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H^1-Convergence of FEM-BDS for Linear Parabolic Interface Problems

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***Abstract*:** In this paper, I have extended the analysis done in the previous research [1] which proposed a standard finite element method with a four step time discretization. The analysis in that paper revealed that almost optimal order of convergence in the $L^2(\Omega)$-norm is obtainable when the interface cannot be fitted exactly. I have also derived almost optimal error estimate in $H^1(\Omega)$-norm. Numerical experiments are presented in this research to support the theoretical result.

***Keywords*:** finite element method, interface, almost-optimal, parabolic equation, implicit scheme.

1. Introduction

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega\backslash\bar{\Omega}_1$ be another open domain contained in Ω with boundary $\Gamma \cup \partial\Omega$ (see Figure 1). The parabolic interface problem is considered in this research

$$u_t - \nabla\cdot(a(x,t)\,\nabla u) + b(x,t)u = f(x,t) \text{ in } \Omega\times(0,T] \quad (1)$$

with initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x) & \text{in} \quad \Omega \\ u(x,t) = 0 & \text{on} \quad \partial\Omega \times [0,\mathrm{T}] \end{cases} \quad (2)$$

and interface conditions

$$\begin{aligned} u_1(x,t)|_\Gamma - u_2(x,t)|_\Gamma &= 0 \\ [a_1\nabla u_1(x,t) - a_2\nabla u_2(x,t)]\cdot n &= g(x,t) \ \text{ on } \Gamma \end{aligned} \quad (3)$$

where $0 < T < \infty$ and n is the unit outward normal to the boundary $\partial\Omega_1$. u_i, a_i, b_i, and f_i stand for the restriction of u, a, b and f respectively to $\Omega_i, i = 1,2$. Input functions a, b and f are assumed continuous on each domain but discontinuous across the interface for $t \in [0,T]$.

A typical example of (1) – (3) is the heat (or diffusion) equation when the heat transfer (or diffusion) involves more than one material medium with different properties such as the conductivities, diffusion constraints, etc. This kind of problems have higher regularities in each individual material region than in the entire physical domain because of the discontinuities across the interface [2,3]. Thus, achieving higher order accuracy may be difficult.

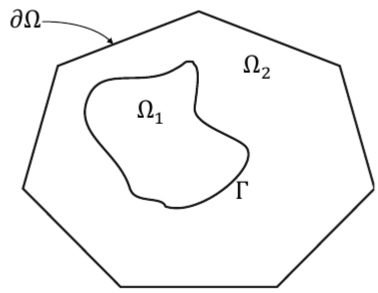

Figure 1. A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$with interface Γ

The study of interface problems by finite element method (FEM) was first carried out by Babuska [2]. The attention of researchers has since been drawn to the implementation and analysis of FEMs for interface problems. [4−15] contain recent development in the implementation and analysis of FEMs for interface problem. Recently [16] presented a residual-based a posteriori error analysis for a modified Crank-Nicolson time-step in finite element method for a linear parabolic interface problem. Convergence rate of almost optimal order was proved using a space-time reconstruction that is piecewise quadratic in time and Clement-type interpolation estimates.

It is known that spatial and time discretization are the sources of errors in FEM, however, research has largely focused on the use of FEM for linear parabolic interface problems with emphasis on the improvement of the spatial discretization. Recently, standard finite element method with time discretization based on four-step backward difference scheme (BDS) was proposed and analyzed in [1]. It was shown that the method is numerically stable and that higher-order accuracy in time could be obtained. The analysis further revealed that almost optimal order of convergence in the $L^2(\Omega)$-norm is obtainable when the interface cannot be fitted exactly. In this paper, we extend this analysis and derive almost optimal error estimate in $H^1(\Omega)$-norm. Numerical experiments are presented to support the theoretical result.

In this study, the linear theories of interface and non-interface problems, Sobolev imbedding inequality are used. Other technical tools used in this paper are approximation properties for linear interpolation operator and projection operator. We use the standard notations for Sobolev spaces and norms as contained in [17].

We shall need the following space
$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$
equipped with the norm

$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \; \forall\, v \in X$

The paper is organized as follows. In Section 2, we describe a finite element discretization of the problem and state auxiliary results needed for our analysis. In Section 3, we prove a convergence rate of almost optimal order in $H^1(\Omega)$ -norm for the fully discrete scheme. Numerical examples are presented in Section 4 and conclusion is made in Section 5. Throughout this paper, C is a generic positive value at different occurrences.

