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APPROXIMATE SOLUTION OF NONLINEAR HYPERBOLIC EQUATIONS WITH HOMOGENEOUS JUMP CONDITIONS

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Abstract. We present the error analysis of a class of second order nonlinear hyperbolic interface problems where the spatial and time discretizations are based on a finite element method and linearized backward difference scheme respectively. Both semi discrete and fully discrete schemes are analyzed with the assumption that the interface is arbitrary but smooth. Almost optimal convergence rate in the H^1-norm is obtained. Numerical examples are given to support the theoretical result.

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Keywords. Almost optimal, nonlinear hyperbolic equation, linearized backward difference.

1. INTRODUCTION

In this work, we study finite element solution of the nonlinear hyperbolic equation of the form

$$u_{tt} - \nabla \cdot (a(x,u)\nabla u) + b(x,u)u = f(x,t) \quad \text{in} \quad \Omega \times (0,T] \tag{1.1}$$

with initial and boundary conditions

$$\begin{cases} u(x,0) &= u_0(x) & \text{in } \Omega \\ u_t(x,0) &= u_1(x) & \text{in } \Omega \\ u(x,t) &= 0 & \text{on } \partial\Omega \times [0,T] \end{cases} \tag{1.2}$$

and interface conditions

$$\begin{cases} [u]_\Gamma &= 0 \\ \left[a(x,u)\frac{\partial u}{\partial n}\right]_\Gamma &= 0 \end{cases} \tag{1.3}$$

where $0 < T < \infty$ and Ω is a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$. $\Omega_1 \subset \Omega$ is an open domain with smooth boundary $\Gamma = \partial\Omega_1$, $\Omega_2 = \Omega \setminus \bar{\Omega}_1$ is another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$, see Figure 1.1. The symbol $[u]$ is the jump of a quantity u across the interface Γ and n is

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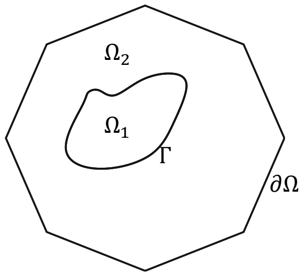

Fig. 1.1. A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ.

the unit outward normal to the boundary $\partial\Omega_1$. The interface conditions are defined as the difference of the limiting values from each side of the interface ie

$$[u]_{m\in\Gamma} := \lim_{x\to m^+} u_1(x,t) - \lim_{x\to m^-} u_2(x,t)$$

and

$$\left[a(x,u)\tfrac{\partial u}{\partial n}\right]_{m\in\Gamma} := \left[\lim_{x\to m^+} a_1\nabla u_1(x,t) - \lim_{x\to m^-} a_2\nabla u_2(x,t)\right]\cdot n$$

where $u_i(x,t)$, $a_i(x,u)$, $b_i(x,u)$ and $f_i(x,t)$ are the restrictions of $u(x,t)$, $a(x,u)$, $b(x,u)$ and $f(x,t)$ to Ω_i, $i = 1,2$. The input functions $a_i(x,u)$, $b_i(x,u)$ and $f_i(x,t)$ are assumed continuous on Ω_i, $i = 1,2$ for $t \in [0,T]$. We impose the following

Assumption 1.1. (i) Ω is a bounded convex polygonal domain in $\mathbb{R}^2$, the interface $\Gamma \subset \Omega$ and the boundary $\partial\Omega$ are piecewise smooth, Lipschitz continuous and 1-dimensional.

(ii) $f(x,t) \in H^1(0,T;L^2(\Omega))$. Functions a and b satisfy

$$a_i(x,\xi) \geq \mu_1, \qquad b_i(x,\xi) \geq \mu_1, \qquad \|a_i(x,0)\|_{L^\infty(\Omega)} + \|b_i(x,0)\|_{L^\infty(\Omega)} \leq \mu_2,$$

$$|a_i(x,\xi) - a_i(x,\psi)| + |b_i(x,\xi) - b_i(x,\psi)| \leq \mu_3\|\xi - \psi\|_{L^2(\Omega_i)},$$

for $\xi, \psi \in \mathbb{R}$, $x \in \Omega_i$, $t \in \mathbb{R}^+$ with positive constants μ_1, μ_2 and μ_3 independent of t, x, ξ, ψ.

Hyperbolic partial differential equations arise in many physical problems such as vibrating string, vibrating membrane, shallow water waves, etc [13, 23, 24] and become interface problems when medium or materials with different properties are involved [10, 16, 15]. The solutions of interface problems have low regularity globally but may have higher regularities in each individual material region because of the discontinuities across the interface [21]. Thus, obtaining exact solutions or approximate solutions with higher order accuracy may be difficult.

Finite element solutions of non-interface hyperbolic problems have been extensively discussed in [7, 8, 9, 17, 19, 22, 25]. The convergence of finite element solutions of linear hyperbolic interface problems has been considered in [3, 4, 14, 15, 16]. In [16], the authors assumed that the interface can be fitted exactly using interface elements with curved edges and established convergence rates of optimal order for both semi and full discretizations. Time discretization was based on symmetric difference approximation around the nodal points. In [15], approximation properties of interpolation and projection operators were used to establish convergence rates of optimal order for finite element solution of an homogenous hyperbolic interface problem. Their time discretization was also based on symmetric difference approximation around the nodal points. Linear finite element with time discretization based on implicit scheme was presented for wave equation with discontinuous coefficient in [14]. In [3], we investigated the error contributed by semi discretization to the finite element solution of linear hyperbolic interface problems. With low regularity assumptions on the solution across the interface and with the assumption that the interface could not be fitted exactly, almost optimal convergence rates in $L^2(\Omega)$ and $H^1(\Omega)$ norms were established. In [4], we proposed finite element solution of a linear hyperbolic interface problem where the interface was approximated by straight lines. Quasi-uniform triangular elements were used for the spatial discretization and time discretization was based on a three-step implicit scheme. The proposed scheme was proved to be stable and preserves the discrete maximum principle under certain conditions on the input data. Almost optimal convergence rates in $L^2(\Omega)$ and $H^1(\Omega)$ norms were obtained. Inspite of the wide applicability of nonlinear hyperbolic equations, the discussion on finite element solutions of nonlinear hyperbolic interface problems of the form (1.1)–(1.3) is scarce in literature.

