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ON FINITE ELEMENT METHOD FOR LINEAR HYPERBOLIC INTERFACE PROBLEMS

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ABSTRACT. We investigate the error contributed by semi discretization to the finite element solution of linear hyperbolic interface problems. With low regularity assumption on the solution across the interface, almost optimal convergence rates in $L^2(\Omega)$ and $H^1(\Omega)$ norms are obtained. We do not assume that the interface could be fitted exactly. Numerical experiments are presented to support the theoretical results.

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1. INTRODUCTION

Interface problems have varieties of applications in scientific computing. The most well-known linear hyperbolic PDE is the wave equation, which becomes an interface problem when materials with different properties are involved [8, 12]. The solutions of interface problems may have higher regularities in each individual material region than in the entire physical domain because of the discontinuities across the interface [4, 9]. Thus, achieving higher order accuracy may be difficult using the classical method.

The study of interface problems by Finite Element Method (FEM) was first carried out by Babuska [4] who studied finite element approximation to elliptic interface problems on smooth domains with a smooth interface. Finite element solution of interface problem has since gained attention of researchers. For recent works on elliptic and parabolic interface problems, see [2, 3, 18, 20, 21, 23, 24] and the references therein, and [5, 6, 7, 13, 15, 17, 19] for works on hyperbolic non-interface problems.

Let Ω be a convex polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ and $\Omega_1 \subset \Omega$ be an open domain with smooth boundary $\Gamma = \partial\Omega_1$. Let $\Omega_2 = \Omega \setminus \overline{\Omega}_1$ be another open domain contained in Ω with boundary

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$\partial\Omega_1 \cup \partial\Omega$, see Fig. 1. We consider the hyperbolic interface problem

$$u_{tt} - \nabla \cdot (a(x,t)\nabla u) + b(x,t)u = f(x,t) \quad \text{in} \quad \Omega \times (0,T] \tag{1}$$

with boundary condition

$$\left\{ \begin{array}{rcl} u(x,t) & = & 0 \quad \text{on } \partial\Omega \times [0,T] \,, \end{array} \right. \tag{2}$$

initial conditions

$$\begin{array}{rcl} u(x,0) & = & u_0(x) \quad \text{in } \Omega \\ u_t(x,0) & = & u_1(x) \quad \text{in } \Omega \end{array} \tag{3}$$

and interface conditions

$$\left\{ \begin{array}{rcl} \lim\limits_{x\to m^+} u_1(x) - \lim\limits_{x\to m^-} u_2(x) & = & 0 \\ \left[\lim\limits_{x\to m^+} a_1 \nabla u_1(x) - \lim\limits_{x\to m^-} a_2 \nabla u_2(x)\right] \cdot n & = & g(x,t) \end{array} \right. \tag{4}$$

for $m \in \Gamma$ and $T \in (0,\infty)$ and n is the unit outward normal to the boundary $\partial\Omega_1$. The input functions $a(x,t)$, $b(x,t)$ and $f(x,t)$ are assumed continuous on each domain but discontinuous across the interface for $t \in [0,T]$.

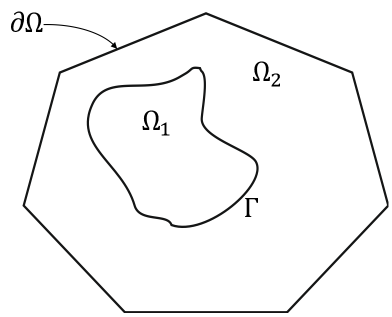

Fig. 1. A polygonal domain $\Omega = \Omega_1 \cup \Omega_2$ with interface Γ.

The convergence of finite element solution of problem (1) satisfying conditions (2)−(4) has been considered in [12]. With the assumption that the interface can be fitted exactly using interface elements with curved edges, the authors established convergence rates of optimal order in L^2- and H^1-norms for both semi and full discretizations. For the fully discrete scheme, the time discretization was based on symmetric difference approximation around the nodal points. Discrete projection operators were used in their analysis. Deka and Ahmed [11] investigated the convergence of finite element solution of an homogenous hyperbolic interface problem. Convergence rates of optimal order were obtained for both semi and full discretizations. Their time discretization was based on symmetric difference approximation around the nodal points and approximation properties of interpolation as well as projection operators were used in their analysis.

In practice, the use of curved interface elements may be computationally difficult or impossible particularly when the interface is irregular in shape and therefore has to be approximated. In this work, we do not assume that the interface could be fitted exactly. Under certain regularity assumptions on the data of the problem, we obtain almost optimal order of convergence in the $L^2(\Omega)$- and $H^1(\Omega)$-norms for spatial discretization. In our analysis, the linear theories of interface problems, Sobolev imbedding inequalities and approximation properties of elliptic projection operator are used with the assumption that $g(x,t) \in H^{1/2}(\Gamma) \cap H^2(\Gamma)$ and $f_i(x,t) \in H^2(\Omega_i)$ for $i = 1, 2$, $t \in [0,T]$.