2. Finite Element Discretization

T_h denotes a partition of Ω into disjoint K (called elements) such that no vertex of any triangle lies on the interior or side of another triangle. The domain Ω_1 is approximated by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries respectively.

Let h_K be the diameter of an element $K \in T_h$ and $h = \max_{K\in T_h}$. Let T_h^* denote the set of all elements that are intersected by the interface Γ (see Figure 2);

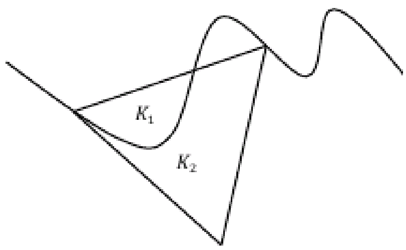

Figure 2. A typical interface element

$T_h^* = \{K \in T_h : K \cap \Gamma \neq \emptyset\}$

$K \in T_h^*$ is called an interface element and we write $\Omega_h^* = \cup_{K\in T_h^*} K$.

The triangulation T_h of the domain Ω satisfies the following conditions

(i) $\bar{\Omega} = \bigcup_{K\in T_h} \bar{K}$

(ii) $\bar{K}_1, \bar{K}_2 \in T_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.

(iii) Each $K \in T_h$ is either in Ω_1^h or Ω_2^h, and has most two vertices lying on Γ_h.

(iv) For each element $K \in T_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles respectively. It is assumed that, for some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1, independent of h, such that

$$C_{0r_K} \le h \le C_{1\bar{r}_K} \quad \forall\, h \in (0, h_0)$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on T_h vanishing on $\partial\Omega$. The FE solution $u_h(x,t) \in S_h$ is represented as

$$u_h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x),$$

where each basis function $\phi_j, (j = 1,2,\dots,N_h)$ is a pyramid function with unit height. For the approximation g_h of g, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation T_h that lie on the interface Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h, then

$$g_h(x,t) = \sum_{j=1}^{n_h} \beta_j(t)\psi_j(x).$$

Let $\pi_h: C(\bar{\Omega}) \to S_h$ be the Lagrange interpolation operator corresponding to the space S_h. We have (cf [1])

Lemma 2.1. For the linear interpolation operator $\pi_h: C(\bar{\Omega}) \to S_h$, we have, for $m = 0,1$, and $0 < h < 1$

$$\|u - \pi_h u\|_{H^m(\Omega)} \le Ch^{2-m}\left(1 + \frac{1}{|\ln h|}\right)^{1/2} \|u\|_X \quad \forall\, u \in X$$

For the approximation property of g_h to the interface function g, we have the following (cf [3])

Lemma 2.2. Assume that $g \in H^2(\Gamma)$. Then we have

$$|\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}|$$

$$\le Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_h^*)} \quad \forall\, v_h \in S_h$$

We recall some results which will be used for our analysis. See [4, 18] for proofs.

Lemma 2.3 Let Ω_h^* be the union of all interface triangles and $f \in H^2(\Omega)$ for $t \in [0,T]$, we have

$$\|v\|_{H^1(\Omega_h^*)} \le Ch^{1/2}\|v\|_X \quad \forall\, v \in X$$

$$|(f,v) - (f,v)_h| \le Ch^2\|f\|_{H^2(\Omega)}\|v\|_{H^1(\Omega)}$$

3. Error Estimate

We discuss a fully discrete scheme based on four-step backward difference approximation. The weak form of (1) - (3) is given as

$$(u_t, v) + A(u,v) = f(u,v) + (g,v)_\Gamma$$

$$\forall\, v(t) \in H_0^1(\Omega), \text{a.e.}\, t \in [0,T] \qquad (4)$$

where

$$(\phi,\psi) = \int_\Omega \phi\psi\, dx, \qquad \langle\phi,\psi\rangle_\Gamma = \int_\Gamma \phi\psi\, d\Gamma,$$

$$A(\phi,\psi) = \int_\Omega [a(x,t)\nabla_\phi \cdot \nabla_\psi + b(x,t)\phi\psi]\, dx$$