The objective of this paper is to establish convergence in the H^1-norm for the approximate solution of nonlinear hyperbolic interface problems of the form (1.1)–(1.3) on finite elements. Both semi discrete and fully discrete schemes are analyzed. Full discretization of (1.1)–(1.3) results to a system of nonlinear equations due to the presence of $a(x, u)$ and $b(x, u)$. We propose a linearized scheme in order to avoid this difficulty, and for practicability of the scheme, we do not assume that the interface could be perfectly fitted. The interface is first approximated by piecewise continuous straight lines and the mesh is fitted to this approximation. In this study, we use the standard notations and properties of Sobolev spaces as contained in [1]. Other tools used in this paper are the linear theories of interface and non-interface problems, as well as approximation properties of the elliptic projection operator.

Let v_i be the restriction of v to Ω_i, $i = 1, 2$, we shall need the following spaces for the convergence analysis

$$X = \left\{ v : v \in H^1(\Omega), v_i \in H^2(\Omega_i) \right\} , \quad Y = \left\{ v : v \in L^2(\Omega), v_i \in H^1(\Omega_i) \right\}$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v_1\|_{H^2(\Omega_1)} + \|v_2\|_{H^2(\Omega_2)} \quad \forall\, v \in X,$$

$$\|v\|_Y = \|v\|_{L^2(\Omega)} + \|v_1\|_{H^1(\Omega_1)} + \|v_2\|_{H^1(\Omega_2)} \quad \forall\, v \in Y.$$

The weak form of (1.1)–(1.3) is to find $u(t) \in H_0^1(\Omega)$, $t \in (0,T]$ such that

$$(u_{tt}, v) + A(u : u, v) = (f, v) \quad \forall\, v(t) \in H_0^1(\Omega),\ t \in (0,T] \tag{1.4}$$

where

$$(\phi,\psi) = \int_\Omega \phi\psi\, dx \quad A(\xi : \phi, \psi) = \int_\Omega \left[a(x,\xi)\nabla\phi \cdot \nabla\psi + b(x,\xi)\phi\psi\right]\, dx.$$

For (1.4), we have the following energy estimate

Theorem 1.2. *Suppose that the conditions of Assumption 1.1 are satisfied for $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $b : \Omega \times \mathbb{R} \to \mathbb{R}$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}^+$. Then there exists a $C > 0$ such that*

$$\begin{aligned} &\|u\|_{L^2(0,T;X)} + \|u_t\|_{L^2(0,T;Y)} + \|u_{tt}\|_{L^2(0,T;L^2(\Omega_1)\cap L^2(\Omega_2))} \le \\ &\qquad \le C\left(\|f\|_{H^1(0,T;L^2(\Omega))} + \|f(x,0)\|_{L^2(\Omega)} + \|u_0\|_X + \|u_1\|_Y\right) \end{aligned} \tag{1.5}$$

Proof. Let $v = u_t$ in (1.4). For $t \in [0,T]$, a simple calculation shows that

$$\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H^1(\Omega)} \le C\left[\|u_0\|^2_{H^1(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,t;L^2(\Omega))}\right]. \tag{1.6}$$

Let

$$a_1(x,u)\frac{\partial u}{\partial n}\Big|_\Gamma = g_1 \qquad \text{and} \qquad a_2(x,u)\frac{\partial u}{\partial n}\Big|_\Gamma = -g_2.$$

It is clear from (1.3) that $g_1 + g_2 = 0$. From (1.1)–(1.3), we have

$$\int_{\Omega_i} u_{ttt}v + \int_{\Omega_i}(a\nabla u_t \cdot \nabla v + b u_t v) = \int_{\Omega_i} f_t v + \int_\Gamma g_{it} v\,, \qquad i = 1, 2.$$

We take $v = u_{tt}$ and obtain

$$\tfrac{d}{dt}\|u_{tt}\|^2_{L^2(\Omega_i)} + \mu_1 \tfrac{d}{dt}\|u_t\|^2_{H^1(\Omega_i)} - \|u_{tt}\|^2_{L^2(\Omega_i)} \;\le\; \tfrac{1}{4}\|f_t\|^2_{L^2(\Omega_i)} + \int_\Gamma g_{it}u_{tt}$$

which implies

$$\begin{aligned} &\|u_{tt}\|^2_{L^2(\Omega_i)} + \|u_t\|^2_{H^1(\Omega_i)} \le \\ &\quad \le C\left(\|u_0\|^2_{H^2(\Omega_i)} + \|u_1\|^2_{H^1(\Omega_i)} + \int_0^t \|f_t\|^2_{L^2(\Omega_i)}dt + \|f(x,0)\|^2_{L^2(\Omega_i)}\right) \\ &\quad + \int_0^t \exp(-s)\int_\Gamma g_{it}u_{tt}\, dt. \end{aligned} \tag{1.7}$$

It follows directly that

$$\begin{aligned} &\int_0^T \left(\|u_{tt}\|^2_{L^2(\Omega_1)} + \|u_{tt}\|^2_{L^2(\Omega_2)} + \|u_t\|^2_{H^1(\Omega_1)} + \|u_t\|^2_{H^1(\Omega_2)}\right)\, dt \le \\ &\quad \le C\left[\|u_0\|^2_X + \|u_1\|^2_Y + \|f(x,0)\|^2_{L^2(\Omega)} + \int_0^T \|f_t\|^2_{L^2(\Omega)}\, dt\right] \end{aligned} \tag{1.8}$$