For a given Banach space B, we define

$$W^{m,p}(0,T;B)$$

$$= \begin{cases} u(t) \in B \text{ for a.e. } t \in (0,T) & \text{and } \displaystyle\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt < 0 \\ & \text{for } 1 \leq p < \infty \\ u(t) \in B \text{ for a.e. } t \in (0,T) & \text{and } \displaystyle\sum_{i=0}^{m} \operatorname*{ess\,sup}_{0\leq t\leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B < 0 \\ & \text{for } p = \infty \end{cases}$$

equipped with the norms

$$\|u\|_{W^{m,p}(0,T;B)} = \begin{cases} \left[\displaystyle\sum_{i=0}^{m} \int_0^T \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B^p dt \right]^{1/p} & 1 \leq p < \infty \\ \displaystyle\sum_{i=0}^{m} \operatorname*{ess\,sup}_{0\leq t\leq T} \left\| \frac{\partial^i u}{\partial t^i}(t) \right\|_B & p = \infty \end{cases}$$

We write $L^2(0,T;B) = W^{0,2}(0,T;B)$ and $H^m(0,T;B) = W^{m,2}$ $(0,T;B)$. $H^{1/2}(\partial\Omega)$ is the space

$$\left\{ v \in L^p(\partial\Omega) \mid \frac{|v(x) - v(y)|^2}{\|x-y\|^{1+n}} < \infty \right\}$$

with the norm

$$\|u\|_{H^{1/2}(\partial\Omega)} = \left[\|v\|^2_{L^p(\partial\Omega)} + \int_{\partial\Omega\times\partial\Omega} \frac{|v(x) - v(y)|^2}{\|x-y\|^{1+n}} \, dxdy \right]^{1/2}$$

We also use the following spaces

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)\,, \quad Y = L^2(\Omega) \cap H^1(\Omega_1) \cap H^1(\Omega_2)$$

equipped with the norms

$$\|v\|_X = \|v\|_{H^1(\Omega)} + \|v\|_{H^2(\Omega_1)} + \|v\|_{H^2(\Omega_2)} \quad \forall\, v \in X$$
$$\|v\|_Y = \|v\|_{L^2(\Omega)} + \|v\|_{H^1(\Omega_1)} + \|v\|_{H^1(\Omega_2)} \quad \forall\, v \in Y.$$

For $f_i(x,t) \in H^2(\Omega_i)$, $i = 1,2$, we define

$$\|f\|_{H^2(\Omega)} = \|f_1\|_{H^2(\Omega_1)} + \|f_2\|_{H^2(\Omega_2)}\,, \quad t \in [0,T].$$

We recall that for $u \in H^1(\Omega)$, the boundary value of u (ie $u_{|\partial\Omega}$) is defined on $H^{1/2}(\partial\Omega)$ the trace space of $H^1(\Omega)$. Similarly, the trace space on the interface Γ is $H^{1/2}(\Gamma)$. The trace operator from $H^1(\Omega)$ to $H^{1/2}(\partial\Omega)$ is continuous and satisfies the embedding

$$\|z\|_{H^{1/2}(\partial\Omega)} \le c_0 \|z\|_{H^1(\Omega)} \quad \forall\, z \in H^1(\Omega)\,.$$

See [1, 14] for more information on trace operator.

The weak form of (1)$-$(4) is to find $u \in H_0^1(\Omega)$ such that

$$(u_{tt}, v) + A(u,v) = (f,v) + \langle g, v\rangle_\Gamma \quad \forall\, v(t) \in H_0^1(\Omega), \text{ a.e. } t \in [0,T] \tag{5}$$

with $u(0) = u_0$ and $u_t(0) = u_1$. Here

$$(\phi,\psi) = \int_\Omega \phi\psi\, dx \qquad A(\phi,\psi) = \int_\Omega [a(x,t)\nabla\phi \cdot \nabla\psi + b(x,t)\phi\psi]\, dx$$
$$\langle \phi,\psi\rangle_\Gamma = \int_\Gamma \phi\psi\, d\Gamma\,.$$

Regarding the regularity of the solutions of the interface problem (1)$-$(4), we have the following result:

Theorem 1: Let $f \in H^1(0,T;L^2(\Omega))$, $g \in H^1(0,T;H^{1/2}(\Gamma))$ and $u_0, u_1 \in H_0^1(\Omega)$. Then problem (1) together with conditions (2)$-$(4) has a unique solution

$$u \in L^2(0,T;X \cap H_0^1(\Omega)) \cap H^1(0,T;H^2(\Omega_1) \cap H^2(\Omega_2)) \cap H^2(0,T;Y)$$

Proof: See [12].

The remaining part of the paper is organized as follows. In Section 2, we describe a finite element discretization of the problem, establish an error estimate for the interpolation operator and state approximations across the interface. In Section 3, we prove convergence rates of almost optimal order for the semi discrete scheme. Numerical examples are presented in Section 4 and conclusion is made in section 5. Throughout this paper, C is a generic positive constant which is independent of the mesh parameter h and may take on different values at different occurrences.