The spatially discrete approximation of (4) could be posed as: find $u_h: [0,T] \to S_h$ such that $u_h(0) = u_{h,0}$ and satisfies

$$(u_{h,t}, v_h)_h + A_h(u_h, v_h) = (f(x,t), v_h)_h + \langle g_h, v_h\rangle_{\Gamma_h}$$

$$\forall v_h \in S_h, \text{a.e}\, t \in [0,T] \qquad (5)$$

For the fully discrete approximation, let the interval $[0,T]$ be divided into M equally spaced (for simplicity) subintervals: $0 = t_0 < t_1 < \cdots < t_M = T$

with $t_n = nk,\ k = T/M$ being the time step. Let

$u^n = u(x,t_n), -f^n = f(x,t_n),$ and $u^n = g(x,t_n)$

For a given sequence $\{w_n\}_{n=0}^M \subset L^2(\Omega)$, we have the backward difference quotients defined by

$$\partial^1 w^n = \frac{w^n - w^{n-1}}{\tau_1} \qquad n = 1,2,\dots,M$$

$$\partial^2 w^n = \frac{3w^n - 4w^{n-1} + w^{n-2}}{2\tau_2} \qquad n = 2,3\ \dots, M$$

$$\partial^3 w^n = \frac{11w^n - 18w^{n-1} + 9w^{n-2} - 2w^{n-3}}{6\tau_3}$$
$$n = 3,4,\dots,M$$
$$\partial^4 w^n = \frac{25w^n - 48w^{n-1} + 36w^{n-2} - 16w^{n-3} + w^{n-4}}{12k}$$
$$n = 4,5,\dots,M$$

The FEM-BDS approximation to (4) is defined as follows: let $U_h^0 = \pi_h u_0$, find $U_h^n \in S_h$, such that

$$\begin{cases} (\partial^1 U_h^1, v_h)_h + A_h(U_h^1, v_h) &= (f^1, v_h)_h + \langle g_h^1, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \\ (\partial^2 U_h^2, v_h)_h + A_h(U_h^2, v_h) &= (f^2, v_h)_h + \langle g_h^2, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \\ (\partial^3 U_h^3, v_h)_h + A_h(U_h^3, v_h) &= (f^3, v_h)_h + \langle g_h^3, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \\ (\partial^4 U_h^n, v_h)_h + A_h(U_h^n, v_h) &= (f^n, v_h)_h + \langle g_h^n, v_h\rangle_{\Gamma_h} \quad \forall\, v_h \in S_h \quad n = 4,5,\dots,M \end{cases} \tag{6}$$

where $(\phi,\psi)_h: H^1(\Omega)\times H^1(\Omega) \to \mathbb{R}$ and $\langle g(x,t), v_h\rangle_{\Gamma_h}: H^{1/2}(\Gamma)\times H^1(\Omega)\to\mathbb{R}$ are defined as

$$(\psi,\phi)_h = \sum_{K\in T_h}\int_K \psi\phi\, dx,$$

$$A_h(\phi,\psi) = \sum_{K\in T_h}\int_K [a(x,t)\nabla\phi\cdot\nabla\psi + b(x,t)\phi\psi]\, dx$$

$$\langle g(x,t),\phi\rangle_{\Gamma_h} = \int_{\Gamma_h} g(x,t)\phi\, ds$$

$\forall \phi,\psi \in H^1 \in \Omega, g\in H^{1/2}(\Gamma), t\in[0,T]$ and $s\in\Gamma_h$. $(\psi,\phi)_h: H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$, $A_h(\phi,\psi): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ and $\langle g(x,t), v_h\rangle_{\Gamma_h}: H^{1/2}(\Omega)\times H^1(\Omega)\to\mathbb{R}$ are the discrete versions of $(\psi,\phi): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$, $A(\phi,\psi): H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$ and $\langle g(x,t), v_h\rangle_{\Gamma}: H^{1/2}(\Omega)\times H^1(\Omega)\to\mathbb{R}$ respectively and are obtained numerically using quadrature schemes.

We have the following stability (cf [1])

Lemma 3.1. $a_i(x,t), b_i(x,t)$ and $f_i(x,t)$ be continuous on $\Omega_i\times(0,T], i = 1,2$. Suppose $g(x,t)\in L^2\left(0,T; H^{1/2}(\Omega)\right)$, there exists a constant C independent of k *and* h *such that*

$$\|U_h^n\|_{L^2(\Omega)}^2 + k\|U_h^n\|_{H^1(\Omega)}^2 \le C\left[\|U_h^0\|_{L^2(\Omega)}^2 + k\sum_{j=1}^{n}\left(\|f^j\|_{L^2(\Omega)}^2 + \|g_h^j\|_{H^{1/2}(\Gamma_h)}^2\right) + k^3\right]$$

for $n = 2$ .... and $0 < k \le k_0 < 1$.