Now, we multiply (1.1) by $-u_{tt}$, integrate over Ω_i then simplify the resulting equation and obtain

$$\mu_1^2\|u\|^2_{H^2(\Omega_i)} \le 2\mu_3^2\|u\|^2_{H^1(\Omega_i)} + 2\|u_{tt}\|^2_{L^2(\Omega_i)} + 2\|f\|^2_{L^2(\Omega_i)} + 2\int_\Gamma b g_i u. \tag{1.9}$$

It follows from (1.6), (1.7) and (1.9) that

$$(1.10)\quad \int_0^T \left(\|u\|^2_{H^2(\Omega_1)} + \|u\|^2_{H^2(\Omega_2)}\right)\, dt \leq$$
$$\leq C\left[\|u_0\|^2_X + \|u_1\|^2_Y + \|f(x,0)\|^2_{L^2(\Omega)} + \int_0^T \left(\|f\|^2_{L^2(\Omega)} + \|f_t\|^2_{L^2(\Omega)}\right) dt\right].$$

(1.5) follows from (1.6), (1.8) and (1.10). □

Remark 1.3. Estimate (1.5) establishes that a weak solution exists. For $u \in L^\infty(0,T;H^{m+1}(\Omega))$, $m \in \mathbb{N}$, standard energy argument for hyperbolic equations requires that $u_0 \in H^{m+1}(\Omega)$ and $u_1 \in H^m(\Omega)$ [18, Theorems 5 and 6, pages 389-391]. However, this level of global regularity is not guaranteed for interface problems as such problems are more regular on the individual domain than the entire domain [6, 21].

This paper is organized as follows. In Section 2, we describe a finite element discretization of the problem and state a result on the elliptic projection operator used for the error analysis. In Section 3, we give the discrete versions of (1.4) then establish the convergence rates of almost optimal order for both semi discrete and fully discrete schemes. We confirm our theoretical analysis with numerical examples in Section 4. Throughout this paper, C is a generic positive constant (which is independent of the mesh parameter h and the time step size k) and may take on different values at different occurrences.

2. FINITE ELEMENT DISCRETIZATION

$\mathcal{T}_h$ denotes a conforming triangulation of Ω. Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ (see Fig 2.1);

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$$

$K \in \mathcal{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$.

The domain Ω_1 is approximated by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries respectively. The triangulation $\mathcal{T}_h$ of the domain Ω is fitted to Ω_1^h and satisfies the following conditions

(i) $\bar{\Omega} = \bigcup_{K\in\mathcal{T}_h} \bar{K}$

(ii) If $\bar{K}_1, \bar{K}_2 \in \mathcal{T}_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.

(iii) Each $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h, and has at most two vertices lying on Γ_h.

(iv) For each element $K \in \mathcal{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles respectively. It is assumed that, for some fixed

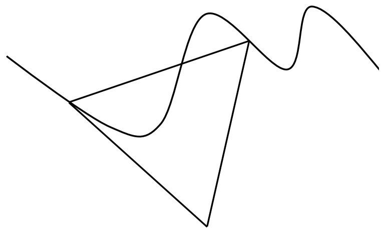

Fig. 2.1. A typical interface element.

$h_0 > 0$, there exists two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \leq h \leq C_1 \bar{r}_K \quad \forall\, h \in (0, h_0)$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$.
The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x) \ ,$$

where each basis function ϕ_j, $(j = 1, 2, \ldots, N_h)$ is a pyramid function with unit height. For the approximation $\hat{g}(t)$, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h.

Let $P_h : X \cap H_0^1(\Omega) \to S_h$ be the elliptic projection of the exact solution ν in S_h defined by

$$A(u : \nu - P_h\nu, \phi) = 0 \quad \forall\, \phi \in S_h,\ t \in [0,T]. \tag{2.1}$$

For this projection, we have

Lemma 2.1. *Let $a = a(x,u)$, $b = b(x,u)$ satisfy Assumption 1.1 and let a_{tt}, b_{tt} be continuous on $\Omega_i \times (0,T]$, $i = 1,2$. Assume that $u \in X \cap H_0^1$ and let $P_h u$ be defined as in (2.1), then*

$$\begin{aligned}
\|\tfrac{\partial^n}{\partial t^n}(P_h u - u)\|_{H^1(\Omega)} &\leq Ch\left(1 + \tfrac{1}{|\ln h|}\right)^{1/2} \sum_{i=1}^{n} \|\tfrac{\partial^i u}{\partial t^i}\|_X \\
\|\tfrac{\partial^n}{\partial t^n}(P_h u - u)\|_{L^2(\Omega)} &\leq Ch^2\left(1 + \tfrac{1}{|\ln h|}\right) \sum_{i=1}^{n} \|\tfrac{\partial^i u}{\partial t^i}\|_X
\end{aligned}$$

for $n = 0, 1, 2$.

Proof. It can be proved using the interpolation error estimate [2, Lemma 2.1] and a similar argument to the proof of [5, Lemmas 2.4 and 2.5] but with little modification due to different assumptions on $a(x,u)$ and $b(x,u)$. □

REMARK 2.2. The term $|\ln h|$ in Lemma 2.1 is due to the fact that the mesh in Section 2 cannot perfectly fit the interface. However, with the use of interface elements with curved edges along the interface, convergence rate of optimal is obtainable (see [16] for example). In practice, the use of curved interface elements that perfectly fits the interface may be computationally difficult or impossible particularly when the interface is irregular in shape [12]. $\square$

3. ERROR ESTIMATES

3.1. Continuous-in-Time Approximation. We may pose the semi discrete problem as: find $u_h : [0,T] \to S_h$ such that $u_h(0) = u_{h,0}$ and

$$(u_{h,tt}, v_h) + A(u_h : u_h, v_h) = (f, v_h) \quad \forall\, v_h \in S_h, \text{ a.e } t \in [0,T] \tag{3.1}$$

Below is the main results concerning the convergence of (3.1) to the exact solution in the $L^\infty(0,T;H^1(\Omega))$-norm.