2. FINITE ELEMENT DISCRETIZATION

We adopt the discretization used in [2, 9]. $\mathcal{T}_h$ denotes a partition of Ω into disjoint triangles K (called elements) such that no vertex of any triangle lies on the interior or side of another triangle. The domain Ω_1 is approximated by a domain Ω_1^h with a polygonal boundary Γ_h whose vertices all lie on the interface Γ. Ω_2^h represents the domain with $\partial\Omega$ and Γ_h as its exterior and interior boundaries respectively.

Let h_K be the diameter of an element $K \in \mathcal{T}_h$ and $h = \max_{K\in\mathcal{T}_h} h_K$. Let $\mathcal{T}_h^\star$ denote the set of all elements that are intersected by the interface Γ;

$$\mathcal{T}_h^\star = \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$$

$K \in \mathcal{T}_h^\star$ is called an interface element and we write $\Omega_h^\star = \bigcup_{K\in\mathcal{T}_h^\star} K$. The triangulation $\mathcal{T}_h$ of the domain Ω satisfies the following conditions

- $\bar{\Omega} = \bigcup\limits_{K\in\mathcal{T}_h} \bar{K}$
- If $\bar{K}_1, \bar{K}_2 \in \mathcal{T}_h$ and $\bar{K}_1 \neq \bar{K}_2$, then either $\bar{K}_1 \cap \bar{K}_2 = \emptyset$ or $\bar{K}_1 \cap \bar{K}_2$ is a common vertex or a common edge.
- Each $K \in \mathcal{T}_h$ is either in Ω_1^h or Ω_2^h, and has at most two vertices lying on Γ_h.
- For each element $K \in \mathcal{T}_h$, let r_K and $\bar{r}_K$ be the diameters of its inscribed and circumscribed circles respectively. It is assumed that, for some fixed $h_0 > 0$, there exist two positive constants C_0 and C_1, independent of h, such that

$$C_0 r_K \le h \le C_1 \bar{r}_K \quad \forall\, h \in (0, h_0)$$

Let $S_h \subset H_0^1(\Omega)$ denote the space of continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$.
The finite element solution $u^h(x,t) \in S_h$ is represented as

$$u^h(x,t) = \sum_{j=1}^{N_h} \alpha_j(t)\phi_j(x)\ ,$$

where each basis function ϕ_j, $(j = 1, 2, \ldots, N_h)$ is a pyramid function with unit height. For the approximation $g_h(x,t)$, let $\{z_j\}_{j=1}^{n_h}$ be the set of all nodes of the triangulation $\mathcal{T}_h$ that lie on the interface

Γ and $\{\psi_j\}_{j=1}^{n_h}$ be the hat functions corresponding to $\{z_j\}_{j=1}^{n_h}$ in the space S_h.

Lemma 1: For the linear interpolation operator $\pi_h : C(\bar{\Omega}) \to S_h$, we have, for $m = 0, 1$, and $0 < h < 1$

$$\|u - \pi_h u\|_{H^m(\Omega)} \leq Ch^{2-m}\left(1 + \frac{1}{|\ln h|}\right)^{1/2} \|u\|_X \qquad \forall\, u \in X \quad (6)$$

Proof: See [2].

The results in Lemma 2 take the effect of the interface approximation into account

Lemma 2: Assume that $g \in H^2(\Gamma)$, $f \in H^2(\Omega)$ and $\nu_h, \omega_h \in S_h$. Then we have

$$\begin{aligned} |\langle g, v_h\rangle_\Gamma - \langle g_h, v_h\rangle_{\Gamma_h}| &\leq Ch^{3/2}\|g\|_{H^2(\Gamma)}\|v_h\|_{H^1(\Omega_h^\star)} && (7)\\ \|v\|_{H^1(\Omega_h^\star)} &\leq Ch^{1/2}\|v\|_X \quad \forall\, v \in X && (8)\\ |(f, v) - (f, v)_h| &\leq Ch^2\|f\|_{H^2(\Omega)}\|v\|_{H^1(\Omega)} && (9) \end{aligned}$$

Proof: See [9] for (7), [22] for (8) and [10] for (9).

3. CONTINUOUS TIME ERROR ESTIMATES

In this section, we establish the error estimates of the finite element solution of problem (1) with conditions (2)−(4). The semidiscrete version of (5) is stated as:

find $u^h : [0, T] \to S_h$ such that $u^h(0) = u_0^h$, $u_t^h(0) = u_1^h$ and satisfies

$$(u_{tt}^h, v_h)_h + A_h(u^h, v_h) = (f, v_h)_h + \langle g_h, v_h\rangle_{\Gamma_h} \qquad \forall\, v_h \in S_h,\ \text{a.e } t \in [0, T] \quad (10)$$

where, $A_h(\phi, \psi)$ and $(\xi, v_h)_h$ are defined as

$$A_h(\phi, \psi) = \sum_{K\in\mathcal{T}_h} \int_K [a\nabla\phi \cdot \nabla\psi + b\phi\psi]\, dx$$

$$(\xi, \phi)_h = \sum_{K\in\mathcal{T}_h} \int_K \xi\phi\, dx \quad \forall\, \phi, \psi \in H^1(\Omega),\ t \in [0, T]$$