The result below establishes the convergence of the fully discrete solution to the exact solution $H^1(\Omega)$-norm.

Theorem 3.2. *Let* u^n *and* U_h^n *be the solutions of* (4) *and* (6) *respectively. Suppose* $a_i(x,t), b_i(x,t)$ *and* $f_i(x,t)$ *be continuous on* $\Omega_i\times(0,T], i = 1,2$ *and* $g(x,t)\in L^2\big(0,T;H^2(\Omega)\big)$. *There exists a positive constant* B_n *independent of h and k such that*

$$\|u^n - U_h^n\|_{H^1(\Omega)} \le \left[k^4 + h\left(1+\frac{1}{|\ln h|}\right)^{1/2}\right]B_n$$

For the proof of this result, we shall need the following (cf [1])

Let $P_h: X\cap H^1(\Omega)\to S_h$ be the elliptic projection of the exact solution u in S_h defined by

$$A_h(P_h v,\phi) = A(v,\phi)\quad \forall\,\phi\in S_h,\ t\in[0,\mathrm{T}]. \tag{7}$$

We therefore have

Lemma 3.3. *Let* $a_i(x,t), b_i(x,t)$ *be continuous on* $\Omega_i\times(0,T], i = 1,2$. *Assume that* $u\in X\cap H_0^1$ *and let* $P_h u$ *be defined as in* (7), *then*

$$\|P_h u - u\|_{H^1(\Omega)} \le Ch\left(1+\frac{1}{|\ln h|}\right)^{1/2}\|u\|_X$$

$$\|P_h u - u\|_{L^2(\Omega)} \le Ch^2\left(1+\frac{1}{|\ln h|}\right)\|u\|_X$$

$$\|(P_h u - u)_t\|_{H^1(\Omega)} \le Ch\left(1+\frac{1}{|\ln h|}\right)^{1/2}(\|u\|_X + \|u_t\|_X)$$

$$\|(P_h u - u)_t\|_{L^2(\Omega)} \le Ch^2\left(1+\frac{1}{|\ln h|}\right)(\|u\|_X + \|u_t\|_X)$$

Theorem 3.5. *Let* u *and* u_h *be the solutions of* (4) *and* (5) *respectively. Suppose* $a_i(x,t), b_i(x,t)$ *and* $f_i(x,t)$ *are continuous on* $\Omega_i\times(0,T], i = 1,2$ *and* $g(x,t)\in L^2\big(0,T;H^2(\Gamma)\big)$. *There exists a positive constant* C *independent of h such that*

$$\|u - u_h\|_{H^1(\Omega)} \le h\left(1+\tfrac{1}{|\ln h|}\right)^{1/2} C(u,f,g) \tag{8}$$

Proof. Subtract (5) from (4)

$$(u_t - u_h) + (u,u_h) = (u_{h,t},u_h)_h + A_h(u_h,v_h) + (f,v_h) - (f,v_h)_h + \langle (g,v_h)\rangle_{\Gamma_h} \quad \forall\, v_h\in S_h$$

Let $e(t) = u - u_h$, choose $v_h = P_h u - u_h$ and use (8)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e(t)\|_{L^2(\Omega)}^2 + A_h\big(e(t),e(t)\big) &= \big(u_{h,t} - u_t, P_h u - u\big)_h + A_h(e(t), u - P_h u) \\ &\quad + A_h(u, P_h u - u_h) - A_h(P_h u, P_h u - u_h) \\ &\quad + (f, P_h u - u_h) - (f, P_h u - u_h)_h \\ &\quad + \langle g, P_h u - u_h\rangle_\Gamma - \langle g_h, P_h u - u_h\rangle_{\Gamma_h} \\ &\quad + (u_t, P_h u - u_h)_h - (u_t, P_h u - u_h) \\ &\le B_1 + B_2 + B_3 + B_4 + B_5 \end{aligned} \tag{9}$$

where

$B_1 = \left|(u_t - u_{h,t}, P_h u - u)_h\right|$, $B_2 = |A_h(e(t), u - P_h u)|$