THEOREM 3.1. *Suppose that the conditions of Assumption 1.1 are satisfied for $a : \Omega\times\mathbb{R}\to\mathbb{R}$, $b : \Omega\times\mathbb{R}\to\mathbb{R}$ and $f : \Omega\times\mathbb{R}^+\to\mathbb{R}$ and let u and u_h be the solutions of (1.4) and (3.1) respectively, then for $u_0 \in H_0^1(\Omega)\cap X$, $u_1 \in H_0^1(\Omega)$ and $0 < h < h_0$, there exists a positive constant C, independent of h, such that*

$$\max_{0\le t\le T} \|u - u_h\|_{H^1(\Omega)} \;\le\; Ch\left(1 + \tfrac{1}{|\ln h|}\right)^{1/2}$$

Proof. Subtract (3.1) from (1.4)

$$(u_t - u_{h,tt}, v_h) + A(u : u, v_h) - A(u_h : u_h, v_h) = 0 \qquad \forall\, v_h \in S_h$$

Let $e(t) = u - u_h$, $v_h = (P_h u - u_h)_t$

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)} + \tfrac{\mu_1}{2}\tfrac{d}{dt}\|e(t)\|^2_{H^1(\Omega)} &= \\
&= (u_{h,tt} - u_{tt}, (P_h u - u)_t) + A(u_h; e(t), (u - P_h u)_t) \\
&\quad + A(u_h : u, (P_h u - u)_t) - A(u : u, (P_h u - u)_t) \\
&\le I_1 + I_2 + I_3 \qquad (3.2)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= |(u_{tt} - u_{h,tt}, (P_h u - u)_t)|, \qquad I_2 = |A(u_h : e(t), (u - P_h u)_t)|, \\
I_3 &= |A(u_h : u, (P_h u - u)_t) - A(u : u, (P_h u - u)_t)|
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
I_1 &= |\tfrac{d}{dt}(e'(t), (P_h u - u)_t) - (e'(t), (P_h u - u)_{tt})| \\
&\le \tfrac{1}{4}\tfrac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)} + \tfrac{d}{dt}\|(P_h u - u)_t\|^2_{L^2(\Omega)} + \tfrac{1}{4}\|e'(t)\|^2_{L^2(\Omega)} \\
&\quad + \|(P_h u - u)_{tt}\|^2_{L^2(\Omega)} \\
&\le \tfrac{1}{4}\tfrac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)} + \tfrac{1}{4}\|e'(t)\|^2_{L^2(\Omega)} + \|(P_h u - u)_t\|^2_{L^2(\Omega)} \\
&\quad +2\|(P_h u - u)_{tt}\|^2_{L^2(\Omega)} \qquad (3.3)
\end{aligned}$$

$$
\begin{aligned}
I_2 &\le (\mu_3\|u_h\|_{L^2(\Omega)}+\mu_2)\|e(t)\|_{H^1(\Omega)}\|(u-P_hu)_t\|_{H^1(\Omega)}\\
&\le \tfrac{\mu_1}{4}\|e(t)\|^2_{H^1(\Omega)}+\tfrac{1}{\mu_1}(\mu_3\|u_h\|_{L^2(\Omega)}+\mu_2)^2\|(P_hu-u)_t\|^2_{H^1(\Omega)} \qquad (3.4)
\end{aligned}
$$

For I_3, we use Young's inequality and obtain

$$
\begin{aligned}
I_3 &\le \mu_3\|e(t)\|_{L^2(\Omega)}\|u\|_{H^1(\Omega)}\|(u-P_hu)_t\|_{H^1(\Omega)}\\
&\le \tfrac{\mu_1}{4}\|e(t)\|^2_{H^1(\Omega)}+\tfrac{\mu_3^2}{\mu_1}\|u\|^2_{H^1(\Omega)}\|(u-P_hu)_t\|^2_{H^1(\Omega)} \qquad (3.5)
\end{aligned}
$$

We substitute (3.3)–(3.5) into (3.2) and obtain

$$
\begin{aligned}
&\tfrac{1}{4}\tfrac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)}+\tfrac{\mu_1}{2}\tfrac{d}{dt}\|e(t)\|^2_{H^1(\Omega)}\le\\
&\le\tfrac{1}{4}\|e'(t)\|^2_{L^2(\Omega)}+\tfrac{\mu_1}{2}\|e(t)\|^2_{H^1(\Omega)}+2\|(P_hu-u)_{tt}\|^2_{L^2(\Omega)}\\
&\quad+\ \|(P_hu-u)_t\|^2_{L^2(\Omega)}+\tfrac{1}{\mu_1}(\mu_3\|u_h\|_{L^2(\Omega)}+\mu_2)^2\|(P_hu-u)_t\|^2_{H^1(\Omega)}\\
&\quad+\ \tfrac{\mu_3^2}{\mu_1}\|u\|^2_{H^1(\Omega)}\|(u-P_hu)_t\|^2_{H^1(\Omega)}.
\end{aligned}
$$

It is obvious that

$$
h^2\left(1+\tfrac{1}{|\ln h|}\right)\le h\left(1+\tfrac{1}{|\ln h|}\right)^{1/2} \quad\Leftrightarrow\quad 0<h<0.58857838891.
$$