Let $P_h : X \cap H_0^1(\Omega) \to S_h$ be the elliptic projection of the exact solution u in S_h defined by

$$A_h(P_h\nu, \phi) = A(\nu, \phi) \quad \forall\, \phi \in S_h,\ t \in [0, T] \qquad (11)$$

For this projection, we have

Lemma 3: Let $a_{tt}(x, t)$, $b_{tt}(x, t)$ be continuous on $\Omega_i \times (0, T]$, $i =$

$1, 2$. Assume that $u \in X \cap H_0^1$ and let $P_h u$ be defined as in (11), then

$$\begin{aligned} \|\frac{\partial^n}{\partial t^n}(P_h u - u)\|_{H^1(\Omega)} &\leq Ch\left(1 + \frac{1}{|\ln h|}\right)^{1/2} \sum_{i=1}^{n} \|\frac{\partial^i u}{\partial t^i}\|_X \\ \|\frac{\partial^n}{\partial t^n}(P_h u - u)\|_{L^2(\Omega)} &\leq Ch^2\left(1 + \frac{1}{|\ln h|}\right) \sum_{i=1}^{n} \|\frac{\partial^i u}{\partial t^i}\|_X \end{aligned}$$

for $n = 0, 1, 2$.
Proof: See [2].
Below are the main results concerning the convergence of the semi-discrete solution to the exact solution in the $H^1(\Omega)$-norm and $L^2(\Omega)$-norm respectively:
Theorem 2: Let u and u^h be the solutions of (5) and (10) respectively with $u_0 \in X \cap H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Suppose $a_i(x,t)$ and $b_i(x,t)$ are continuous on $\Omega_i \times (0,T]$, $i = 1, 2$, $g(x,t) \in H^1(0,T; H^2(\Gamma))$, and $f_i(x,t) \in H^1(0,T; H^2(\Omega_i))$. There exists a positive constant C independent of h such that

$$\max_{0 \leq t \leq T} \|u - u^h\|_{H^1(\Omega)} \leq h\left(1 + \frac{1}{|\ln h|}\right)^{1/2} C$$

Proof: Subtract (10) from (5)

$$\begin{aligned} &(u_{tt} - u_{tt}^h, v_h) + A(u, v_h) \\ &\quad = A_h(u^h, v_h) + (f(x,u), v_h) - (f(x,u^h), v_h)_h \\ &\qquad + \langle g, v_h \rangle_\Gamma - \langle g_h, v_h \rangle_{\Gamma_h} + (u_{tt}^h, v_h)_h - (u_{tt}^h, v_h) \quad \forall\, v_h \in S_h \end{aligned}$$

Let $e(t) = u - u^h$, $v_h = (P_h u - u^h)_t$ and use (11)

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|e'(t)\|_{L^2(\Omega)}^2 + \frac{\mu}{2}\frac{d}{dt}\|e(t)\|_{H^1(\Omega)}^2 \\ &\qquad \leq (u_{tt}^h - u_{tt}, (P_h u - u)_t) + A_h(e(t), (u - P_h u)_t) \\ &\qquad\quad + A_h(u, (P_h u - u^h)_t) - A_h(P_h u, (P_h u - u^h)_t) \\ &\qquad\quad + (f(x,t), (P_h u - u^h)_t) - (f(x,t), (P_h u - u^h)_t)_h \\ &\qquad\quad + \langle g, (P_h u - u^h)_t \rangle_\Gamma - \langle g_h, (P_h u - u^h)_t \rangle_{\Gamma_h} \\ &\qquad \leq B_1 + B_2 + B_3 + B_4 \end{aligned} \tag{12}$$

where

$$\begin{aligned} B_1 &= |(u_{tt} - u_{tt}^h, (P_h u - u)_t)| \qquad B_2 = |A_h(e(t), (u - P_h u)_t)| \\ B_3 &= |A_h(u - P_h u, (P_h u - u^h)_t)| \\ B_4 &= |(f(x,t), (P_h u - u^h)_t) - (f(x,t), (P_h u - u^h)_t)_h| \\ &\qquad + |\langle g, (P_h u - u^h)_t \rangle_\Gamma - \langle g_h, (P_h u - u^h)_t \rangle_{\Gamma_h}| \end{aligned}$$

For B_1, we have

$$\begin{aligned}
B_1 &= |\frac{d}{dt}(e'(t),(P_hu-u)_t)-(e'(t),(P_hu-u)_{tt})| \\
&\leq \frac{1}{4}\frac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)}+\frac{d}{dt}\|(P_hu-u)_t\|^2_{L^2(\Omega)}+\frac{1}{4}\|e'(t)\|^2_{L^2(\Omega)} \\
&\quad +\|(P_hu-u)_{tt}\|^2_{L^2(\Omega)} \\
&\leq \frac{1}{4}\frac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)}+\frac{1}{4}\|e'(t)\|^2_{L^2(\Omega)}+\|(P_hu-u)_t\|^2_{L^2(\Omega)} \\
&\quad +2\|(P_hu-u)_{tt}\|^2_{L^2(\Omega)} \qquad (13)
\end{aligned}$$

$$\begin{aligned}
B_2 &\leq \max\{a,b\}\|e(t)\|_{H^1(\Omega)}\|(u-P_hu)_t\|_{H^1(\Omega)} \\
&\leq \max\{a,b\}\left[\frac{1}{4}\|e(t)\|^2_{H^1(\Omega)}+\|(P_hu-u)_t\|^2_{H^1(\Omega)}\right] \qquad (14)
\end{aligned}$$

