx = \frac{x}{16}, Ty = \frac{y}{16}, Tz = \frac{z}{16}$, we also have that $\alpha(Tx, Ty, Tz) = \alpha(\frac{x}{16}, \frac{y}{16}, \frac{y}{16}) > 2$. Suppose $\alpha(x, a, a) > 2$ and $\alpha(a, y, z) > 2$ for all $x, y, z, a \in X$, it implies that $x, y, z, a \in [0, 1]$, it follows that $\alpha(x, y, z) > 2$. Thus, we have that T is triangular admissible type S mapping. More so, for any $x_0 \in [0, 1]$, we have that $\alpha(x_0, Tx_0, Tx_0) \geq 2$. Let $\{x_n\}$ be sequence in X with $\alpha(x_n, x_{n+1}, x_{n+1}) \geq 2$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, using the definition of α, we must have that $\{x_n\} \subset [0, 1]$ and thus $x \in [0, 1]$. Hence $\alpha(x_n, x, x) \geq 2$. Since $\alpha(x, y, z) > 2$ if $x, y, z \in [0, 1]$, we need to show that T is α_s^δ-Suzuki type rational mapping for any $x, y, z \in [0, 1]$ with $\frac{1}{3s^2}G(x, Tx, Tx) \leq G(x, y, z)$. Let $x, y, z \in [0, 1]$ and without loss of generality, we suppose that $x \leq y, x \leq z$ and $y \leq z$. It is easy to see that for all $x, y, z \in [0,1]$, $\frac{1}{12}G(x,Tx,Tx) \leq G(x,y,z)$ Now, observe that

$$\begin{aligned}
\psi(s^3 G(Tx,Ty,Tz)) &= \psi\left(\frac{8}{256}[|x-y|+|y-z|+|x-z|]^2\right) \\
&\leq \frac{16}{256}[|x-y|+|y-z|+|x-z|]^2 \\
&\leq [|x-y|+|y-z|+|x-z|]^2 \\
&= 2[|x-y|+|y-z|+|x-z|]^2 - [|x-y|+|y-z|+|x-z|]^2 \\
&= \psi(G(x,y,z)) - \phi(G(x,y,z)) \\
&= F(\psi(G(x,y,z)), \phi(G(x,y,z))) \\
&\leq F(\psi(M(x,y,z)), \phi(M(x,y,z)))
\end{aligned}$$

Thus T satisfy all the hypothesis of Corolary 2.10 and $x = 0$ is the unique fixed point of T. □

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