Therefore using Lemma 2.1 for $0<h<0.58857838891$, it follows that

$$
\begin{aligned}
&\tfrac{1}{4}\tfrac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)}+\tfrac{\mu_1}{2}\tfrac{d}{dt}\|e(t)\|^2_{H^1(\Omega)}\le\\
&\le\tfrac{1}{4}\|e'(t)\|^2_{L^2(\Omega)}+\tfrac{\mu_1}{2}\|e(t)\|^2_{H^1(\Omega)}\\
&\quad+\ Ch^2\left(1+\tfrac{1}{|\ln h|}\right)\left[\left(1+\|u\|^2_{H^1(\Omega)}+(\mu_3\|u_h\|_{L^2(\Omega)}+\mu_2)^2\right)\right.\\
&\quad\times\left.\left(\|u\|^2_X+\|u_t\|^2_X\right)+\|u_{tt}\|^2_X\right].
\end{aligned}
$$

After a simple calculation, we have

$$
\begin{aligned}
\|e(t)\|^2_{H^1(\Omega)} \le\ & \exp(T)\|e'(0)\|^2_{L^2(\Omega)}+\exp(T)\|e(0)\|^2_{H^1(\Omega)}\\
&+Ch^2\int_0^t\exp(t-s)\left(1+\tfrac{1}{|\ln h|}\right)\\
&\times\left[\left(1+\|u\|^2_{H^1(\Omega)}+(\mu_3\|u_h\|_{L^2(\Omega)}+\mu_2)^2\right)\right.\\
&\times\left.\left(\|u\|^2_X+\|u_t\|^2_X\right)+\|u_{tt}\|^2_X\right]ds.
\end{aligned}
$$

The result follows by taking $u_{0,h}=P_hu_0$ and $u_{1,h}=P_hu_1$. □

3.2. Discrete-in-Time Approximation. In this section, we propose a linearized scheme for the solution of (1.4) due to the presence of the nonlinear terms. An almost optimal order error estimate in the $H^1(\Omega)$-norm is analyzed.

The interval $[0,T]$ is divided into M equally spaced (for simplicity) subintervals:

$$
0=t_0<t_1<\ldots<t_M=T
$$

with $t_n = nk$, $k = T/M$ being the time step. Let

$$u^n = u(x, t_n) \quad \text{and} \quad f^n = f(x, t_n) \ .$$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward difference quotient defined by

$$\partial^2 w^n \ = \ \frac{w^n - 2w^{n-1} + w^{n-2}}{k^2} \ , \quad n = 2, 3, \ldots, M.$$

The fully discrete finite element approximation to (1.4) is to find $U_h^n \in S_h$, such that

$$(\partial^2 U_h^n, v_h) + A(U_h^n : U_h^n, v_h) = (f^n, v_h) \quad \forall\, v_h \in S_h \quad n = 2, 3, \ldots, M. \tag{3.6}$$

Scheme (3.6) has the disadvantage that a nonlinear system of algebraic equations has to be solved at each time step due to the presence of $a(x, U_h^n)$ and $b(x, U_h^n)$. We therefore propose a linearized modification of the scheme in which this difficulty is avoided by replacing U_h^n by U_h^{n-1} in these two places. Thus the linearized fully discrete finite element approximation to (1.4) is to find $U_h^n \in S_h$, such that

$$(\partial^2 U_h^n, v_h) + A(U_h^{n-1} : U_h^n, v_h) = (f^n, v_h) \quad \forall\, v_h \in S_h \quad n = 2, 3, \ldots, M. \tag{3.7}$$

For the analysis of linearized schemes for nonlinear parabolic interface problems, see [5, 26].

The result below establishes the convergence of the scheme (3.7) to the exact solution in $H^1(\Omega)$-norm.

Theorem 3.2. *Let u^n and U_h^n be the solutions of (1.4) and (3.7) respectively at t_n with $U_h^0 = P_h u_0$ and $U_h^1 = U_h^0 + kP_h u_1$. Suppose that the conditions of Assumption 1.1 are satisfied for $a : \Omega \times \mathbb{R} \to \mathbb{R}$, $b : \Omega \times \mathbb{R} \to \mathbb{R}$ and $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$. There exists a positive constant C independent of $h \in (0, h_0)$ and $k \in [0, k_0)$ such that*

$$\|u^n - U_h^n\|_{H^1(\Omega)} \ \le \ C \left[k + h \left(1 + \frac{1}{|\ln h|} \right)^{1/2} \right] \ , \quad n = 2, 3, \ldots, M \ .$$

Proof. Let $z^n = P_h u^n - U_h^n$. From (1.4) and (3.7) using (2.1), we have

$$\begin{aligned}(\partial^2 z^n, v_h) + A(U_h^{n-1}; z^n, v_h) \ &= \ (\partial^2 (P_h u^n - u^n), v_h) + (\partial^2 u^n - u_{tt}^n, v_h) \\ &\quad + A(U_h^{n-1} : P_h u^n, v_h) - A(u^n : P_h u^n, v_h)\end{aligned}$$

After a simple calculation using Young's inequality with $v_h = \partial^2 z^n$, we have

$$\begin{aligned}\|\partial^2 z^n\|_{L^2(\Omega)}^2 + \frac{\mu_1}{4k}\|z^n\|_{H^1(\Omega)}^2 \ &\le \ \frac{\mu_1}{4k}\|z^{n-1}\|_{H^1(\Omega)}^2 + \frac{\mu_1}{4k}\|z^{n-2}\|_{H^1(\Omega)}^2 \\ &\quad + \frac{9\mu_1}{4k}\|z^{n-1} - z^{n-2}\|_{H^1(\Omega)}^2 + B_1 + B_2\end{aligned} \tag{3.8}$$

where

$$\begin{aligned} B_1 \ &= \ (\partial^2 (P_h u^n - u^n), \partial^2 z^n) + (\partial^2 u^n - u_{tt}^n, \partial^2 z^n) \\ B_2 \ &= \ A(U_h^{n-1} : P_h u^n, \partial^2 z^n) - A(u^n : P_h u^n, \partial^2 z^n)\end{aligned}$$

$$B_1 \leq 2\|\partial^2(P_h u^n - u^n)\|^2_{L^2(\Omega)} + \tfrac{1}{4}\|\partial^2 z^n\|^2_{L^2(\Omega)} + 2\|\partial^2 u^n - u^n_{tt}\|^2_{L^2(\Omega)} \tag{3.9}$$