By Holder's and Young's inequalities , we obtain

$$\begin{aligned}
B_3 \;\leq\; \max\{a,b\}&\left[\frac{1}{2}\|P_hu-u\|^2_{H^1(\Omega)}+\|(P_hu-u)_t\|^2_{H^1(\Omega)}\right. \\
&\left.+\frac{1}{2}\|e(t)\|^2_{H^1(\Omega)}+\frac{d}{dt}\left(\varepsilon\|P_hu-u\|^2_{H^1(\Omega)}+\frac{1}{4\varepsilon}\|e(t)\|^2_{H^1(\Omega)}\right)\right] \qquad (15)
\end{aligned}$$

Using Lemma 2,

$$\begin{aligned}
B_4 &\leq Ch^2\|f\|_{H^2(\Omega)}\|(P_hu-u)_t\|_{H^1(\Omega)}+\frac{d}{dt}\left[Ch^2\|f\|_{H^2(\Omega)}\|e(t)\|_{H^1(\Omega)}\right] \\
&\quad +Ch^2\|f'\|_{H^2(\Omega)}\|e(t)\|_{H^1(\Omega)}+Ch^2\|g\|_{H^2(\Gamma)}\|(P_hu-u)_t\|_{H^1(\Omega)} \\
&\quad +\frac{d}{dt}\left[Ch^2\|g\|_{H^2(\Gamma)}\|e(t)\|_{H^1(\Omega)}\right]+Ch^2\|g'\|_{H^2(\Gamma)}\|e(t)\|_{H^1(\Omega)} \\
&\leq Ch^4\left[\|f\|^2_{H^2(\Omega)}+\|f'\|^2_{H^2(\Omega)}+\|g\|^2_{H^2(\Gamma)}+\|g'\|^2_{H^2(\Gamma)}\right] \\
&\quad +\|(P_hu-u)_t\|^2_{H^1(\Omega)}+\frac{1}{2}\|e(t)\|^2_{H^1(\Omega)} \\
&\quad +\frac{d}{dt}\left[Ch^4\varepsilon(\|f\|^2_{H^2(\Omega)}+\|g\|^2_{H^2(\Gamma)})+\frac{1}{2\varepsilon}\|e(t)\|_{H^1(\Omega)}\right] \qquad (16)
\end{aligned}$$

We substitute (13)−(16) into (12), use Lemma 3 and simplify the resulting expression taking $\varepsilon = 3/\mu$. We obtain, for h sufficiently small,

$$\begin{aligned}
&\frac{1}{4}\frac{d}{dt}\|e'(t)\|^2_{L^2(\Omega)} + \frac{\mu}{4}\frac{d}{dt}\|e(t)\|^2_{H^1(\Omega)} \\
&\quad\leq \frac{1}{4}\|e'(t)\|^2_{L^2(\Omega)} + \gamma\|e(t)\|^2_{H^1(\Omega)} \\
&\qquad + Ch^2\left(1+\frac{1}{|\ln h|}\right)\Big(\|f\|^2_{H^2(\Omega)} + \|f'\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} \\
&\qquad + \|g'\|^2_{H^2(\Gamma)} + \|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X\Big) \\
&\qquad + Ch^4\frac{d}{dt}\left[\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right]
\end{aligned}$$

where $\gamma = 1 + \frac{1}{4}\max\{a,b\}$. It follows that

$$\begin{aligned}
&\frac{\mu}{4}\frac{d}{dt}\left[\exp\left(-\frac{4\gamma}{\mu}t\right)\|e(t)\|^2_{H^1(\Omega)}\right] \\
&\quad\leq \exp\left(-\frac{4\gamma}{\mu}t\right)Ch^2\left(1+\frac{1}{|\ln h|}\right)\Big(\|f\|^2_{H^2(\Omega)} + \|f'\|^2_{H^2(\Omega)} \\
&\qquad + \|g\|^2_{H^2(\Gamma)} + \|g'\|^2_{H^2(\Gamma)} + \|u\|^2_X + \|u_t\|^2_X + \|u_{tt}\|^2_X\Big) \\
&\qquad + \exp\left(-\frac{4\gamma}{\mu}t\right)Ch^4\frac{d}{dt}\left[\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right] .
\end{aligned}$$

It follows by integration that

$$\begin{aligned}
\|e(t)\|^2_{H^1(\Omega)} \;&\leq\; \exp\left(\frac{4\gamma}{\mu}t\right)\|e(0)\|^2_{H^1(\Omega)} \\
&\quad + Ch^2\left(1+\frac{1}{|\ln h|}\right)\int_0^t\left[\exp\left(\frac{4\gamma}{\mu}(t-s)\right)\Big(\|f\|^2_{H^2(\Omega)}\right. \\
&\quad + \|f'\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)} + \|g'\|^2_{H^2(\Gamma)} + \|u\|^2_X + \|u_t\|^2_X \\
&\quad \left. + \|u_{tt}\|^2_X\Big)\right] ds + Ch^4\left[\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right] .
\end{aligned}$$

The result follows by taking $u_0^h = \pi_h u_0$ and using Lemma 1.