$B_3 = |A_h(u, P_h u - u_h) - A_h(P_h u, P_h u - u_h)|$

$B_4 = |(f, P_h u - u_h) - (f, P_h u - u_h)_h| + |(u_t, P_h u - u_h)_h - (u_t, P_h u - u_h)|$

$B_5 = \left|\langle g, P_h u - u_h\rangle_\Gamma - \langle g_h, P_h u - u_h\rangle_{\Gamma_h}\right|$

For B_1, we have

$$\begin{aligned} B_1 &= \left|\frac{d}{dt}(e(t), P_h u - u)_h - (e(t), (P_h u - u)_t)_h\right| \\ &\le \frac{1}{2}\frac{d}{dt}\|e(t)\|_{L^2(\Omega)}^2 + \frac{1}{2}\frac{d}{dt}\|P_h u - u\|_{L^2(\Omega)}^2 \\ &\quad + \frac{1}{4\varepsilon}\|e(t)\|_{L^2(\Omega)}^2 + \varepsilon\|P_h u - u_t\|_{L^2(\Omega)}^2 \\ &\le \frac{1}{2}\frac{d}{dt}\|e(t)\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon}\|e(t)\|_{L^2(\Omega)}^2 \\ &\quad + \tfrac{1}{2}\|P_h u - u\|_{L^2(\Omega)}^2 + C(\varepsilon)\|P_h u - u_t\|_{L^2(\Omega)}^2 \end{aligned} \tag{10}$$

$B_2 \leq \|e(t)\|_{H^1(\Omega)}\|u - P_h u\|_{H^1(\Omega)}$
$\leq \frac{1}{4\varepsilon}\|e(t)\|^2_{H^1(\Omega)} + \varepsilon\|P_h u - u_t\|^2_{H^1(\Omega)}$ (11)

For B_3, we obtain
$B_3 \leq c_1\|u - P_h u\|_{H^1(\Omega)}\|P_h u - u_h\|_{H^1(\Omega)}$
$\leq (c_1 + c_1^2\varepsilon)\|P_h u - u_t\|^2_{H^1(\Omega)} + \frac{1}{4\varepsilon}\|e(t)\|^2_{H^1(\Omega)}$ (12)
$B_4 \leq Ch^2\|f\|_{H^2(\Omega)}\|P_h u - u_h\|_{H^1(\Omega)}$
$+ Ch^2\|u_t\|_X\|P_h u - u_h\|_{H^1(\Omega)}$
$\leq C(\varepsilon)h^2\left(1 + \frac{1}{|\ln h|}\right)\left(\|f\|^2_{H^2(\Omega)} + \|u_t\|^2_X + \|u\|^2_X\right) +$
$\frac{1}{4\varepsilon}\|e(t)\|^2_{H^1(\Omega)}$ (13)

Using Lemma 2.2,
$B_5 \leq Ch^{3/2}\|g\|_{H^2(\Gamma)}\|P_h u - u_h\|_{H^1(\Omega)}$
$\leq Ch^3(\varepsilon + 1)\|g\|^2_{H^2(\Gamma)} + Ch^2\left(1 + \frac{1}{|\ln h|}\right)\|u\|^2_X$ (14)

We substitute (10)-(14) into (9) and simplify the resulting expression taking $\varepsilon = \frac{5}{2c_1}$ we obtain, for h sufficiently small,

$$\frac{c_1}{2}\|e(t)\|^2_{H^1(\Omega)} \leq Ch^2\left(1 + \frac{1}{|\ln h|}\right)\left(\|g\|^2_{H^2(\Gamma)} + \|f\|^2_{H^2(\Omega)} + \|u\|^2_X + \|u_t\|^2_X\right)$$

(8) follows immediately.

Proof of Theorem 3.2 Subtract the last equation in (6) from (5)

$$\left(u_{h,t}(t_n) - \partial^4 U_h^n, u_h\right)_h + A_h(u_h(t_n) - U_h^n, u_h) = 0$$

Let $v_h = u_h(t_n) - U_h^n$, it is easy to see that

$$\|u_h(t_n) - U_h^n\|_{H^1(\Omega)} \leq C\left\|u_{h,t}(t_n) - \partial^4 U_h^n\right\|_{L^2(\Omega)} \leq Ck^4\left\|\frac{\partial^5 u_h}{\partial t^5}(t_n)\right\|_{L^2(\Omega)} \quad n = 4,5,\ldots$$