By Taylor's expansion, there exists $\lambda > 0$, such that

$$\begin{aligned}
B_2 &\leq \mu_3\|U_h^{n-1} - u^n\|_{L^2(\Omega)}\|P_h u^n\|_{H^1(\Omega)}\|\partial^2 z^n\|_{H^1(\Omega)} \\
&\leq \mu_3\lambda k\|u^n_t\|_{L^2(\Omega)}\|P_h u^n\|_{H^1(\Omega)}\|\partial^2 z^n\|_{H^1(\Omega)} \\
&\quad + \mu_3\|z^{n-1}\|_{L^2(\Omega)}\|P_h u^n\|_{H^1(\Omega)}\|\partial^2 z^n\|_{H^1(\Omega)} \\
&\quad + \mu_3\|P_h u^{n-1} - u^{n-1}\|_{L^2(\Omega)}\|P_h u^n\|_{H^1(\Omega)}\|\partial^2 z^n\|_{H^1(\Omega)} \\
&\leq Ck^2\|u^n_t\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} + \mu_3^2\|z^{n-1}\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} \\
&\quad + \mu_3^2\|P_h u^{n-1} - u^{n-1}\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} + \tfrac{3}{4}\|\partial^2 z^n\|^2_{H^1(\Omega)} \qquad (3.10)
\end{aligned}$$

Substitute (3.9) and (3.10) into (3.8), and use inverse estimate [11, Theorem 4.5.11],

$$\begin{aligned}
\tfrac{\mu_1}{4k}\|z^n\|^2_{H^1(\Omega)} &\leq \tfrac{\mu_1}{4k}\|z^{n-1}\|^2_{H^1(\Omega)} + \tfrac{\mu_1}{4k}\|z^{n-2}\|^2_{H^1(\Omega)} + \tfrac{9\mu_1}{4k}\|z^{n-1} - z^{n-2}\|^2_{H^1(\Omega)} \\
&\quad + 2\|\partial^2(P_h u^n - u^n)\|^2_{L^2(\Omega)} + 2\|\partial^2 u^n - u^n_{tt}\|^2_{L^2(\Omega)} \\
&\quad + Ck^2\|u^n_t\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} + \mu_3^2\|z^{n-1}\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} \\
&\quad + Ch^4\left(1 + \tfrac{1}{|\ln h|}\right)^2\|u^n\|^2_{H^1(\Omega)}\|u^{n-1}\|^2_X.
\end{aligned}$$

We used Lemma 2.1 to obtain the last inequality. Therefore,

$$\begin{aligned}
(1 - ck)\|z^n\|^2_{H^1(\Omega)} &\leq \|z^{n-1}\|^2_{H^1(\Omega)} + \|z^{n-2}\|^2_{H^1(\Omega)} + 9\|z^{n-1} - z^{n-2}\|^2_{H^1(\Omega)} \\
&\quad + C\left[k\|\partial^2(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^2 u^n - u^n_t\|^2_{L^2(\Omega)}\right] \\
&\quad + Ck^2\|u^n_t\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} \\
&\quad + Ch^4 k\left(1 + \tfrac{1}{|\ln h|}\right)^2\|u^n\|^2_{H^1(\Omega)}\|u^{n-1}\|^2_X
\end{aligned}$$

where $c = \frac{4\mu_3^2}{\mu_1}\|u^n\|^2_{H^1(\Omega)}$. For $0 < k < \min\left\{\frac{1}{2}, \frac{1}{2c}\right\}$, there is a $C > 0$ such that $(1 - ck)^{-1} \leq C$, and therefore

$$\begin{aligned}
\|z^n\|^2_{H^1(\Omega)} &\leq C\|z^{n-1}\|^2_{H^1(\Omega)} + C\|z^{n-2}\|^2_{H^1(\Omega)} + C\|z^{n-1} - z^{n-2}\|^2_{H^1(\Omega)} \\
&\quad + C\left[k\|\partial^2(P_h u^n - u^n)\|^2_{L^2(\Omega)} + k\|\partial^2 u^n - u^n_{tt}\|^2_{L^2(\Omega)}\right] \\
&\quad + Ck^2\|u^n_t\|^2_{L^2(\Omega)}\|u^n\|^2_{H^1(\Omega)} \\
&\quad + Ch^4 k\left(1 + \tfrac{1}{|\ln h|}\right)^2\|u^n\|^2_{H^1(\Omega)}\|u^n\|^2_X, \quad \text{for } n = 2, 3, \ldots, M.
\end{aligned}$$

By iteration on n, we have

$$\begin{aligned}\|z^n\|^2_{H^1(\Omega)} \quad &\leq \quad C\sum_{i=0}^{1}\|z^i\|^2_{H^1(\Omega)} + Ch^4k\left(1+\tfrac{1}{|\ln h|}\right)^2\sum_{j=2}^{n}\|u^j\|^2_X\|u^j\|^2_{H^1(\Omega)} \\ &\quad + C\sum_{j=2}^{n}\|z^{j-1}-z^{j-2}\|^2_{H^1(\Omega)} + Ck\sum_{j=2}^{n}\|\partial^2 u^j - u^j_{tt}\|^2_{L^2(\Omega)} \\ &\quad + Ck\sum_{j=2}^{n}\|\partial^2(u^j - P_h u^j)\|^2_{L^2(\Omega)} + Ck^3\sum_{j=2}^{n}\|u^j_t\|^2_{L^2(\Omega)}\|u^j\|^2_{H^1(\Omega)}\end{aligned}$$