Theorem 3: Let u and u^h be the solutions of (5) and (10) respectively with $u_0, u_1 \in X \cap H_0^1(\Omega)$. Suppose $a_i(x,t)$ and $b_i(x,t)$ are continuous on $\Omega_i \times (0,T]$, $i = 1,2$, $g(x,t) \in H^1(0,T;H^2(\Gamma))$, and $f_i(x,t) \in H^1(0,T;H^2(\Omega_i))$. There exists a positive constant C independent of h such that

$$\max_{0\leq t\leq T}\|u-u^h\|_{L^2(\Omega)} \;\leq\; h^2\left(1+\frac{1}{|\ln h|}\right)C$$

Proof: Using (11), we have

$$
\begin{aligned}
&((u^h - P_h u)_{tt}, v_h)_h + A_h(u^h - P_h u, v_h) \\
&\quad = ((u - P_h u)_{tt}, v_h) + (f(x,t), v_h)_h - (f(x,t), v_h) + \langle g_h, v_h\rangle_{\Gamma_h} \\
&\qquad - \langle g, v_h\rangle_\Gamma + ((P_h u)_{tt}, v_h) - ((P_h u)_{tt}, v_h)_h \qquad (17)
\end{aligned}
$$

We take $v_h = (u^h - P_h u)_t$ and make use of Lemma 2

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(u^h - P_h u)_t\|^2_{L^2(\Omega)} + \frac{\mu}{2}\frac{d}{dt}\|u^h - P_h u\|^2_{H^1(\Omega)} \\
&\quad \leq \|(u^h - P_h u)_t\|_{L^2(\Omega)}\|(u - P_h u)_{tt}\|_{L^2(\Omega)} \\
&\qquad + Ch^2\|u_{ttt}\|_{H^1(\Omega)}\|u^h - P_h u\|_{H^1(\Omega)} \\
&\qquad + Ch^2\frac{d}{dt}\left[\|u_{tt}\|_{H^1(\Omega)}\|u^h - P_h u\|_{H^1(\Omega)}\right] \\
&\qquad + Ch^2\|f'\|_{H^2(\Omega)}\|u^h - P_h u\|_{H^1(\Omega)} \\
&\qquad + Ch^2\frac{d}{dt}\left[\|f\|_{H^2(\Omega)}\|u^h - P_h u\|_{H^1(\Omega)}\right] \\
&\qquad + Ch^2\|g'\|_{H^2(\Gamma)}\|P_h u - u^h\|_{H^1(\Omega)} \\
&\qquad + Ch^2\frac{d}{dt}\left[\|g\|_{H^2(\Gamma)}\|P_h u - u^h\|_{H^1(\Omega)}\right]
\end{aligned}
$$

Simplifying this using Young's inequality and Lemma 3, we have

$$
\begin{aligned}
\|(u^h - P_h u)_t\|^2_{L^2(\Omega)} \leq\; & Ch^4\left(1 + \frac{1}{|\ln h|}\right)^2 \int_0^t \Big(\|u\|^2_X + \|u_t\|^2_X \\
& + \|u_{tt}\|^2_X + \|f'\|^2_{H^2(\Omega)} + \|g'\|^2_{H^2(\Gamma)}\Big)\, ds \\
& + Ch^4\left(\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right) \\
& + C\|(u_1^h - (P_h u)_t(x,0)\|^2_{L^2(\Omega)}
\end{aligned}
$$

Using Lemma 1 with $u_1^h = \pi_h u_1$, we obtain

$$
\begin{aligned}
\|(u^h - P_h u)_t\|^2_{L^2(\Omega)} \leq\; & Ch^4\left(1 + \frac{1}{|\ln h|}\right)^2 \int_0^t \Big(\|u\|^2_X + \|u_t\|^2_X \\
& + \|u_{tt}\|^2_X + \|f'\|^2_{H^2(\Omega)} + \|g'\|^2_{H^2(\Gamma)}\Big)\, ds \\
& + Ch^4\left(\|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\right) \\
& + Ch^4\left(1 + \frac{1}{|\ln h|}\right)^2 \|u_1\|^2_X \qquad (18)
\end{aligned}
$$

Now, we take $v_h = u^h - P_h u$ in (17) and make use of Lemma 2

$$
\begin{aligned}
&\frac{1}{2}\frac{d^2}{dt^2}\|u^h - P_h u\|^2_{L^2(\Omega)} + \mu\|u^h - P_h u\|^2_{H^1(\Omega)} \\
&\qquad \le \|u^h - P_h u\|_{L^2(\Omega)}\|(u - P_h u)_{tt}\|_{L^2(\Omega)} \\
&\qquad\quad + Ch^2\|(P_h u)_{tt}\|_{H^1(\Omega)}\|u^h - P_h u\|_{H^1(\Omega)} \\
&\qquad\quad + Ch^2\|f\|_{H^2(\Omega)}\|u^h - P_h u\|_{H^1(\Omega)} \\
&\qquad\quad + Ch^2\|g\|_{H^2(\Gamma)}\|P_h u - u^h\|_{H^1(\Omega)} + \|(u^h - P_h u)_t\|^2_{L^2(\Omega)}
\end{aligned}
$$