A similar approach to the other equations in (6) gives

$$\|u_h(t_1) - U_h^1\|_{H^1(\Omega)} \leq C\tau_1\left\|\frac{\partial^2 u_h}{\partial t^2}(t_1)\right\|_{L^2(\Omega)}$$

$$\|u_h(t_2) - U_h^2\|_{H^1(\Omega)} \leq C\tau_2^2\left\|\frac{\partial^3 u_h}{\partial t^3}(t_2)\right\|_{L^2(\Omega)}$$

$$\|u_h(t_3) - U_h^3\|_{H^1(\Omega)} \leq C\tau_3^3\left\|\frac{\partial^4 u_h}{\partial t^4}(t_3)\right\|_{L^2(\Omega)}$$

Taking τ_1, τ_2, τ_3 small enough such that $\tau_1 \leq k^4$, $\tau_2 \leq k^2$, $\tau_3 \leq k^{4/3}$, we have

$$\|u_h(t_n) - U_h^n\|_{H^1(\Omega)} \leq C(u)k^4, \quad n = 1,2,\ldots. \quad (15)$$

The result follows from (8) and (15).

4. Numerical Results

For the numerical experiment, globally continuous piecewise linear finite element functions based on quasi-uniform triangulation described in Section 2 are used. The mesh generation and computation are done with FreeFEM ++ [19].

Example 4.1. We discuss the result of a two-dimensional linear parabolic interface problem in the domain $\Omega = (-2,2) \times (-2,2)$ where Γ is a semicircle centered at (2,0) with radius 2. $\Omega_1 = \{(x,y) \in \mathbb{R}^2 : (x-2)^2 + y^2 < 4\}$ $\Omega_2 = \Omega \backslash \Omega_1$.

Consider the problem (1) – (3) in $\Omega \times (0,T]$, $T < \infty$. For the exact solution, we choose

$$u = \begin{cases} \frac{1}{2}(x^3 - 6x^2 + xy^2 + 8x - 2y^2)\sin t & \text{in } \Omega_1 \times (0,\mathrm{T}] \\ (4x - x^2 - y^2)\cos(0.25\pi x)\cos(0.25\pi y)t\exp(-t) & \text{in } \Omega_1 \times (0,\mathrm{T}] \end{cases}$$

We choose a and b as

$$a = \begin{cases} x^2 & \text{in } \Omega_1 \\ 2 & \text{in } \Omega_2 \end{cases} \qquad b = \begin{cases} 1 & \text{in } \Omega_1 \\ 2 & \text{in } \Omega_2 \end{cases}$$

The source function f, the interface function g and the initial data u_0 are determined from the choice of u. The H^1-norm errors a $T = 2$ for various step size k and mesh parameter h are presented in Table 1.

Table 1. Numerical results for Example 4.1

h	Error ($k = 0.0001$)
0.4721640	8.32632×10^{-1}
0.2555920	4.00683×10^{-1}
0.1244050	2.00008×10^{-1}
0.0646922	9.91988×10^{-2}

k	Error ($h = 0.0253896$)
0.200	3.35081×10^{-2}
0.125	3.34680×10^{-2}
0.100	3.34640×10^{-2}
0.080	3.34618×10^{-2}

For a fixed h and varying k, the error is almost constant which shows the error is mainly due to refinement of the domain, however the second graph of figure 3 shows that error $\cong c_1 + c_2k^{3.944}$ for a fixed h where $c_1, c_2 > 0$. It can be seen from Table 1 that

$$\text{Error} \cong O\left(k^{3.944} + h^{0.943}\left(1 + \frac{1}{|\ln h|}\right)^{1/2}\right)$$

Table 2 shows the case where both k and h vary simultaneously. To achieve this, we choose $h \approx k^4$.

Table 2. Numerical results for Example 4.1 where both k and h vary simultaneously

k	H	Error	Rate
$\frac{1}{2}$	0.472164	8.34867×10^{-1}	
$\frac{1}{16}$	0.228813	3.76739×10^{-1}	1.0984
$\frac{1}{65536}$	0.124405	2.00008×10^{-1}	1.0391

Although the analysis was carried out for the case $u(x,t) = 0$ on $\partial\Omega$, the error estimate and the stability result also apply to the case $u(x,t) \neq 0$ on $\partial\Omega$. We demonstrate this with the next example.