Using the discrete version of Gronwall's inequality and simplifying the resulting expression, we obtain

$$\begin{aligned}\|z^n\|^2_{H^1(\Omega)} \quad &\leq \quad \sum_{i=0}^{1}\|z^i\|^2_{H^1(\Omega)} + C\int_0^{t_n}\|(u-P_hu)_{tt}\|^2_{L^2(\Omega)}\, dt \\ &\quad + Ck^2\int_0^{t_n}\|\tfrac{\partial^3 u}{\partial t^3}\|^2_{L^2(\Omega)}\, dt + Ck^2\int_0^{t_n}\|u_t\|^2_{L^2(\Omega)}\|u\|^2_{H^1(\Omega)}\, dt \\ &\quad + Ch^4\left(1+\tfrac{1}{|\ln h|}\right)^2\int_0^{t_n}\|u\|^2_X\|u\|^2_{H^1(\Omega)}\, dt \\ &\leq \quad Ck^2\int_0^{t_n}\left(\|u_t\|^2_{L^2(\Omega)}\|u\|^2_{H^1(\Omega)} + \|\tfrac{\partial^3 u}{\partial t^3}\|^2_{L^2(\Omega)}\right) dt \\ &\quad + C\sum_{i=0}^{1}\|z^i\|^2_{H^1(\Omega)} + Ch^4\left(1+\tfrac{1}{|\ln h|}\right)^2 \\ &\quad \times\int_0^{t_n}\left[\|u\|^2_{H^1(\Omega)}\|u\|^2_X + \|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X\right]\, dt.\end{aligned}$$

By triangle inequality and Lemma 2.1,

$$\begin{aligned}\|u^n - U^n_h\|^2_{H^1(\Omega)} \quad &\leq \quad 2\|u^n - P_h u^n\|^2_{H^1(\Omega)} + 2\|z^n\|^2_{H^1(\Omega)} \\ &\leq \quad Ch^2\left(1+\tfrac{1}{|\ln h|}\right)\|u^n\|_X \\ &\quad + C\sum_{i=0}^{1}\left(\|u^i - U^i_h\|^2_{H^1(\Omega)} + \|u^i - P_h u^i\|^2_{H^1(\Omega)}\right) \\ &\quad + Ck^2\int_0^{t_n}\left(\|u_t\|^2_{L^2(\Omega)}\|u\|^2_{H^1(\Omega)} + \|\tfrac{\partial^3 u}{\partial t^3}\|^2_{L^2(\Omega)}\right) dt \\ &\quad + Ch^4\left(1+\tfrac{1}{|\ln h|}\right)^2 \\ &\quad \times\int_0^{t_n}\left[\|u\|^2_X\|u\|^2_{H^1(\Omega)} + \|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X\right]\, dt.\end{aligned}$$

It is obvious that

$$h^4\left(1+\tfrac{1}{|\ln h|}\right)^2 \leq h^2\left(1+\tfrac{1}{|\ln h|}\right) \quad \Leftrightarrow \quad 0 < h < 0.58857838891.$$

The result follows taking $U_h^0 = P_h u_0$ and $U_h^1 = U_h^0 + kP_h u_1$. □

REMARK 3.3. In the proof of Theorem 3.2, we used $v_h = \partial^2 z^n$. If we choose $v_h = z^n$, by a similar argument, one can obtain

$$\|u^n - U_h^n\|_{L^2(\Omega)} \le C\left[k + h^2\left(1 + \frac{1}{|\ln h|}\right)\right], \quad n = 2, 3, \ldots, M. \quad \square$$

4. EXAMPLES

Here, we present examples to verify Theorem 3.2. Globally continuous piecewise linear finite element functions based on triangulation described in Section 2 are used. The mesh generation and computation are done with FreeFEM++ [20].

EXAMPLE 4.1. The problem is defined on the domain $\Omega = (-1, 1) \times (-1, 1)$ where the interface Γ is a circle centered at $(0, 0)$ with radius 0.5. $\Omega_1 = \{(x, y) : x^2 + y^2 < 0.25\}$, $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.
On $\Omega \times (0, 50]$, we consider the nonlinear problem (1.1)−(1.3) whose exact solution is

$$u = \begin{cases} \frac{1}{8}(1 - 4r^2)t\sin(t) & \text{in} \quad \Omega_1 \times (0, 50] \\ \frac{1}{4}(1 - x^2)(1 - y^2)(1 - 4r^2)t\sin(2t) & \text{in} \quad \Omega_2 \times (0, 50] \end{cases},$$

where $r^2 = x^2 + y^2$. The source function f and the initial data u_0, u_1 are determined from the choice of u with $b = 0$ and

$$a = \begin{cases} 1 + u & \text{in} \quad \Omega_1 \times (0, T] \\ \frac{1}{1+u^2} & \text{in} \quad \Omega_2 \times (0, T] \end{cases}.$$

We allow k and h to vary simultaneously by choosing $k = O(h)$. Errors in H^1-norm at $t = 1$ and convergence rates are presented in Table 4.1. To verify the agreement of the numerical experiment with the theoretical results, we use the formula

$$\text{Order of convergence} = \frac{\ln(e_{i+1}/e_i)}{\ln(\mathfrak{h}_{i+1}/\mathfrak{h}_i)},$$

where e_i is the error at the i-th iteration corresponding to the mesh size h_i and $\mathfrak{h}_i = h_i\left(1 + \frac{1}{|\ln h_i|}\right)^{1/2}$. □

The next example demonstrates that the error estimates apply even when the domain is not polygonal.