It follows after a simple calculation using Young's inequality and Lemma 3 that

$$
\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\|u^h - P_h u\|^2_{L^2(\Omega)} \le\ & \frac{1}{2}\|u^h - P_h u\|^2_{L^2(\Omega)} + \|(u^h - P_h u)_t\|^2_{L^2(\Omega)} \\
& + Ch^4\left(1 + \frac{1}{|\ln h|}\right)^2 \Big[\, \|u\|^2_X + \|u_t\|^2_X \\
& + \|u_{tt}\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\Big] .
\end{aligned}
$$

There exists a positive constant γ such that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^h - P_h u\|^2_{L^2(\Omega)} \\
&\qquad \le \frac{1}{2}\gamma T\|u^h - P_h u\|^2_{L^2(\Omega)} + \gamma T\|(u^h - P_h u)_t\|^2_{L^2(\Omega)} \\
&\qquad\quad + \gamma T C h^4\left(1 + \frac{1}{|\ln h|}\right)^2 \Big[\, \|u\|^2_X + \|u_t\|^2_X \\
&\qquad\quad + \|u_{tt}\|^2_X + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\Big] \\
&\qquad\quad + \|u^h_0 - P_h u_0\|^2_{L^2(\Omega)} + \|u^h_1 - P_h u_1\|^2_{L^2(\Omega)} .
\end{aligned}
$$

We take $u^h_0 = \pi_h u_0$, $u^h_1 = \pi_h u_1$ and integrate using (18)

$$
\begin{aligned}
\|u^h - P_h u\|^2_{L^2(\Omega)} \le\ & Ch^4\left(1 + \frac{1}{|\ln h|}\right)^2 \int_0^t \big(\|u\|^2_X + \|u_t\|^2_X \\
& + \|u_{tt}\|^2_X + \|f'\|^2_{H^2(\Omega)} + \|g'\|^2_{H^2(\Gamma)} \\
& + \|f\|^2_{H^2(\Omega)} + \|g\|^2_{H^2(\Gamma)}\Big) \, ds \\
& + Ch^4\left(1 + \frac{1}{|\ln h|}\right)^2 \big(\|u_0\|^2_X + \|u_1\|^2_X\big) \quad (19)
\end{aligned}
$$

By triangle and Young's inequalities,

$$
\|u - u^h\|^2_{L^2(\Omega)} \le 2\|u - P_h u\|^2_{L^2(\Omega)} + 2\|P_h u - u^h\|^2_{L^2(\Omega)} \tag{20}
$$

The result follows from (19), (20) and Lemma 3.

4. NUMERICAL EXPERIMENT

Here, we present examples to verify our results. Globally continuous piecewise linear finite element functions based on the triangulation described in Section 2 are used. The mesh generation and computation are done with FreeFEM++ [16]. For this experiment, our time discretization is based on centered difference scheme with sufficiently small time step.

Example 1: Consider the domain $\Omega = (-1,1)\times(-1,1)$ where the interface Γ is a circle centered at $(0,0)$ with radius 0.5. $\Omega_1 = \{(x,y) : x^2+y^2 < 0.25\}$, $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.
On $\Omega \times (0,T]$, $0 < T < \infty$, we consider the problem (1)$-$(4) whose exact solution, is

$$u = \begin{cases} (0.25 - x^2 - y^2)\sin(0.75t) & \text{in} \quad \Omega_1 \times (0,T] \\ (0.25 - x^2 - y^2)(1-x^2)(1-y^2)t^2\exp(-t) & \text{in} \quad \Omega_2 \times (0,T]. \end{cases}$$

The source function f, interface function g and the initial data u_0, u_1 are determined from the choice of u with

$$a = \begin{cases} 3 & \text{in} \quad \Omega_1 \times (0,T] \\ 2 & \text{in} \quad \Omega_2 \times (0,T] \end{cases} \qquad b = \begin{cases} 0.5 & \text{in} \quad \Omega_1 \times (0,T] \\ 1 & \text{in} \quad \Omega_2 \times (0,T] \end{cases}$$

Errors in L^2 and H^1 norms at $t = 1$ for various step size h are presented in Table 1. The error values indicate that

$$\|\text{Error}\|_{L^2(\Omega)} = O\left(h^{2.026}\left(1 + \frac{1}{|\ln h|}\right)\right)$$

$$\|\text{Error}\|_{H^1(\Omega)} = O\left(h^{1.024}\left(1 + \frac{1}{|\ln h|}\right)^{1/2}\right)$$

This is in agreement with the theoretical results.

