LINEAR ALGEBRAII

COURSE CODE: MTH202

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Course Contents

Chapter 1

Systems of Linear Equation Change of Basis

1.1. Matrix Representation of a Linear Operator

The matrix representation of a linear operator (transformation) T is written in the form

$$M_s[T] = [T]_s = [[T(u)_1]_s\,, [T(u)_2]_s\,, \cdots, [T(u)_n]_s]$$

That is the column of $M(T)$ are the coordinate vectors of $T(u)_1,\ T(u)_2,\ \cdots, T(u)_n$ respectively

Where

$$[T(u)_1] = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$[T(u)_2] = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$

$$\vdots$$

$$[T(u)_n] = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

Example 1

Let $F = R^2 \to R^2$ be the linear operator defined by $F(x, y) = (2x + 3y,\ 4x - 5y)$

i. find the matrix representation of F relative to the basis $S = \{u_{1\,,}u_2\} = \{(1, 2),\ (2, 5)\}$
ii. find the matrix representation of F relative to the (usual) basis

$$S = \{e_{1\,,}e_2\} = \{(1, 0), (0, 1)\}$$

Example 2

Let V be the vector space of functions with basis $S = \{\sin t,\ \ \cos t,\ \ e^{3t}\}$ and let $\boldsymbol{D}: V \to V$ be the differential operator defined by $\boldsymbol{D}(f(t)) = {}^{d(f,t)}/_{dt}$. Find the matrix representing $\boldsymbol{D}$ in the basis S.

1.2. Matrix Mapping and their Matrix Representation.

Example 3

Consider the following matrix A which may be viewed as a linear operator on R^2, and basis S of R^2

$A = \begin{bmatrix} 3 & -2 \\ 4 & -5 \end{bmatrix}$ $and\ S = \{u_1,\ u_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix},\ \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$. Find the matrix representation of S relative to the basis S

Exercise 1

Find the matrix representation of A relative to the usual basis $S = \{e_{1}, e_2\} = \{(1,0), (0,1)\}$

Note:

$[A]_E$ is the original matrix A. This result is true in general.

The matrix representation of any n X n square matrix A over a field K relative to the usual basis E of K^n is the matrix A itself, that is $[A]_E = A$

1.3. Properties of Matrix Representation

Theorem 1.1

Let $T: V \to V$ be a linear operator and let S be a (finite) basis of V. Then, for any vector $v\ in\ V$,

$$[T]_s[v]_s = [[T(v)]_s]$$

Example 4

Consider the linear operator F on R^2 and the basis S given as $F(x, y) = (2x + 3y,\ 4x - 5y)$ and $S = \{u_{1}, u_2\} = \{(1, -2),\ (2, -5)\}$

Let

$$v = (5, -7)$$

Show that the action of an individual linear operator F on a vector v is preserved by its matrix representation.

Example 5

Let G be a linear operator on R^3 defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$

(a) Find the matrix representation of G relative to the basis

$$S = \{w_1, w_2, w_3\} = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

(b) Verify that $[G][v] = [G(v)]$ for any vector $v\ in\ R^3$.

Theorem 1.2

Let V be an n-dimensional vector space over K, let S be a basis of V, and let M be the algebra of n X n matrices over K. Then the mapping:

$m: A(V) \to M$ defined by $m(T) = [T]_s$

Is a vector space isomorphism.

That is, for any $F, G \in A(V)$ and any $k \in K$,

I. $m(F + G) = m(F) + m(G)\ or\ [F + G] = [F] + [G]$
II. $m(kF) = km(F) or\ [kF] = k[F]$
III. m is bijective (one-to-one and onto)

Theorem 1.3

For any linear operators $F, G \in A(V)$

$$m(G \circ F) = m(G)m(F)\ or\ [G \circ F] = [G][F]$$

1.4. CHANGE OF BASIS

In this section, we shall answer the question "how do our representation change if we select another basis"

Definition: Let $S = \{u_1, u_2, \dots, u_n\}$ be a basis of vector space V, and let $S' = \{v_1, v_2, \dots, v_n\}$ be another basis. (for reference, we will call S the "old" basis and S' the "new" basis.) since S is a basis, each vector in the new basis S' can be written uniquely as a linear combination of the vectors in S; say,

$$v_1 = a_{11}u_1 + a_{12}u_2 + \cdots + a_{1n}u_n$$

$$v_2 = a_{21}u_1 + a_{22}u_2 + \cdots + a_{2n}u_n$$

$$\vdots$$

$$v_n = a_{n1}u_1 + a_{n2}u_2 + \cdots + a_{nn}u_n$$

Let P be the transpose of the above matrix of coefficients; that is, let $P = [p_{ij}]$, where $p_{ij} = a_{ji}$ Then P is called the change-of-basis matrix (or transition matrix) from the "old" basis S to the "new" basis S' .

The following remarks are in order.

Remark 1: The above change-of-basis matrix P may also be viewed as the matrix whose columns are, respectively, the coordinate column vectors of the "new" basis vectors v_i relative to the "old" basis S; namely,

$$P = [[v_1]_s, [v_2]_s, \dots, [v_n]_s]$$

Remark 2: Analogously, there is a change-of-basis matrix Q from the "new" basis S' to the "old" basis S. Similarly, Q may be viewed as the matrix whose columns are, respectively, the coordinate column vectors of the "old" basis vectors u_i relative to the "new" basis S' ; namely,

$$Q = [[u_1]_{s'}, [u_2]_{s'}, \dots, [u_n]_{s'}]$$

Remark 3: Since the vectors $v_1, v_2, \dots, v_n$ in the new basis S' are linearly independent, the matrix P is invertible. Similarly, Q is invertible. In fact, we have the following proposition.

Proposition 1: Let P and Q be the above change-of-basis matrices. Then $Q = P^{-1}$.

Example 6

Consider the following two bases of R^2

$S = \{u_{1\,,}u_2\} = \{(1,2),\ (3,5)\}$ and $S' = \{v_{1\,,}v_2\} = \{(1,-1),\ (1,-2)\}$

a) Find the change of basis matrix P from S to the new basis S'
b) Find the change of basis matrix Q from the new basis S' to the old basis S

Example 7

Consider the following bases of R^3

$E = \{e_1, e_2, e_3\} = \{(1,0,0),(0,1,0),(0,0,1)\},\ S = \{u_1, u_2, u_3\} = \{(1,0,1),(2,1,2),(1,2,2)\}.$

a) Find the change of basis matrix P from the basis E to the basis S

b) Find the change of basis matrix Q from the basis S to the basis E

1.5.