Example 4.2. We consider problem (1) – (3) in $\Omega \times (0,\mathrm{T}]$ where $T < \infty$ and $\Omega = (-1,1) \times (-1,1)$. $\Omega_1 = \{(x,y) \in$

$\Omega: x^2 + y^2 < 0.25\}, \Omega_2 = \Omega \backslash \Omega_1$ and the interface Γ is a circle centered ate (0,0) with radius 0.5. For the exact solution, we chose

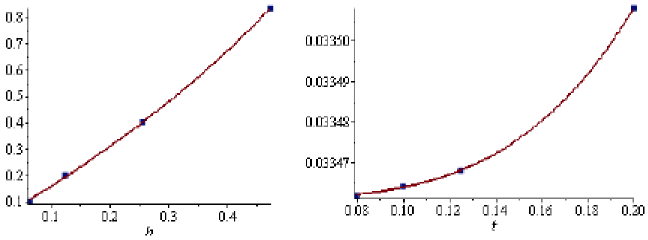

Figure 3. The graphs show the convergence behaviour as given in Table 1

$$u = \begin{cases} (0.25 - x^2 - y^2)\ln(t+1) + 0.75\sin(t) \text{ in } \Omega_1 \times (0,\mathrm{T}] \\ (1 - x^2 - y^2)\sin t \qquad\qquad\qquad\qquad \text{in } \Omega_1 \times (0,\mathrm{T}] \end{cases} \quad (15)$$

and

$$a = \begin{cases} 2 & \text{in } \Omega_1 \\ 1 & \text{in } \Omega_2 \end{cases} \qquad b = \begin{cases} 3 & \text{in } \Omega_1 \\ 0 & \text{in } \Omega_2 \end{cases}$$

The source function f, interface function g, initial data u_0 and the boundary conditions are determined from the choice of u. The H^1-norm errors at $T = 3$ for various step size k and mesh parameter h are presented in the Table 3.

Table 3. Numerical results for Example 4.1

h	Error ($k = 0.0005$)
0.2481840	1.43037×10^{-1}
0.1267240	6.86941×10^{-2}
0.0695941	3.44080×10^{-2}
0.0646922	1.69453×10^{-2}

k	Error ($h = 0.0140586$)
0.30	6.47582×10^{-3}
0.25	6.47249×10^{-3}
0.20	6.46947×10^{-3}
0.15	6.46923×10^{-3}

It can be seen from Table 3 that

$$\text{Error} \cong O\left(k^{3.901} + h^{0.981}\left(1 + \frac{1}{|\ln h|}\right)^{1/2}\right)$$

Table 4 shows the case where both k and h vary simultaneously.

Table 4. Numerical results for Example 4.2 where both k and h vary simultaneously

k	h	Error	Rate
$\frac{1}{2}$	0.2223820	1.20306×10^{-1}	
$\frac{1}{16}$	0.1092940	5.97438×10^{-2}	0.985
$\frac{1}{65536}$	0.0615149	2.98554×10^{-2}	1.207

To give a visual understanding of results for example 4.2, Figure 4 illustrates the solution of example 4.2 with $h = 0.0311204$, $k = 0.001$.

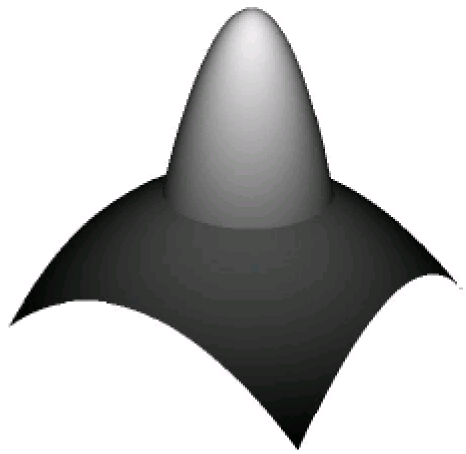

Figure 4. Solution of Example 4.2 with $h = 0.0311204$, $k = 0.001$

5. Conclusion

We established that the scheme proposed in [1] converges in $H^1(\Omega)$-norm. In the analysis, it was assumed that $\frac{\partial^5 u}{\partial t^5}$ exists, however if the regularity of the solutions with respect to time is very low, the result obtained from the method may not be different from other low-order time discretization methods. It was also assumed that the mesh cannot perfectly fit the interface, however, with the assumption that the interface can be fitted exactly using interface elements with curved edges, optimal convergence rate is possible (see [20] for example).

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