EXAMPLE 4.2. We consider (1.1)−(1.3) on the domain $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ where $\Omega_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \frac{1}{4}\}$, $\Omega_2 = \Omega \setminus \Omega_1$ and the interface Γ is the circle $x^2 + y^2 = \frac{1}{4}$.
For the exact solution, we choose

$$u = \begin{cases} (2 - 5x^2 - 5y^2)\sin^2 t & \text{in} \quad \Omega_1 \times (0, T] \\ (1 - x^2 - y^2)\sin^2 t & \text{in} \quad \Omega_2 \times (0, T] \end{cases}$$

k	h	Error	Convergence rate
0.008	0.1518120	2.02004×10^{-1}	
0.004	0.0793667	1.06102×10^{-1}	0.889
0.002	0.0403482	5.24888×10^{-2}	0.965
0.001	0.0206032	2.63159×10^{-2}	0.973

Table 4.1. Error estimates in H^1-norm for Example 4.1.

h	Error ($k = 0.001$)
0.196096	2.9256627×10^{-1}
0.101640	1.4581576×10^{-1}
0.0519419	7.8258517×10^{-2}
0.0267211	5.0963435×10^{-2}

k	Error ($h = 0.0267211$)
0.0050	7.5769633×10^{-2}
0.0025	5.9446024×10^{-2}
0.0020	5.6473694×10^{-2}
0.0010	5.0963435×10^{-2}

Table 4.2. Error estimates in H^1-norm for Example 4.2.

The source function f, interface function g and the initial data u_0 are determined from the choice of u with

$$a = \begin{cases} x^2 + y^2 & \text{in} \quad \Omega_1 \times (0, T] \\ 1 + u & \text{in} \quad \Omega_2 \times (0, T] \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{1}{1+u^2} & \text{in} \quad \Omega_1 \times (0, T] \\ 1 & \text{in} \quad \Omega_2 \times (0, T] \end{cases} .$$

Figures 4.1 and 4.1 show the computed solution of Example 4.2. Errors in H^1-norm at $t = 1$ for various step size h time step k are presented in Table 4.2. The data show that the error is linear both in h and k.

$$\|\text{Error}\|_{H^1(\Omega)} \approxeq 2.262 \times 10^{-2} + 0.8781\, \mathfrak{h}^{1.025} \qquad \text{when } k \text{ is constant}$$

and

$$\|\text{Error}\|_{H^1(\Omega)} \approxeq 4.664 \times 10^{-2} + 15.58 \times 10^{-3} k^{1.186} \qquad \text{when } h \text{ is constant}$$

where $\mathfrak{h} = h\left(1 + \frac{1}{|\ln h|}\right)^{1/2}$.

It can be observed that the numerical results in Tables 4.1 and 4.2 match the convergence rate as given in Theorem 3.2. □

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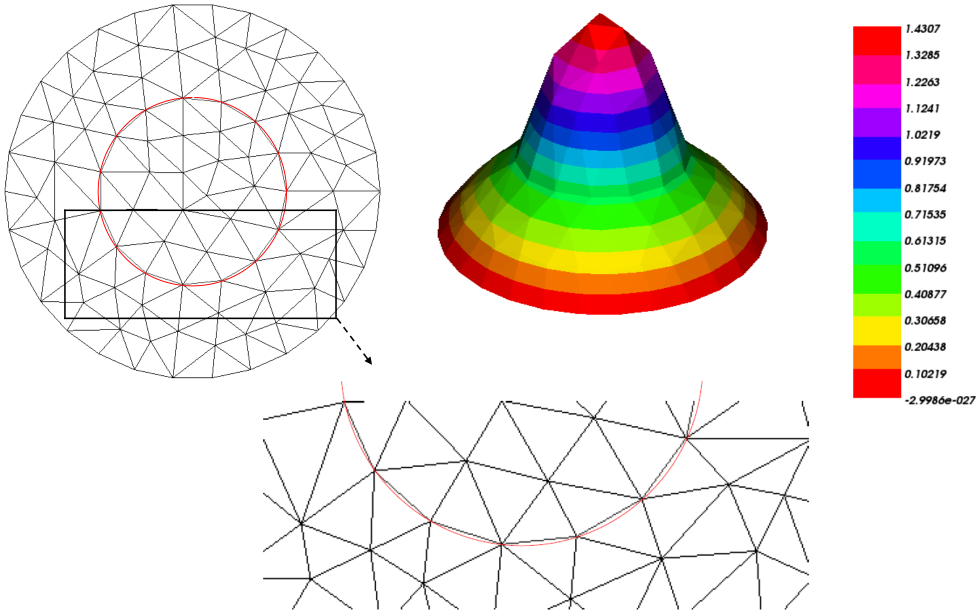

Fig. 4.1. Computed solution of Example 4.2 at $t = 1$ with $h = 0.3568$, $k = 0.001$

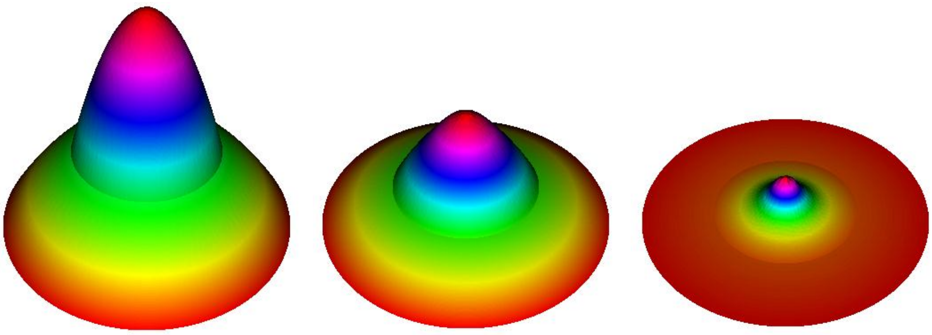

Fig. 4.2. Computed solution of Example 4.2 at $t = 2$, 2.5 and 3 with $h = 0.052$, $k = 0.001$

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