Table 1. Error estimates for Example 1.

h	$\|\text{Error}\|_{L^2(\Omega)}$	$\|\text{Error}\|_{H^1(\Omega)}$
2.43762×10^{-1}	8.79308×10^{-3}	1.59443×10^{-1}
8.79955×10^{-2}	1.02248×10^{-3}	5.24571×10^{-2}
6.80150×10^{-2}	6.17282×10^{-4}	3.94249×10^{-2}
4.40113×10^{-2}	3.20224×10^{-4}	2.61162×10^{-2}
3.35494×10^{-2}	2.22805×10^{-4}	1.95413×10^{-2}

Example 2: Consider the domain $\Omega = (-2,2)\times(-2,2)$ where the interface Γ is a semicircle centered at $(2,0)$ with radius 2. $\Omega_1 = \{(x,y) : (x-2)^2 + y^2 < 4\}$, $\Omega_2 = \Omega \setminus \overline{\Omega}_1$.

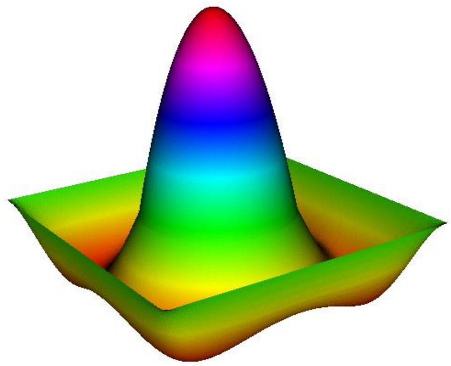

Fig. 2. Solution of Example 1 with $h = 3.35494 \times 10^{-2}$ at $t = 1$.

On $\Omega \times (0, T]$, $0 < T < \infty$, we consider the problem (1)$-$(4) whose exact solution, is

$$u = \begin{cases} (x^3 + xy^2 - 6x^2 - 2y^2 + 8x)t^2 \exp(-t) & \text{in} \quad \Omega_1 \times (0, T] \\ 0.4(4x - x^2 - y^2)\sin(0.5\pi x)\sin(0.5\pi y)t^2 \exp(-t) & \text{in} \quad \Omega_2 \times (0, T]. \end{cases}$$

The source function f, interface function g and the initial data u_0, u_1 are determined from the choice of u with

$$a = \begin{cases} 5 & \text{in} \quad \Omega_1 \times (0, T] \\ 0.5 & \text{in} \quad \Omega_2 \times (0, T] \end{cases} \qquad b = \begin{cases} x^2 + y^2 & \text{in} \quad \Omega_1 \times (0, T] \\ \exp(-t) & \text{in} \quad \Omega_2 \times (0, T] \end{cases}$$

Errors in L^2 and H^1 norms at $t = 2$ for various step size h are presented in Table 2. The error values indicate that

$$\|\text{Error}\|_{L^2(\Omega)} = O\left(h^{2.091}\left(1 + \frac{1}{|\ln h|}\right)\right)$$

$$\|\text{Error}\|_{H^1(\Omega)} = O\left(h^{0.990}\left(1 + \frac{1}{|\ln h|}\right)^{1/2}\right)$$

This is in agreement with the theoretical results.

Table 2. Error estimates for Example 2.

h	$\|\text{Error}\|_{L^2(\Omega)}$	$\|\text{Error}\|_{H^1(\Omega)}$
2.97314×10^{-1}	5.87167×10^{-2}	8.25286×10^{-1}
1.55282×10^{-1}	1.41848×10^{-2}	4.04561×10^{-1}
1.05091×10^{-1}	6.46653×10^{-3}	2.65370×10^{-1}
7.93175×10^{-2}	4.26667×10^{-3}	1.98827×10^{-1}
6.17493×10^{-2}	3.50298×10^{-3}	1.58958×10^{-1}

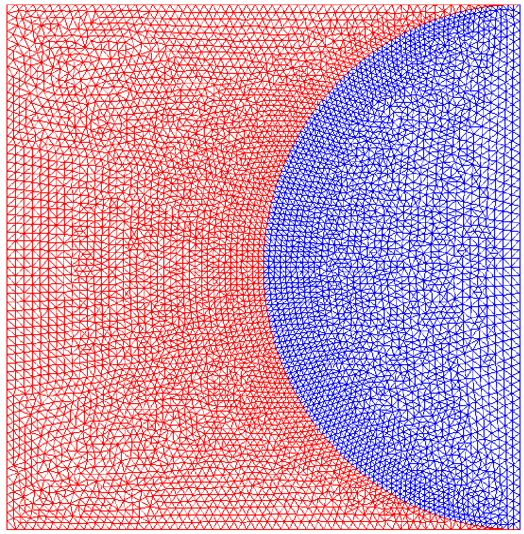

Fig. 3. Solution domain of Example 2 with $h = 0.105091$.

4. CONCLUDING REMARKS

Continuous time approximation of linear hyperbolic interface problems on finite element has been investigated. We assume that the unknown function is of low regularity and obtain convergence rates of almost optimal order in L^2 and H^1 norms. Approximation properties of interpolation and projection operators were used in our analysis. The theoretical results were confirmed numerically.

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